Smart Structures
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Week - 07
Lecture No - 37
Analysis of composite laminate with piezoelectric patches (continued)

In the previous lecture we started with analysis of composite laminates which are piezoelectric patches and we did the analysis using the Galerkin technique. Now, because there are piezoelectric patches there were terms like n p x, n p y, n p s similarly n p x, n p y, n p s which we are yet to define and these terms come because of the free strain in the piezos. Now, to define these terms first we need to define the free strains. Now, free strains are the ones which if we set the piezoelectric patch free and then under some electric field or voltage whatever the strain that happens that we call the as the free stress free strain. And then if we try to block it then the amount of force that is generated is block force or the corresponding stress we can call as block stress. So, here we have a piezo patch which is in two dimensions and we need to find out the free strains in both x and y we have nothing in z direction and that will have normal strain along x y and shear strain along in the x y plane.

So, we have and this we can call epsilon p x y also. So, these are the free strain components and from the constitutive relation if we take only the relevant part it looks like this 0 0 E 3. If we look at the constitutive relation that we defined and then if you take the relevant part of it and remove rest of the components which are leading to other strain components like any strain component the z direction which we do not need then rest of it looks like this. Then finally, this strain components are 0.

$$\begin{Bmatrix} \varepsilon_{px} \\ \varepsilon_{py} \\ \varepsilon_{ps} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{31} \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ E_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} d_{31}E_3 \\ d_{31}E_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_p \\ \varepsilon_p \\ 0 \end{Bmatrix}$$

$$d_{32} = d_{31}$$

$$\varepsilon_p = \varepsilon_{px} = \varepsilon_{py} = d_{31}E_3$$

$$\begin{Bmatrix} \sigma_{px} \\ \sigma_{py} \\ \sigma_{ps} \end{Bmatrix} = \frac{E_c}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_p \\ \varepsilon_p \\ 0 \end{Bmatrix}$$

$$= \frac{E_c}{1-\nu}\begin{Bmatrix} \varepsilon_p \\ \varepsilon_p \\ 0 \end{Bmatrix}$$

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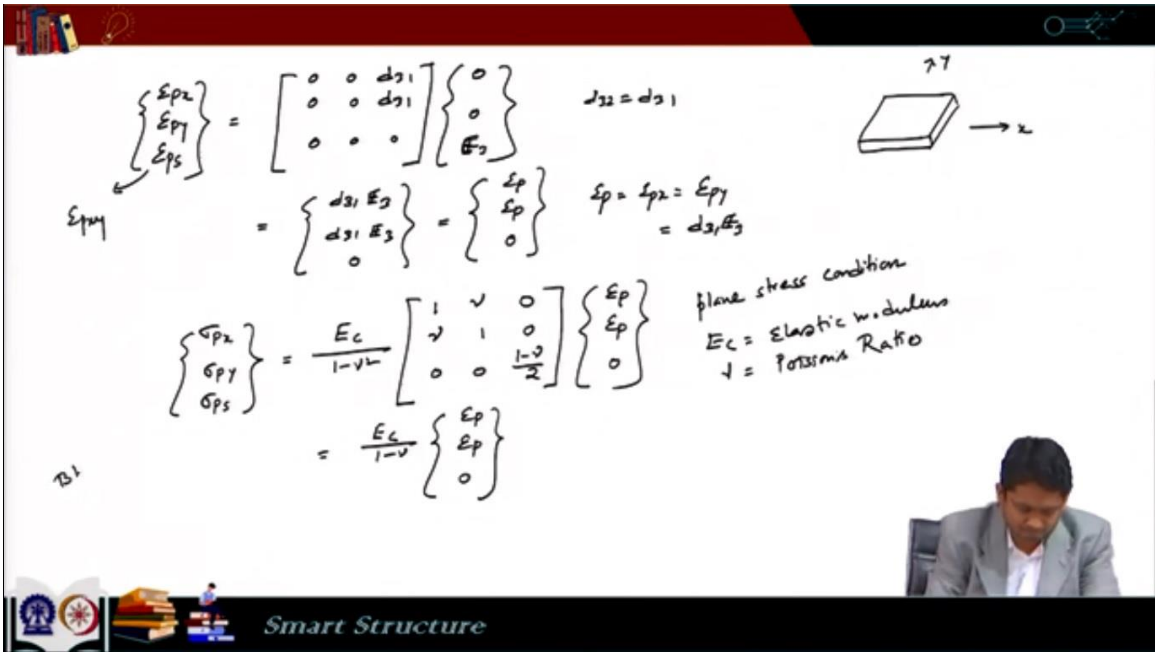

Now please understand this material is isotropic in the x y plane. So, that is why d 3 2 and equal to d 3 1. So, this tells me that the free strain there is no free shear component of the free strain and there are two normal components and they are both same. So, we can define that as just epsilon p as well epsilon p y and that is equal to d 3 1 E 3. So, now we can define the stresses also.

So, if this strain is if this stress is prevented sorry if this strain is prevented if the material is prevented prevented from any deformation then the stress that is generated we can call it a block stress that would look like this. Now this is the constitutive relation assuming that there is a plane the plane strain condition exist. So, plane stress condition not plane strain condition because these are the thin patches or our composite plies are also very thin. So, we consider plane stress condition in our composite plies as well as in the piezo patch. So, there is the constitutive relation for the plane stress condition.

So, nu nu E c is elastic modulus we have already dealt with and nu is Poisson's ratio. Now finally, if we simplify this expression it looks like this. So, these are the stresses. So, we can call these are these as block stresses. So, accordingly there will be block force.

$$\begin{Bmatrix} N_{xp} \\ N_{yp} \\ N_{sp} \end{Bmatrix} = \int \begin{Bmatrix} \sigma_{xp} \\ \sigma_{yp} \\ \sigma_{sp} \end{Bmatrix} dz = \begin{Bmatrix} \sigma_{xp} t_c \\ \sigma_{yp} t_c \\ \sigma_{sp} t_c \end{Bmatrix}$$

$$= \frac{E_c}{1-\nu}\begin{Bmatrix} d_{31}E_3t_c \\ d_{31}E_3t_c \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xp} \\ M_{yp} \\ M_{sp} \end{Bmatrix} = \int -z \begin{Bmatrix} \sigma_{xp} \\ \sigma_{yp} \\ \sigma_{sp} \end{Bmatrix} dz$$

$$= \frac{\varepsilon_p E_c d_{31} E_3}{1-\nu}\begin{Bmatrix} t_b + t_c \\ t_b + t_c \\ 0 \end{Bmatrix}$$

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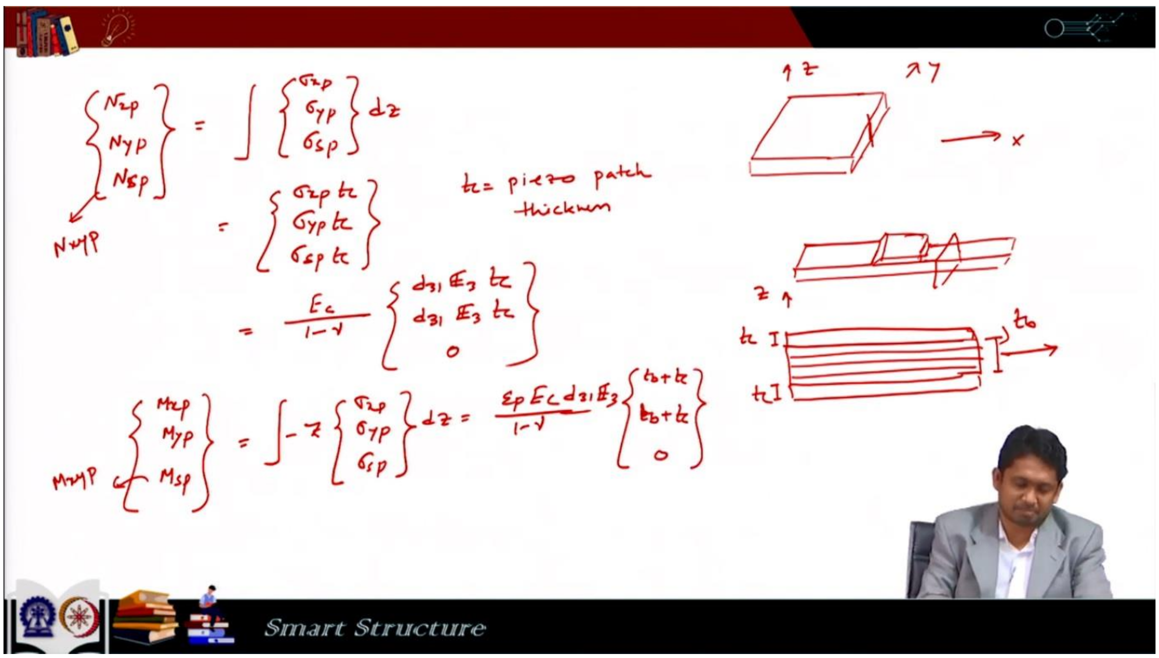

So, the force corresponding to these stresses are called block forces. Now here we will find out the forces per unit width. So, previously we had a beam problem and when we are solving the beam problem our beam was looking like this its width was much smaller than the than the length and other dimension. So, we are finding out the stress resultants N x N y N x M x and everything or the and the corresponding N x p M x p everything at one section at a section like this, but here instead of that we will find it these forces or moments per unit run along the width direction. So, we will integrate these quantities along the thickness only we will not integrate it along the width.

So, if we define the corresponding forces N s p this becomes sigma x p sigma y p sigma s p d z where z is the dimension along the vertical direction. And please understand this composite patch is mounted on a composite laminate we are only integrating over the thickness of the patch. And if we say that the thickness of the patch is T p T c then the block force becomes sigma x p t c sigma y p t c sigma s p t c t is piezo patch thickness. Now, we have already defined these quantities. So, these are 1 minus nu and then we have d 3 1 E 3 t c d 3 1 electric field t c and 0.

So, these are my N x p N y p and N s p which we used in our formulation. Similarly, there can be corresponding momentums and this we can denote as M x p M y p and M s p M s p is also M x y p similarly this is also N x y p. And this quantity is minus z sigma x p sigma y p sigma s p d z. So, if we assume that we have a composite laminate which looks like this in the x z plane and we have a piezoelectric patches at the top. So, these are laminates.

So, it can have several layers. Now, this part is the let us denote this part as t b like we have been doing. So, this part is the thickness of the laminate and this is our thickness of the piezoelectric patch t t c. And then if we assume that they are they have similar property and they are actuated using opposite voltage then accordingly we can find out these quantities and that becomes the corresponding moments. So, this becomes epsilon p E c by 1 minus nu we can multiply d 3 1 E 3 here and then we have t b plus t c t b plus t c and 0.

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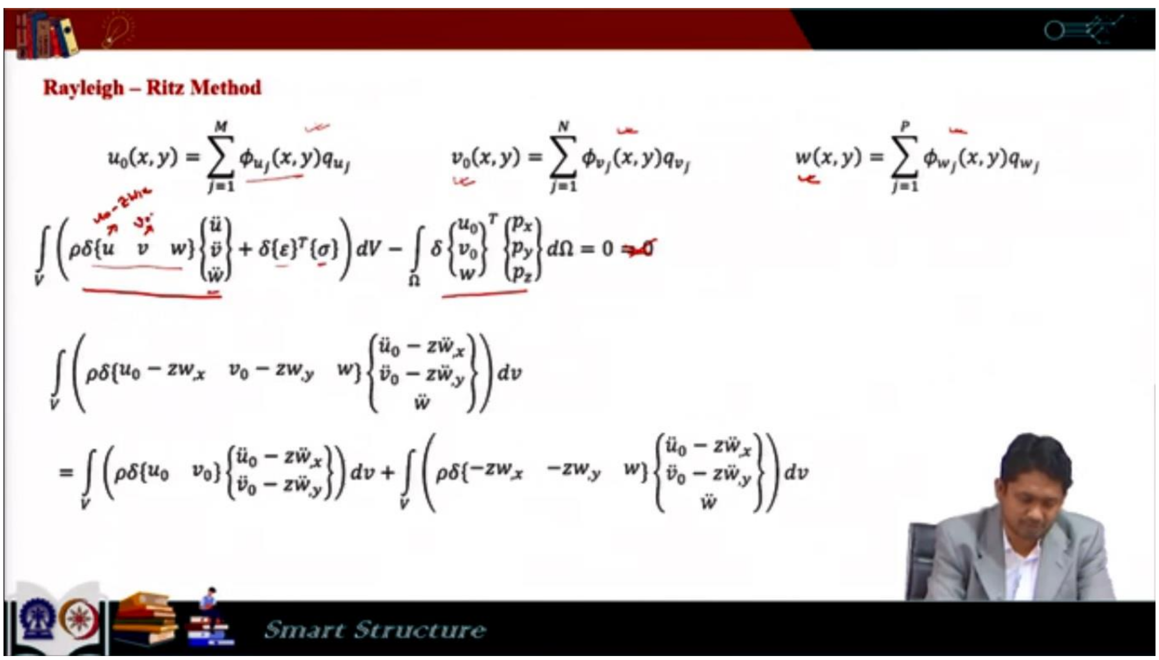

So, these are the corresponding M x p M y p and M s p and this we used in our Galerkin formulation. Now, we will solve the similar problem using the Rayleigh Ritz approach. So, in the Rayleigh Ritz approach, again we assume that our displacement components are approximated by this basis functions. So, u 0 which is the mid plane displacement along x is only a function of x and y and this we get as phi u j multiplied by q j phi u j is a known function of x and y. Similarly, we have v 0 and w 0.

Now, we put this thing in our energy expression. So, these are variational variational indicator this is analogous to the virtual work principle. So, here we have an inertia term. So, this formulation we are doing keeping it more generic. So, that if the problem is time dependent if the dynamic problem this can also be that can also be solved.

So, rho is the density of the material and then we have this variation of the displacement components and multiplied by the accelerations and here we have the variation of strain multiplied by the stress and here we have the virtual work due to the applied forces and this is this is equal to 0 that is what the principle says. Now, u we know it is u 0 multiplied by z into del w by del x. Similarly, v we know it is v 0 multiplied by v 0 v is equal to v 0 multiplied by z into del w by del y. So, that is what we are putting here and that is how all the terms are written. Now, we can write this thing separately.

$$= \int_{\Omega} \Big(m\ddot{u}_0 \delta u_0 + m\ddot{v}_0 \delta v_0 - S\ddot{w}_{1x}\delta u_0 - S\ddot{w}_{1y}\delta v_0 - S\ddot{u}_0 \delta(w_{1x}) - S\ddot{v}_0 \delta(w_{1y})$$

$$+ m\ddot{w}\delta x + I\ddot{w}_{1x}\delta(w_{1x}) + I\ddot{w}_{1y}\delta(w_{1y}) \Big)\, d\Omega$$

$$m = \int \rho dz, S = \int z\rho dz, I = \int z^2 \rho dz$$

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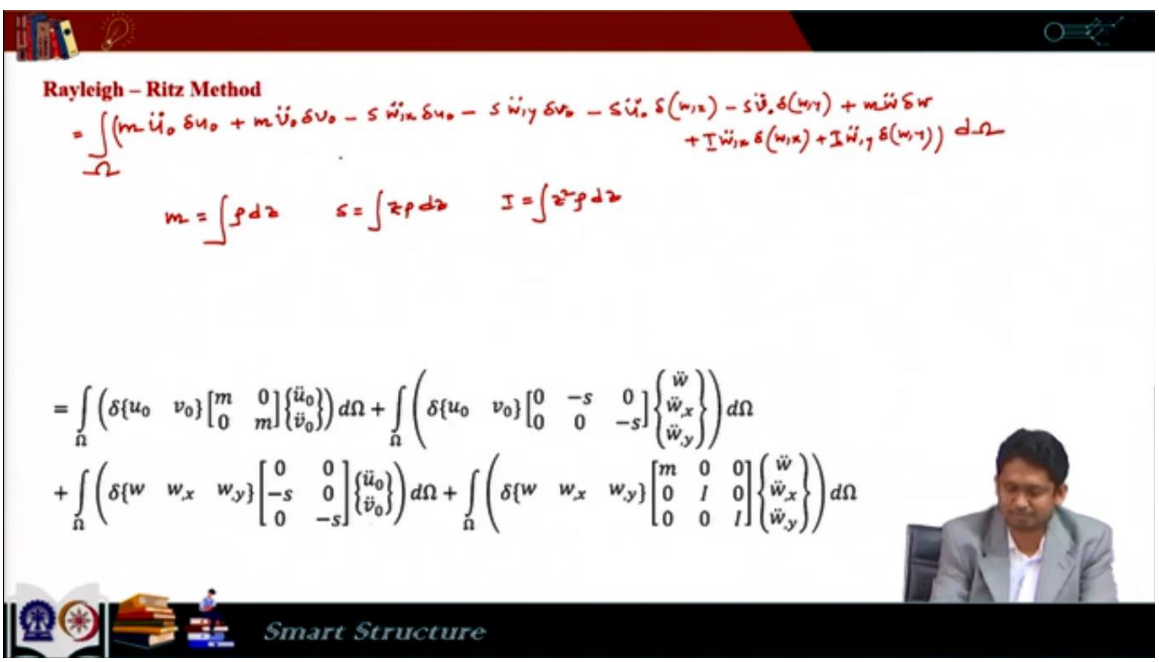

So, u 0 multiplied by this and v 0 multiplied by this this is written separately here and rest of it is put here. So, this term is broken into two parts here for the sake of doing some mathematical manipulations. Then if we evaluate those integrals and then finally, after integrating we get the terms to be m u 0 double dot del u 0 plus m v 0 double dot del v 0 minus s w comma x double dot del u 0 minus s w comma y double dot del v 0 and then we have minus s u 0 double dot variation of del w by del x minus s v 0 double dot variation of del w by del y plus m w double dot del w plus i w comma x delta of w comma x plus i w comma y delta of w comma y and this entire thing is integrated over the two dimensional domain. Now, here m is the integral of rho along z. So, if I if I integrate this density along the z whatever we call we we call it m.

So, in some sense it is mass per unit area. Similarly, we have s and which is z rho dz and we have i which is z square rho dz. So, what we have done is there it was a volume integral. So, after integrating the quantities over the depth we have got rid of the depth z dimension and it is converted to a surface integral. Now, these terms can be rearranged and written in this way.

So, if we can look when this gets multiplied with this matrix we get a term like m u 0 double dot and then it can get multiplied with del u 0. So, m u 0 double dot del u 0 comes from here. Similarly, we get m v 0 double dot and del v 0 that we get here and according we can verify that this term is a compact version of this entire expression. So, here we have the variation of u 0 v 0 here we have the variation of w and it is spatial derivatives. So, this is the same thing written here.

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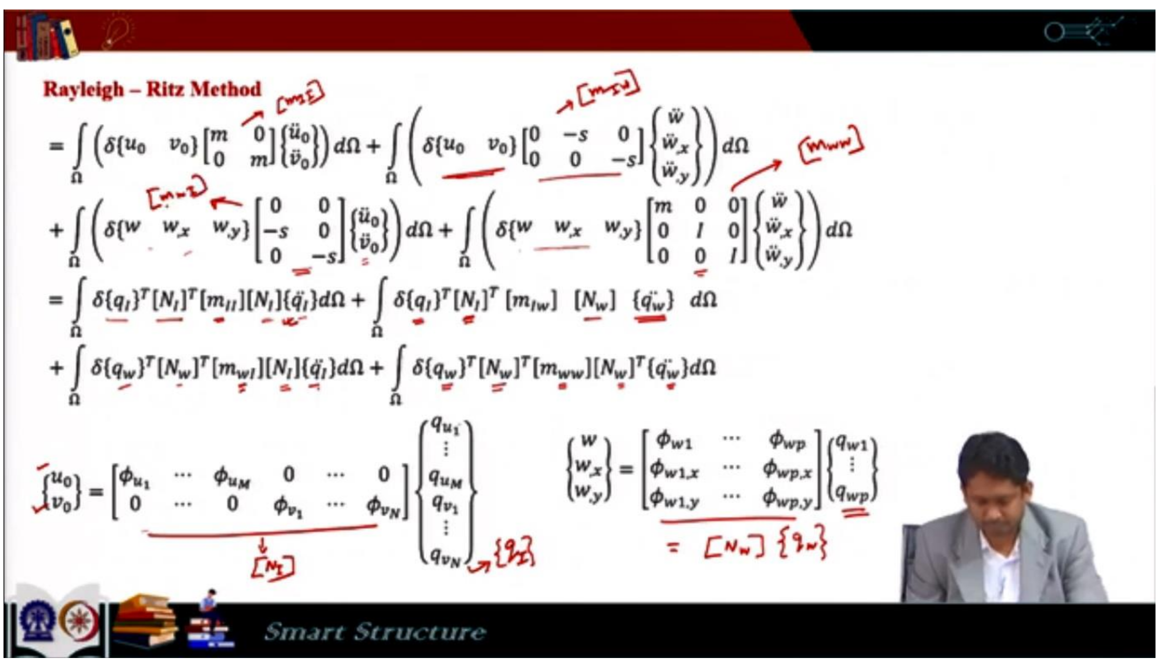

Now, we have already written u 0 and v 0 in terms of this phi's and q u and q v. So, we can write this in a matrix forms as this. So, we can we can take this vector which consists of q

1 to q m and q u 1 to q u m and q v 1 to q v n which we put as one vector and when this gets multiplied with this matrix we get u 0 and v 0. So, the terms from q u 1 to q u m gets multiplied with phi u 1 to phi u m and after being summed up it gives us m u 0 that is our approximation and similarly we get v 0 here. Now, let us define this vector as q i here i stands for in plane.

So, anything that involves u and v or u 0 and v 0 we call it we will put it a put their subscript of i, i means in plane and anything where like here we have components of w we will call that q w and let us call this vector as sorry let us call this matrix as N i. So, we have u 0 v 0 as a row vector here which means it is a transpose of this vector. So, this can be written as delta of q i t multiplied by delta of N i t. So, so this is u 0 v 0 is N i multiplied by q i and if I want to take a variation of it this these terms because these are all known to us it cannot be varied. So, this cannot be this can be varied.

So, it is N i multiplied by variation of q i and then when we take a transpose of this the interchange their place. So, it becomes delta of q it multiplied by N i t and then this matrix let us call it M i i. So, it is M i i because it we are putting two i's because at the both side we have only the in plane component. So, it is M i i and then again this is N i multiplied by q i double dot because u 0 and v 0 is q i multiplied N i multiplied by q i. So, u 0 double dot v 0 double dot is N i multiplied by q i double dot.

$$\int_V \delta\{\varepsilon_0\}\,[Q]\{\varepsilon_0\}dV = \int_\Omega \delta\{\varepsilon_0\}\,[A]\{\varepsilon_0\}d\Omega$$

$$\{\varepsilon_0\} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}$$

$$\{k\} = \begin{bmatrix} \phi_{w1xx} & \cdots\cdots & \phi_{wAxx} \\ \phi_{w1yy} & \cdots\cdots & \phi_{wP,yy} \\ 2\phi_{w1,xy} & \cdots\cdots & 2\phi_{wP,xy} \end{bmatrix} = [B_w]\{q_w\}$$

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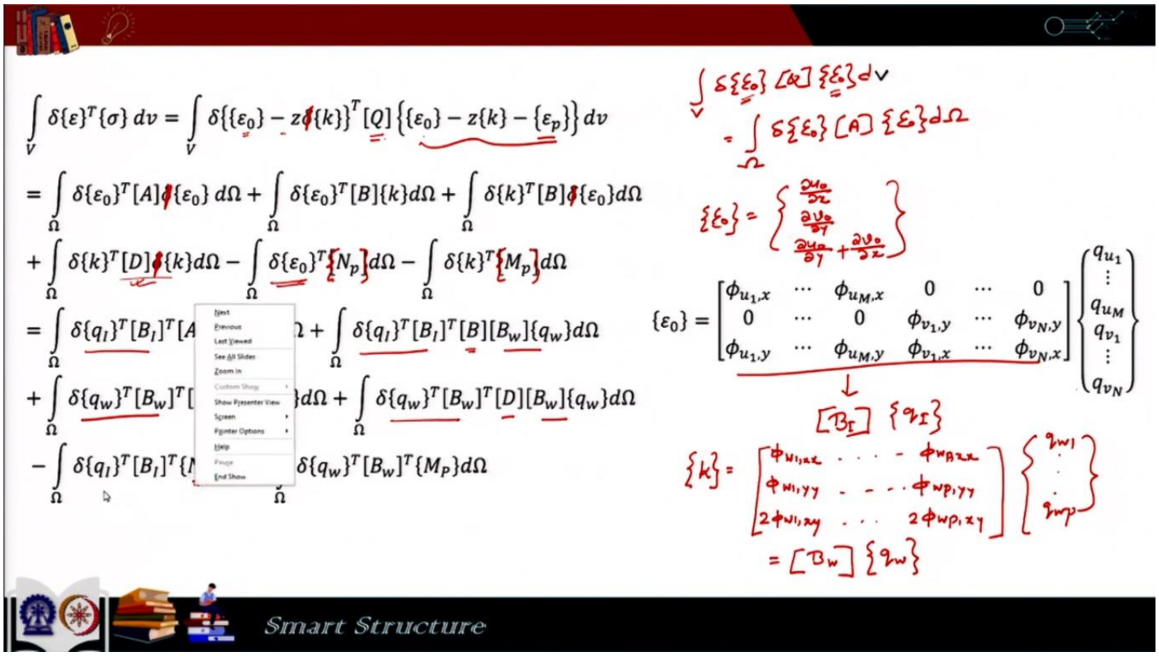

So, that is what we have put here. Now let us come to this term if you look at this term again we have delta of q i multiplied by q i transpose multiplied by N i transpose because of this this matrix let us define this to be M i w because it is at left side we have the in plane components at the right side we have the out of plane w components. So, it is M i w and then now comes this. So, this matrix this vector w w comma x w comma y we can write as this based on our approximation. So, this matrix we call N w and this vector we call q w.

So, that is and similarly when we take double dot here this N w remains N w and a double dot comes in q w. So, it becomes N w multiplied by q w double dot. Now if you look at this. So, here we have w w comma x w comma y as a row vector.

So, this has to be transposed. So, and it is variation has to be taken. So, delta q w transpose comes here N w transpose comes here and then we have this matrix and this we call N w i matrix. The reason is we have at the left hand side we have w terms at the right hand sides we have the in plane terms. So, we call it N w i i matrix. Now we can see that this M w i is transpose of M i w and there in the right hand side u 0 dot u 0 double dot and v 0 double dot.

So, it is N i multiplied by q i double dot. Now let us come here at the last last term. So, here we have w w comma x w comma y it is transposed transpose. So, it is as we know it is variation of q w transpose multiplied by N w transpose and this we call M w w because at it is left we have w terms at the right we have w terms. So, it is M w w and as usual we have N w transpose multiplied by q w double dot.

So, this is our M w w and this is our M w i. So, that is how the inertia related terms comes out to be. Now we need to deal with the terms that has the strain and strain and stress. So, we know that this strain epsilon consists of epsilon 0 multiplied by z into kappa. So, this delta should not be here because we already have a delta.

So, z multiplied by kappa and then it is transposed and then it is multiplied by the stress. So, stress is nothing, but q multiplied by our strain terms and the form the strain terms we need to separate out the free strain terms. So, that is what we see here. Now if we multiply this del epsilon 0 with q and epsilon 0. So, del epsilon 0 multiplied by q into epsilon 0.

Now please understand this q matrix for each ply is different and even we had a q matrix for the for the piezoelectric piezoelectric patch also after applying the plane stress condition which was E E by 1 minus nu square and then a matrix of 1 nu 0 nu 1 0 and 0 0 1 minus 0 0 1 minus nu by 2. So, these those are all q's for each and every layer. Now if we want to convert this to a surface integral we need to integrate these terms over the depth z. Now this term does not depend on z it is independent of z this is independent of z only q is dependent on z and we know that when we integrate q over the thickness of all the layers that gives us a matrix. So, it becomes del of epsilon 0 multiplied by a matrix multiplied by epsilon 0.

So, this is this was wrong excuse me for that this is del V over the volume and then we get this. Similarly when we get a term like delta of epsilon 0 multiplied by q multiplied by minus of z k we get delta of epsilon 0 B and it is a kappa. So, delta epsilon 0 multiplied by q multiplied by minus of z kappa then we get a term like delta epsilon 0 B multiplied by kappa because minus z multiplied by q on being integrated over the thickness gives us the B matrix. Similarly we get another term consisting of B matrix because we have z kappa multiplied by q into epsilon 0 and this term where where we are multiplying z kappa variation of z multiplied by variation of kappa into q into z kappa. So, there is z square into q and that on being integrated over the thickness would give me the D matrix.

So, we get this term. Similarly, we have this epsilon 0 multiplied by q multiplied by epsilon p and that gives us the N p matrix which we already derived. Now our epsilon p is valid only for the piezoelectric patches the in between the composite laminates that the composite plies they are not active. So, they do not have epsilon pi they do not have epsilon p. So, that gives us N p in fact, it is it is a vector and then we have M p because we have minus z and q and epsilon p they are multiplied and on being integrated that will give me M p which we have already seen. Now our job is to relate this epsilon 0s and kappas with the approximation that we made.

So, epsilon 0 as we know is del u 0 by del x del v 0 by del y and del u 0 by del y plus del v 0 by del x. So, if we put the if we use the approximation this entire thing looks like this or this vector looks like this. So, this again let us define this as B i. Now we have a B matrix here and that is which basically that relates that comes from integrating the composite entering the properties along the z direction. So, it it couples the moment and the inclined components.

So, that is that is a stiffness component and here we are putting B with suffix i, but this is nowhere related to this B. So, this B i is relating our strain components with the displacement approximation the constants associated with the displacement approximation. So, it is B i multiplied by this we already know it is Q i. Similarly the kappa matrix becomes if we use the approximations that were made for kappa it looks like this and here we have phi and this entire matrix is multiplied with Q w 1 all the way up to Q w p. So, this is B matrix ah we would like to call it B w because it involves the w components and this as we know it is Q w.

Again if we put all these definitions then we have epsilon 0 transpose and its variation. So, this looks like this A matrix remains A and again we get this here we have epsilon transpose is variation. So, same thing and here we have a B matrix and then here here we have kappa. So, kappa is B w into Q and this is delta k transpose. So, this thing just gets transposed here and one being transpose they change their places this Q w and B w and Q w is has a variation of delta.

$$\int_{\Omega} \delta \begin{Bmatrix} u_0 \\ v_0 \\ w \end{Bmatrix}^T \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} d\Omega = \int_{\Omega} \delta \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} d\Omega + \int_{\Omega} \delta w p_z d\Omega$$

$$= \int_{\Omega} \delta \{q_I\}^T [N_I]^T \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} d\Omega + \int_{\Omega} \delta \{q_w\}^T [N_w]^T d\Omega$$

$$[N_w] = [\phi_{w1} \quad \cdots\cdots \quad \phi_{wp}], w = [\phi_{w1} \quad \cdots \quad \phi_{wp}] \begin{Bmatrix} q_{w1} \\ \cdot\cdot \\ q_{wp} \end{Bmatrix}$$

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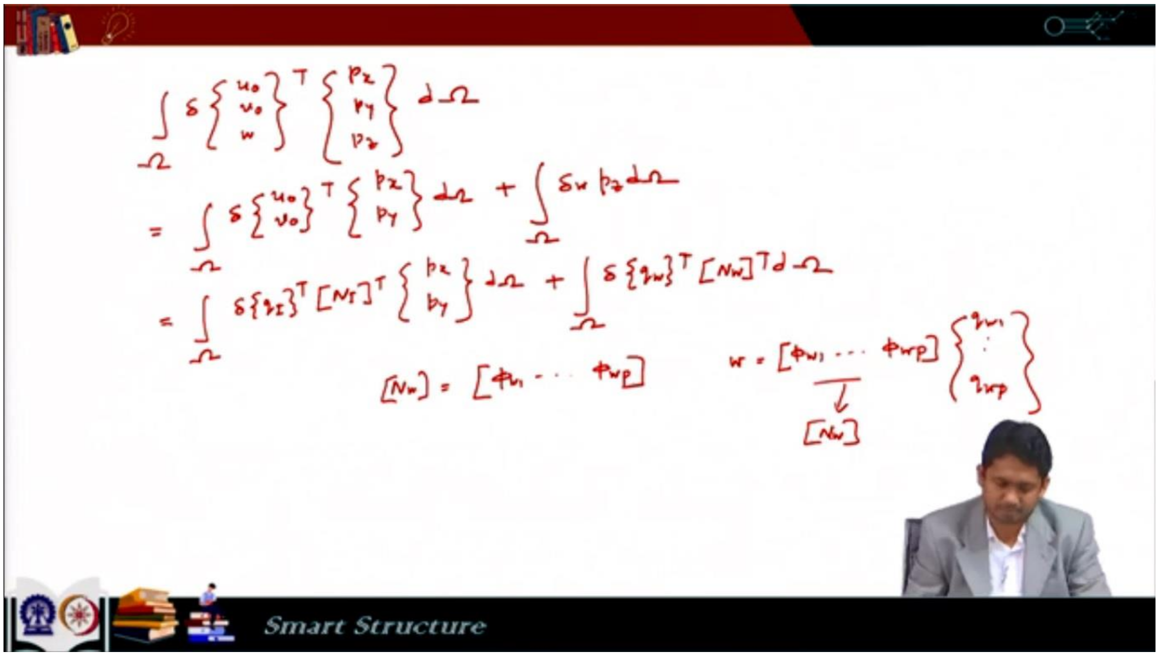

Then B multiplied by the epsilon 0 and here this comes from this term where you have the transpose of kappa D and kappa. So, again please excuse me ah we should not put so many variation symbols here. So, this variation symbol this variation symbol and this variation symbol is not should not be here variation should be only in this terms and accordingly we should not put variation here also. Now here we have the variation of epsilon 0 and its transpose and M p N p. So, N p remains as it is and variation of epsilon 0 comes here and here we have variation of kappa its transpose and this comes here and here we have M p.

So, this is about how it looks if we expand the terms and put all the approximations in delta of epsilon T multiplied by sigma its volume integral that is how it looks. Now we have to take care of the ah force terms the forces are applied at the surface and we assume that we have distributed forces p x p y and p z. So, this is p x p y and p z and this looks like this. So, this term after we break it up it would look like this and we have already made some approximation. So, this is q i T multiplied by N i T and we have p x p y and here ah for del w we can call this as N w 1 transpose of it ok.

We need to put the transpose and here we have N w T. So, where N w T N w is phi dot 1 to phi dot p because we know that our w is this as per our approximation and we are calling this as N w. So, we have taken care of the inertia terms the force terms and the internal virtual work terms. Now we need to put everything together and get the final equation. So, that we will do in the next lecture.

Thank you.