Smart Structures
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Week - 07
Lecture No - 36
Analysis of composite laminate with piezoelectric patches (continued)

Welcome to week 7.

In this week we will discuss about Analyses of Composite Laminates which can have Piezoelectric Patches in it. So, we can have a composite laminate or we call it a laminated composite plate which has several layer supplies and then it can have composite patch. This patch can be localized in a single region or it can be distributed all over it can be a top it can be at bottom. So, composite patch also becomes one more layer to it. So, we will see how we can analyze such laminates.

Now, to do this at first we need to see the governing equation of composites. So, to derive the governing differential equation of composites we can do it in two ways we can take a small part of the composite laminate and we will do the force balance or we can start from the energy equation and from the using the variation and principle we can find out the governing differential equations as well. Here we would look into the force or moment balance approach to find out the governing differential equations. So, let us imagine that we have a one small element of the entire composite laminate and this looks like this.

So, let us denote this as my x axis y axis and vertically up z axis and this small part can have a length of delta x this can have a length of delta y and this laminate can be subjected to various in plane in plane and out of plane forces also. So, in plane forces in plane distributed force let us assume there is p x and let us also assume that there is out of plane distributed forces which is we call p y and then there can be vertical forces distributed vertical force and vertically a force along z direction force along z direction and let us call it p z. Now let us look into the stress resultants. So, this phase can have a normal force N x this phase can have normal force N x plus del N x del x by del N x del N x by del x multiplied by delta x and similarly this force this phase can have N y and this phase can have N y plus its increment which is this. So, these are the normal forces there can be shear force.

So, this is our shear N s and here it is N s plus the increment. Similarly here we have N s and here we have N s plus the increment. Now let us show shear forces. So, the shear forces I am using a different color. So, shear force vertical V x.

So, it is a shear force in either x z or y z plane which we are calling V x and V y. So, this can be V x plus del V x by del x multiplied by delta x and similarly here also V y and here can be V y plus del V y into delta y and at the same time this there can be stress resultants in the form of bending or twisting moments. So, to denote the moments we would use arrow with a double head and this would follow the right hand rule. So, if we say moment like this which is denoted by an arrow which is in this direction denotes a moment like this. So, if we see from this side the moment is an anti-clockwise moment.

Now, if you want to draw the bending and twisting moment for these cases. So, if you draw M x the M x would look like this. So, this is m x again please understand as was said before this M x it acts with respect to y axis, but because we get this moment M x by integrating sigma x after multiplying with z we call it M x. And similarly we have M x plus del M x by del x multiplied by delta x. Similarly, there can be M y and M y is at this phase is M y at this phase is ok.

So, in this case the direction of the arrow is this. So, this is our M y and in this case the direction of the M y is M y plus its increment del y multiplied by del y. And then we have the moment M s that is a twisting moment and the twisting moment is this one M s and then we have m s plus delta M s by delta x del x multiplied by delta x. And similarly there can be m s here and there can be M s plus delta M s by delta y multiplied by delta y. Now we can look into the equilibrium of this entire system.

$$\sum F_x = 0 \Rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_s}{\partial x} + p_x = 0$$

$$\sum F_y = 0 \Rightarrow \frac{\partial N_s}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0$$

$$\sum F_z = 0 \Rightarrow \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_s}{\partial y} + V_x = 0$$

$$\frac{\partial M_s}{\partial x} + \frac{\partial M_y}{\partial y} + V_y = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_s}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - p_z = 0$$

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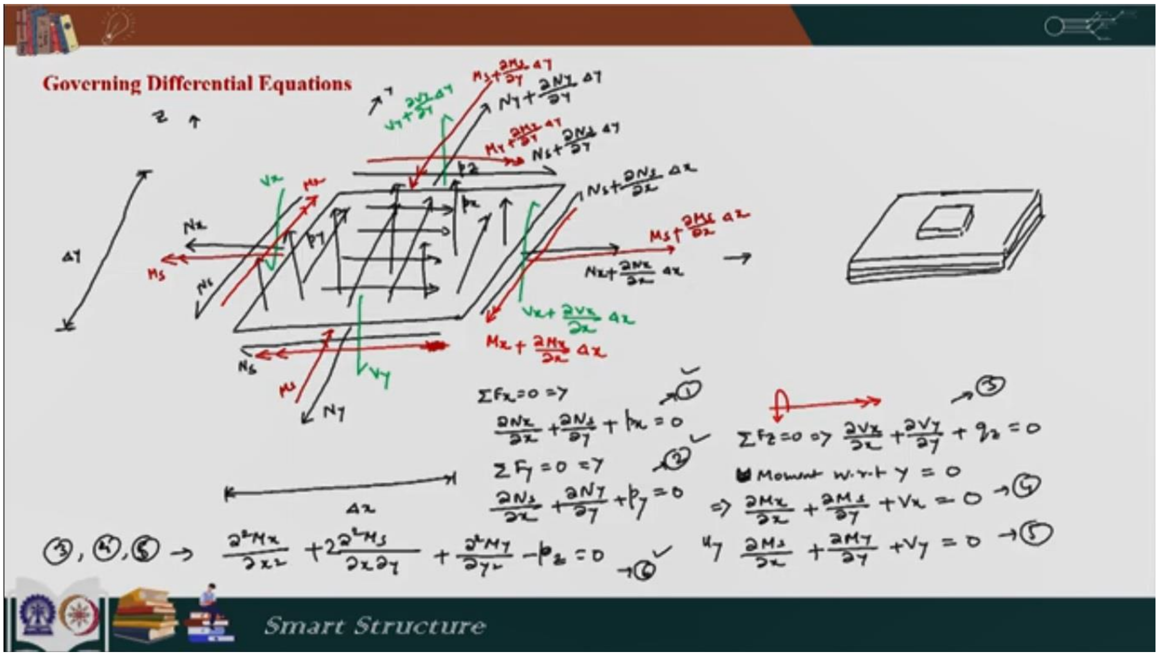

So, if we do summation of all the forces equal to 0 that would give me this equation del M x by del x plus del M s by del y plus p x equal to 0. Similarly if we do the summation of forces along y direction to be 0 that gives me del M s by del x plus del M y by del y plus p y to be 0. And then if we have summation of forces along z direction to be 0 then we get del V x by del x plus del V y by del y plus q z to be 0 and then if we take bending moment 0 anywhere. So, we can take bending moment with respect to y axis let me write moment w with respect to y equal to 0 and then that gives me del M x by del x plus del M s by del y plus V x equal to 0. And similarly ah along y axis we get del M s by del x plus del M y by del y plus V y equal to 0.

So, this we can call equation 1, this we can call equation 2, this equation 3, then we have 4, then 5. So, combining 3, 4,5 so, 3, 4, 5 in combination gives del 2 M x by del x 2 plus twice of del 2 M s by del x del y plus del 2 M y by del y 2 minus p z equal to 0. So, this equation 1 2 and 6 are our governing differential equations shear we have the vertical shear we have eliminated because we are combining equation 3, 4 and 5 and we are getting a new equation 6. So, in terms of the stress result N x, N y, N s, M x, M y and M s we have this 3 equations 1 2 and 6 which needs to be solved to find out the behavior of composite laminate. Now we will solve the laminate problem using various methods.

Now we will solve these laminate problems. So, these are the governing differential equations and we already know that for composites we have M-E-D matrices that relates our strain components in terms of the stress resultants. So, we will substitute those relations in place of N x, N y, N s, M x, M y, M s and also if we assume that ah there is a there are there are piezoelectric materials and it is under induced strain actuation then we get forces like N x p, N y p, N s p and which we need to incorporate and we can write the equation in this form. So, this equation becomes this. So, forces corresponding to free strain are our N x p and N s p and similarly we will have similarly we will have N y p and M x p M y p and M s p.

Now when we have composite ah composite laminates with piezoelectric patches at the top or at the bottom. So, depending on how many patches we have. So, this patch tries to expand or contract. So, previously for the beam problems we had only the expansion or only the expansion or one on expansion contraction only on one direction. Here it can have this x y in both the directions and accordingly it induces ah it induces both block force ah in x direction y direction as well as shear block forces and accordingly it can induce bending moments in M x, M y and also in m s a twisting moment.

So, the expressions for N x p and N y p, N s p we will see it later on. So, just now let us ah assume that there are piezoelectric patches and we know the expressions for N x p, N s p and N y p and if we know them, they are accounted in the differential equation in this form. So, we will see this see what N x p, N y p, N s p are in terms of the free strains later on. So, first equation becomes this the second equation becomes this after you substitute N x, N y, N s by the relations that we get in terms of ABD AB matrices and the third equation is this. So, all these equations so, initially all the all the equations were written.

So, these three equations are written in terms of the stress resultants and finally, these equations are written in terms of the displacement components u 0 v 0 and w. So, this previously for a beam problem there could have been an extension along x direction and there could have been a bending. So, extension along x direction a pure extension we defined as u 0 and vertical displacement due to bending was w. In this problem we have u 0 v 0 and w because there can be extension or contraction both in x and y direction and there is a w the displacement along the z direction. So, we can call it mid plane displacement along x this is mid plane along y and this is vertical displacement.

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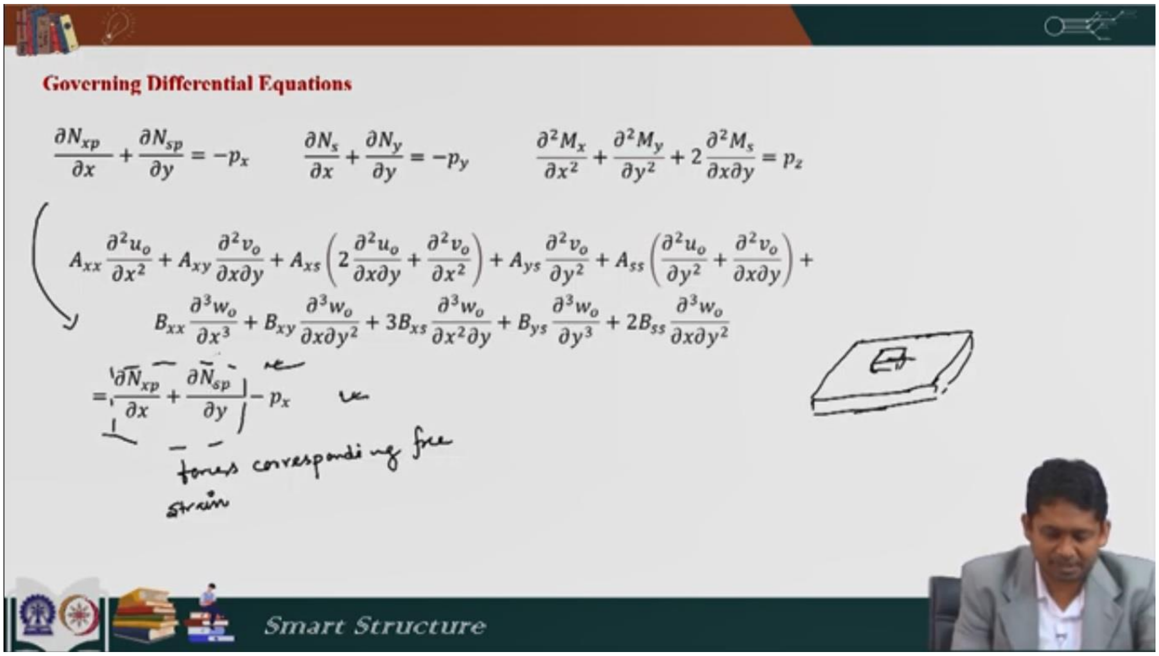

Now, let us write this entire big governing differential equation in a much shortened form. So, here we are defining a matrix of operators and we are defining each operator by this symbol double stroke D. So, D u 1 means this so, when D u 1 operates over something. So, D so, if I look at this entire matrix multiplication this so, here we should have u 0 and v 0. So, u 0 gets multiplied with D 1 D u 1 which means u 0 is operated by this double stroke D u 1.

So, this becomes A x x multiplied by del 2 u 0 by del x 2 plus 2 x x multiplied by del 2 u 0 by del x del y plus A s s multiplied by del 2 u 0 by del y 2 and accordingly D v 1 is ah v 0 is operated by D v 1. So, the operator is here and w is operated by D w 1 in the first equation and likewise all the operators are defined here and they can be written just by looking into these equations. And these are the right hand side terms. So, these are the corresponding force terms that comes due to the block forces and moments from the piezoelectric patches and we will define this in terms of the free strengths later on. Now to apply the Galerkin technique let us imagine that our u 0.

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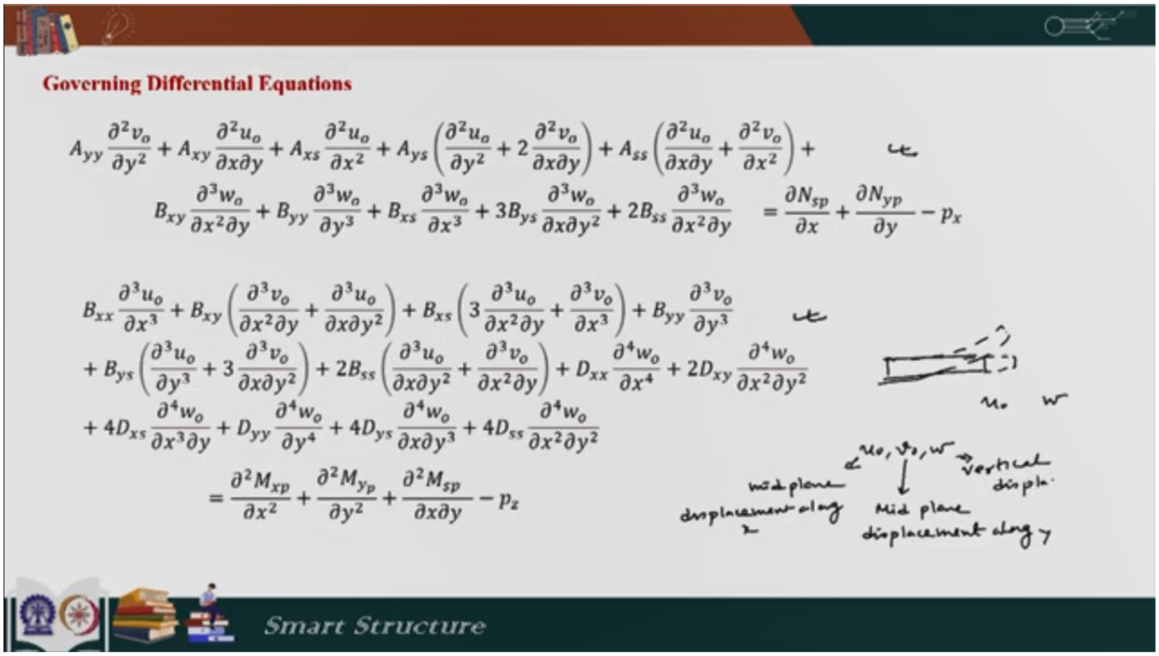

So, these are all u 0 because these are the mid plane displacements u 0 is a varied combination of these basis functions. So, you have phi u j which is a function of x and y multiplied by q u j and then we sum over 1 to M. So, M is the number of terms which is used to approximate u 0. Similarly, v 0 is varied combination of this function where the weight is q v j and these are the unknown's N is the number of terms to approximate v 0, w is this likewise and P is the number of terms to approximate w. So, our unknowns in this problem are q u j q v j and q w j.

So, we can write this in a much-shortened form as this where our capital U matrix capital U is u 0 v 0 and w. So, we write this as phi u 1 up to phi u M then all 0s then we have 0 phi v 1 up to phi v N then all 0s and then we have 0s here phi w 1 phi u 2 p and this gets multiplied with q u 1 up to q u M, q v 1 up to q v N and q w 1 up to u w P and this entire thing we write as abig phi matrix multiplied by q. So, if we use the symbols for matrix and vector this looks like this where phi is this matrix of this basis functions phi's. So, now, rest of the procedure as we discussed for 1D problems we follow the similar procedure and so, we define an error function and the error function is if we go back to the actual equation error function is just this we subtract the right hand side from the left hand side and ideally it should be 0, but because our solutions are approximate there is some error and that is our error function. So, let us define this this matrix as an operator matrix D and this we have already defined as U and let us define this as a matrix F p.

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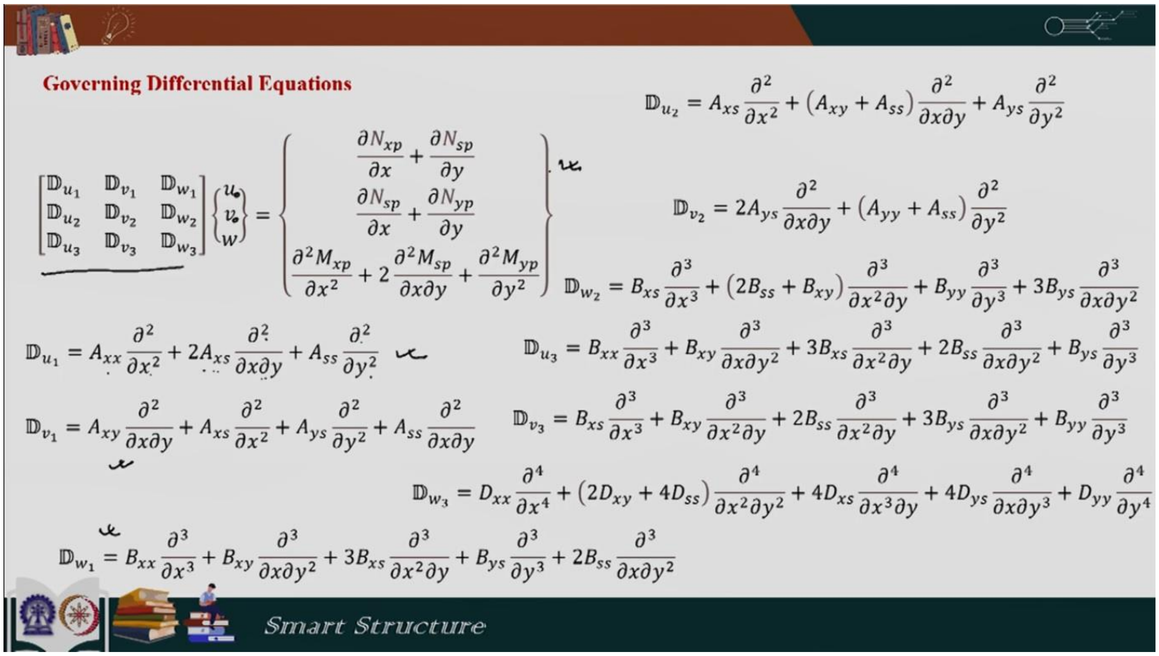

So, our finally, the error function looks like this double stroke D matrix multiplied by U minus F p and this error function is nothing, but again D for U we substitute this phi matrix then we multiply this q minus F p. So, this is our error. So, we let us define it as this epsilon and then what we do is we multiply phi j transpose. So, the j th phi and we transpose it and then we multiply with the error epsilon x y and then we equate that to 0 and we do it for each and every j. So, there are total M plus N plus P number of those j's and that will give us M plus N plus P number of equations and we have M plus N plus P number of unknowns.

So, finally, so here we do it j is equal to 1 to M plus N plus P and then finally, we have following the same procedure that we did for the two one dimensional cases we get the same similar kind of mat matrices and vectors. So, here our K i j is phi i transpose and then we have that operator matrix which we already defined in a much shorter form. So, let us use that instead of writing the whole matrix and then this multiplied by the phi j vector and integrate on omega. So, here we are integrating over omega. So, omega means the domain of the entire surface.

$$U = \begin{Bmatrix} u_0 \\ v_0 \\ w \end{Bmatrix} = \begin{bmatrix} \phi_{u1} & \dots\phi_{uM} & \dots\dots & 0 \\ 0 & \dots\dots\phi_{v1} & \dots\phi_{vN} & \dots 0 \\ . & . & . & . \\ 0 & \dots\dots & 0..\phi_{w1} & \dots\phi_{wp} \end{bmatrix} = \begin{Bmatrix} q_{u1} \\ q_{uM} \\ q_{v1} \\ q_{vN} \\ q_{w1} \\ q_{wP} \end{Bmatrix} = [\phi]\{q\}$$

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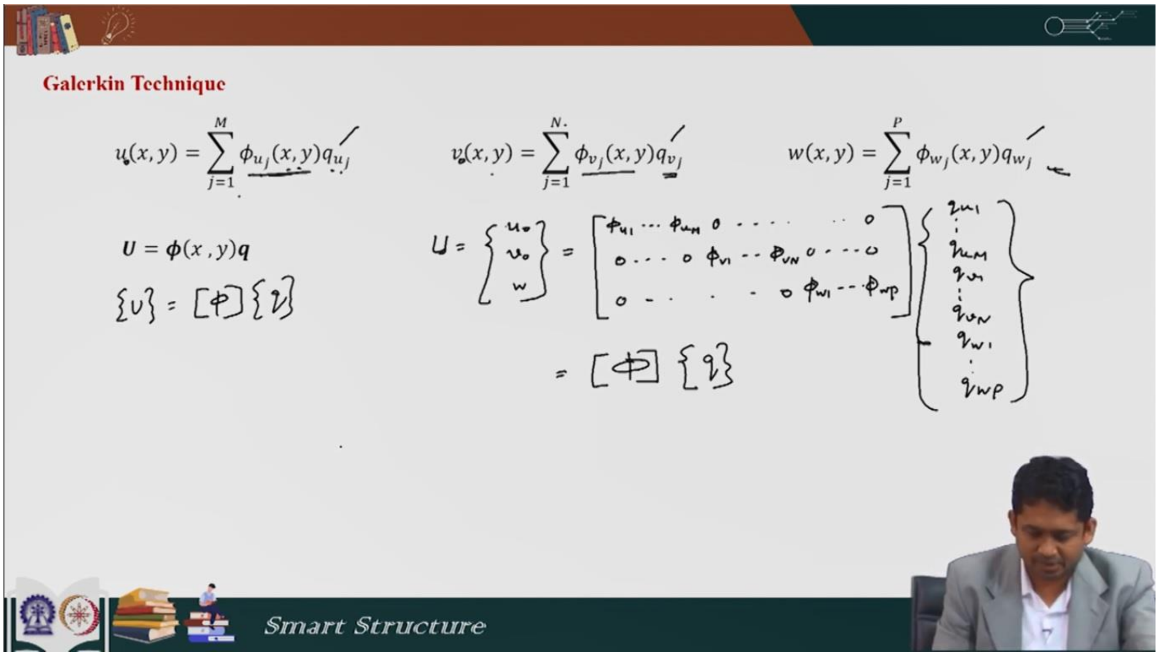

$$\{\varepsilon\} = [D]\{U\} - \{F_P\} = [D][\phi]\{q\} - \{F_P\}$$

So, if our piezoelectric laminate is like this. So, if it goes from x y z if we just look at the 2D surface of it. So, if it goes from suppose its dimension is a here the dimension is b. So, here omega would mean the domain spanned by the it is spanning from 0 x equal to 0 to a and y is equal to 0 to b and again it need not be a rectangular or square it can be any curved domain also.

So, omega takes care of that. So, omega is the 2D surface because we have already written the governing differential equation in terms of 2D we have already get rid of the third dimension because we already wrote it in terms of N x N y N s M y M s and M x. So, those had the stress components integrated over the thickness. So, the thickness is not coming into picture. So, here we are dealing with only the 2D surface domain of it just like that for a beam problem we got rid of the thickness by integration and we dealt with the governing differential equation only on x axis here it is spanned by x and y axis and that is our capital omega. So, accordingly we can now write F p i the ith component of the force vector which is Fp omega.

$$\int_{\Omega} \{\phi_j\}^T \{\varepsilon(x,y)\} d\Omega = 0, j = 1.....(M+N+P)$$

$$[K]\{q\} = \{F_{P\phi}\}, K_{ij} = \int_{\Omega} \{\phi_i\}^T [D]\{\phi_j\} d\Omega = 0$$

$$F_{P\phi_i} = \int_{\Omega} \{\phi_i\}^T \{F_P\} d\Omega$$

$$N_{xyP} = N_{sP}, M_{xyP} = M_{sP}$$

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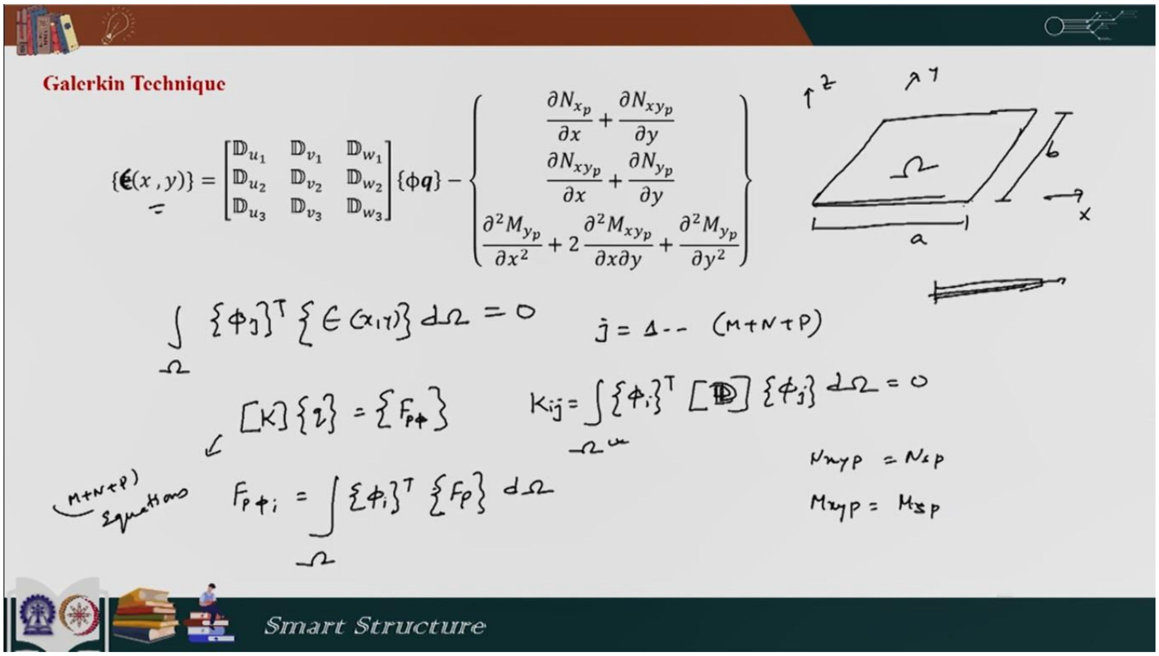

So, this gives rise to total M plus N plus P equations and by solving this M plus N plus P equations we can get those M plus N plus P number of q's which are the unknowns in the problem once we get those M plus N plus p number of q’s we can substitute back and find out our u 0 v 0 and w and that solves the problem. So, here the only thing is that we have not defined our N x p and N x y N x p and N x y p and so and N y p yet which will define in the next class. Now here please understand our N x y p is equal to N s p also and M x y p is equal to M s p also. So, would mostly use the notation where we denote this as N s or M s instead of N x y or M x y. So, this I would conclude this lecture here in the next lecture we will see more into this analysis.

Thank you.