Smart Structures

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Week 06
Lecture No: 34
Mechanics of FRP Composite Laminate Numerical Examples
Part 05

In today's lecture, we will discuss about Composite Laminates.

So, far we have seen the properties of composite plies or lamina. In the last lecture, we defined the properties with respect to the x y, which is a global system. Now, today we will discuss the properties of a laminate with respect to the global system. And we will see, the relation between the stress resultants in the laminates with the strains.

So, our composite laminate looks like this. It is made of several plies. One more ply. And then, we know the properties of the individual plies. We know their stress strain relation with respect to their own fiber orientation direction, and with respect to a global x y direction as well. So now, today we will integrate the quantities along the thickness of the ply, thickness of the laminate, and we relate the stress resultants in terms of the strain components and the strains are already related to the displacement components. So finally, it would relate the stress resultants with respect to the displacement components.

Now, we have sigma x, sigma y and sigma s as our stress components. So, if we integrate this sigma x along the thickness, so z is the thickness direction. So, if we integrate the properties that the sigma x along the thickness direction, that gives us our N_x, which is normal force along x. If, we integrate sigma y along thickness direction that give us N_y, which is normal force along y direction, and if we integrate sigma s along z that gives us N_s, which is shear force. And then, if we multiply sigma x with minus z, and integrate that gives us a moment M_x.

Now, please understand this if we integrate sigma x after multiplying with minus z along the z direction, the moment that we get is with respect to the y direction. So, this moment is acting with respect to y direction. So, it's a moment like this. It's a bending moment like this. It acts with respect to y direction, but still instead of calling it M_γ, we are calling it M_x just because we are getting it by integrating sigma x. Similarly, we get another quantity M_y and this acts with respect to x direction, but still because we are getting it by integrating sigma y we are calling it M_γ and we get a moment M_s which is a twisting moment in the xy plane. So, today we have to get these quantities and relate this quantities N_x, N_γ, N_s, M_x, M_γ, M_s with respect to the strain components and accordingly we this relation comes with respect to the displacement components as well.

$$\int \sigma_x dz = N_x \quad \int -z\sigma_x dz = M_x$$
$$\int \sigma_y dz = N_y \quad \int -z\sigma_x dz = M_x$$
$$\int \sigma_s dz = N_s \quad \int -z\sigma_x dz = M_x$$

Now, before doing that, let us define few things. So, let us look at the stack of plies as this. So, we are looking it just from one side. So, we get to see something like this. So, we have plies. So, this is layer 1, layer 2, we can call it generic layer as layer k and then we have N number of layers and we define the mid plane as our axis.

So, with us we define z with respect to the mid plane. So, this can be x or y axis depending on from which direction we see it and our z goes here. So, this point, the z coordinate of this point we call z_0, the z coordinate of this point we call z_1, z_2 and this is the kth layer. So, here we have z_{k-1}, here we have z_k, here we have z_{N-1}, here we have z_N. So, we can say that the z coordinate, z coordinate of the junction between the k minus 1 and kth layer is z_{k-1} and the z coordinate of the junction between the kth and the k plus 1 th layer is z_k. Now, if we want to find our N vector. So, N vector is just a vector of N_x, N_y and N_s. So, if you want to get that we just have to integrate our sigma vector and sigma is just this. So, this is sigma x, sigma y, sigma s.

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \int \{\sigma\} dz = \int \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} dz$$

Now, this integration is from here to here, along the entire thickness. So, what we will do is - we will break the integration into different, I mean, we will break the integration into different plies. So, we will integrate over each ply and then sum it up. So, we can write this integration as k is equal to 1 to N, z_{k-1} z_k because kth ply spans from z_{k-1} to z_k and then we integrate.

$$\{N\} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \{\sigma\} dz$$

Now, for the kth ply, we can write our stress as the Q matrix with respect to xy system for the kth ply multiplied by epsilon. This relation we already derived in the last lecture.

$$\{N\} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} [Q]_{x,y}^{k} \{\varepsilon\} dz$$

Now, if we write the strain displacement relation, we know that epsilon x is equal to del u by del x minus z multiplied by del 2 w by del x 2 and we call it epsilon x 0 minus kappa x, or let us put the 0 in the superscript as well as epsilon 0 x.

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_{0x} - z\kappa_x$$

Accordingly, we already know, we are, our assumption is - there is no shear deformation. So, as per this we can write epsilon y which is normal strain along y direction in the same way and that becomes epsilon 0 y minus kappa y. So, we are calling del 2 w by del x 2 as kappa x, and del 2 w by del y 2 as kappa y. I have to multiply by minus z.

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_{0y} - z\kappa_y$$

And accordingly, we can write epsilon s is equal to epsilon 0 by epsilon y plus, sorry, del u_0 by del y plus del v_0 by del x minus twice z del 2 w by del x del y. So, this becomes epsilon 0 s minus z into 2 kappa s.

$$\varepsilon_s = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \varepsilon_{0s} - 2z\kappa_s$$

So, we are calling twice del 2 w by del x del y as kappa s. So, this entire epsilon vector can be written as epsilon 0 minus z into kappa. So, epsilon 0 is a vector of epsilon 0x, epsilon 0y, epsilon 0s and kappa is a vector of kappa x, kappa y and kappa s.

$$\{\varepsilon\} = \{\varepsilon_0\} - z\{\kappa\}$$

So, we can bring everything here and write k is equal to 1 to N within each layer we integrate from z_{k-1} to z_k and this can be written as multiplied by epsilon 0 minus z kappa and this entire thing is integrated. So, these are the relations strain displacement relations using which we can write this.

$$\{N\} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} [Q]_{x,y}^k \{\varepsilon_0\} - z[Q]_{x,y}^k \{\kappa\} dz$$

Now we are integrating this Q's over thickness and summing it up. Now this epsilon 0 does not is independent of z. So, it can be taken out of the integral. So, these Q's are integrated over the thickness and summed up. Similarly, here, kappa is independent of z. So, it can be taken out. So, minus z Q can be integrated over the thickness and can be summed up.

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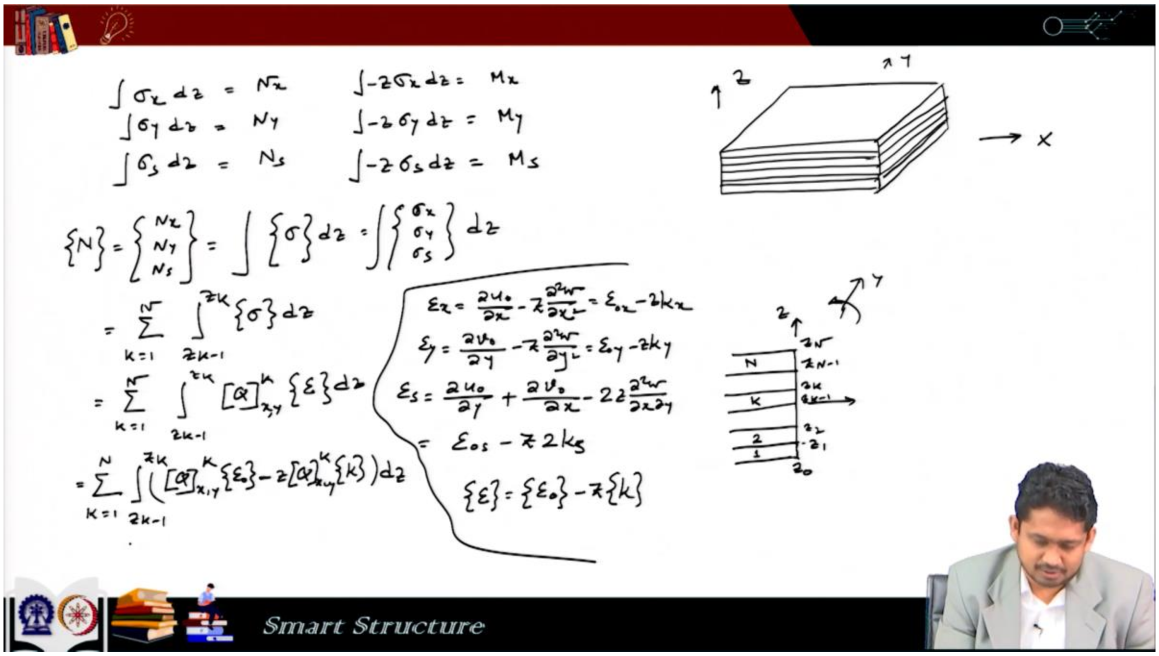

So, finally, we can write a relation like this, where our A is summation of k is equal to 1 to N of the integral of k from z_{k-1} to z_k. And similarly, we can define a quantity B which is similar integration but the integrand is minus z multiplied by Q. So this is one relation. So, our N vector is equal to A epsilon plus B kappa.

$$\{N\} = [A]\{\varepsilon_0\} + [B]\{\kappa\}$$

$$[A] = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} [Q]_{x,y}^{k} dz \qquad [B] = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} -z[Q]_{x,y}^{k} dz$$

Now we define another vector which is M, and M as we know is it obtained by integrating minus z into stress.

$$\{M\} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} -z\{\sigma\} dz$$

So, now we can write same thing, minus z into stress. And this becomes minus z into minus z into Q multiplied by epsilon 0 plus z square multiplied by Q into kappa and this entire thing is integrated and summed over.

$$\{M\} = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left(-z[Q]_{x,y}^{k}\{\varepsilon_0\} + z^2[Q]_{x,y}^{k}\{\kappa\}\right) dz$$

Again epsilon 0 goes out and we are left with minus z into Q and we know that if we integrate minus z into Q and summed up, we get a quantity called B that we have already defined it. So, this is B into epsilon 0 and here, we have z square Q, its integration and its summation and let us call this quantity as D.

$$\{M\} = [B]\{\varepsilon_0\} + [D]\{\kappa\}$$

So, let us define D as k is equal to 1 to N. Integral of z square Q over z_{k-1} to z_k.

$$[D] = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} z^2 [Q]_{x,y}^{k} dz$$

Now, we can always write this entire quantity in a more compact form as this. So, this is how the stress resultants which are forces and moments are related to the strain components.

$$\begin{Bmatrix} \{N\} \\ \cdots \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & \vdots & [B] \\ \cdots & \cdot & \cdots \\ [B] & \vdots & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_0\} \\ \cdots \\ \{\kappa\} \end{Bmatrix}$$

Now we can see that the certain parts of this matrices become 0 or there are certain relations that comes up between these parts of matrices depending on the lamination sequence. So, we will now see that. So, for example, we can see here that this B matrix relates the in plane force components with the out of plane displacements because, these are the in plane force components, these are the bending moments. Here, we have epsilon 0, it has u_0 and v_0 only. So, again it has in plane displacement components and kappa has out of plane displacement which is w. So, u v w are displacement along x y and z, and u_0 v_0 are the mid plane displacement components and w is the out of plane displacement component. So this B matrix signifies that even if there is no in plane displacement components, suppose in plane displacement component is 0, even then, just out of plane displacement components because of this coupling can give rise to something N. Or in other words, if I apply pure in plane load, then also some kappa which contains out of plane displacement can be induced. Similarly, here M, if I do look at this relation, M is equal to B into epsilon 0 plus D. This also means that if I just apply pure bending, I can get in plane displacement components and in plane strength as well. So, A is our in plane stiffness, D is our bending stiffness and B is coupling stiffness. So, A signifies in plane stiffness, B signifies coupling stiffness we can say and D signifies bending stiffness. And also, we can see that this A is a 3 by 3 matrix. So, it has components like A_{xx}, A_{xy}, A_{xs}. So, a quantity like A_{xs}, it couples the normal force components with the shear strength. Similarly, I can have D_{xs} also and that couples bending moment with the twisting moment.

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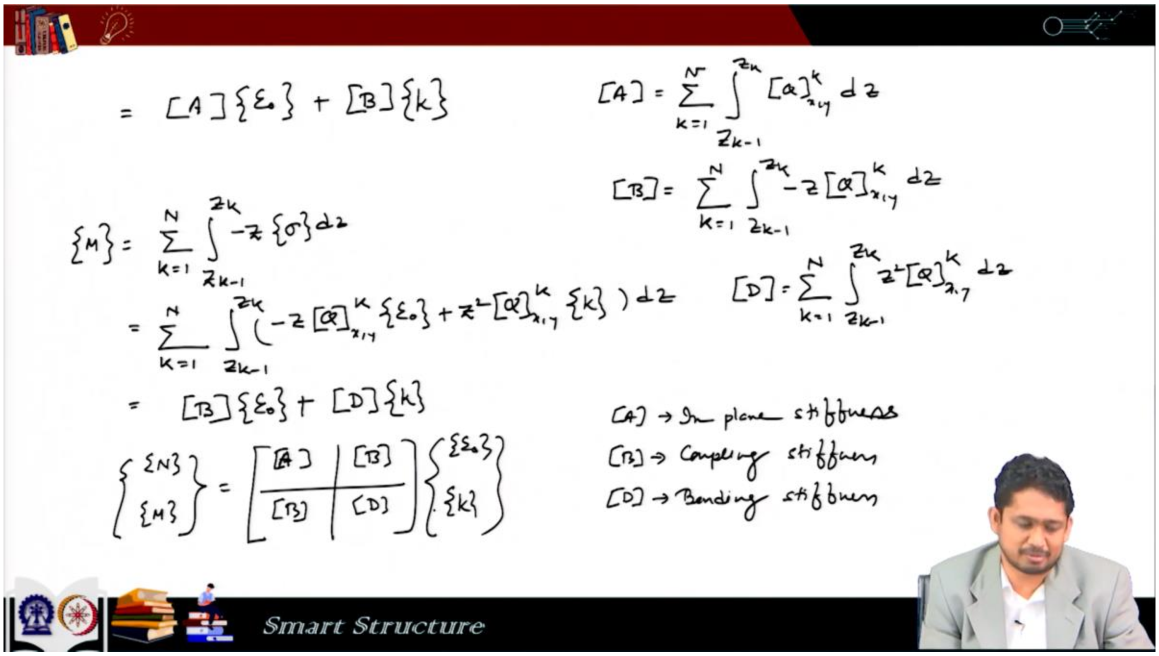

So, we can get rid of some of the couplings by proper lamination sequencing. Sometimes the couplings are desirable, sometimes they are non-desirable. So, let us look into a symmetric laminate. So, a symmetric laminate as the name suggests, has the plies orientation symmetrically placed. So, again let us imagine that we have something like this. So, layer 1, this is layer N. This is layer 2, this is layer N minus 1. And we can call a generic layer k and a symmetrically opposite layer is k prime. So, these are our mid plane z is equal to 0. So, with respect to this mid plane, this ply, I mean, this layer and this layer symmetrically placed. Similarly, this layer and this layer symmetrically placed. This layer and this layer at the symmetrical location. This layer and this layer the symmetrical location. Now, if at these locations, for example, at layer k, and at layer k prime, if the orientation is, if the ply orientation is same, we call it a symmetric laminate. Similarly, if at layer 1 and at layer N, if the ply orientation is same, if the layer 2 and at layer N minus 1, if the ply orientation is same and accordingly, for any k, if the ply orientation at k and k prime location, they are same, we call it a symmetric laminate. So, orientation and ply thickness both have to be same. So, in all our analysis, we are assuming that our ply thickness is all same, all the plies have same thickness. So, for example, a laminate like this, which is generally, we show a lamination sequence in this form – theta 1. Suppose, we consider this, this sequence where I have 1, 2, 3, 4, 5, 6, 7, 8. 8 number of plies. Suppose, our laminate sequencing is this – theta 1, theta 2, theta 3, theta 4, then we have supposed theta 5, theta 6, theta 7, theta 8. So, the lamination sequence is shown in this form: theta 3 theta 4 theta 5 theta 6 theta 7, theta. Theta 1 theta 2 theta 3 theta 4 theta 5 theta 6 theta 7 theta 8. So, this is how the lamination sequence is shown.

$$[\theta_1|\theta_2|\theta_3|\theta_4|\theta_5|\theta_6|\theta_7|\theta_8]$$

Now suppose, in our case theta 5 it is symmetric. If it is symmetric, then we have theta 5 is equal to theta 4, theta 6 is equal to theta 3, theta 7 is equal to theta 2, theta 8 is equal to theta 1. So, in this case we can show it as theta 1 theta 2 theta 3 theta 4 s, s means it is symmetric.

$$[\theta_1|\theta_2|\theta_3|\theta_4]_s$$

So, it repeats itself in a symmetric fashion. So, for a symmetric laminate, it can be shown that the B matrix is always 0 for a symmetric laminate always. It can be proved easily also. So, we know that the property of the Q, Q at kth ply and Q at k prime ply, they are same. So, if we add the contribution to the B matrix from this ply, and if I add the contribution to the B matrix from this ply they are just opposite of each other and on being added they become 0. So, that way if we do it for all the plies the entire B matrix comes to be 0 So, in a symmetric laminate, there is no coupling between the in plane and the outer plane components.

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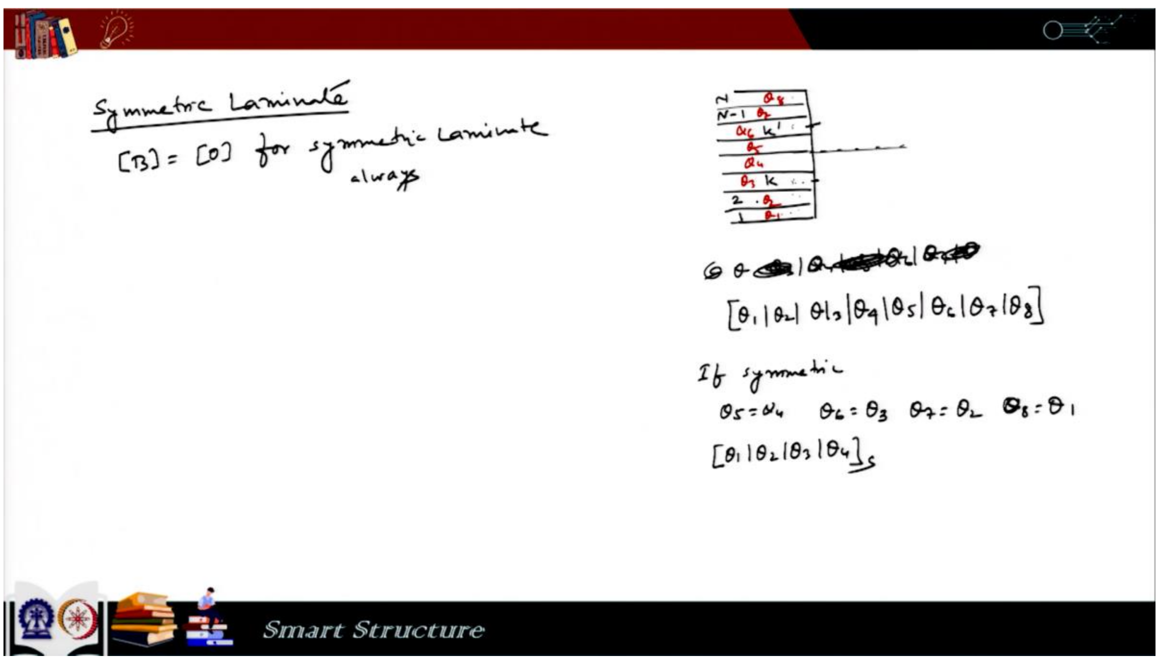

Now there is something called a balanced laminate. In a balanced laminate, it is something where we have number of plies with theta 1 orientation is same as number of or theta i orientation is same as number of plies with minus theta i orientation.

So, a laminate like this: suppose 30 degree, 40 degree, minus 30 degree, minus 40 degree generally, we don't show the degree sign while writing the lamination sequence, we can get rid of, it also it is understood. So, that is a balanced laminate because if I have a 40, I have one minus 40, if I have 30, I have one minus 30.

$$[30|40|-30|-40]$$

Similarly, if I have something like 30, 45, minus 45, 45, minus 45, minus 30, it's also balanced, because I have one 30, I have one minus 30, I have two 45s and I have two minus 45s. So it is balanced.

$$[30|45|-45|45|-45|-30]$$

So, these are for angle ply laminates angle ply means where the angle of orientation is neither 0 nor 90, it is an angle ply laminate. And if the orientation of ply is either 0 or 90, we call it a cross ply laminate. Now in a cross ply laminate, 0 and 90 are supposed to be opposite. So, 90, 0, 90, 0 this is also balanced because, as we said 90 and 0 are opposite in a cross ply laminate. So, if I have two 90s, I have two 0s also. So they are all balanced. So, these are all balanced laminates.

$$[90|0|90|0]$$

Whereas something like this, if I have 30 minus 30 90 90, this is not balanced because, I have one 30. So, I have one minus 30, that is ok, but I have one 90 for this I should have a 0, but that is not there I have another 90. So, it is not balanced.

$$[30|-30|90|90]$$

Now again this balanced laminate can be of different types. So, it can be symmetric and balanced. So, before going into, it can be symmetric. So, symmetric means which is symmetric as well as balanced. So, for example, plus minus theta 1, plus minus theta 2, symmetric, that is a balanced laminate.

$$[\pm\theta_1|\pm\theta_2]_s$$

An example can be 30, minus 30, 45, minus 45, symmetric. So, it is a symmetric as well as balanced. It can be antisymmetric.

$$[30|-30|45|-45]_s$$

Antisymmetric means plus theta 1, plus theta 2, minus theta 2, minus theta 1, that is antisymmetric. Because with respect to mid plane, if I look at two locations which are symmetrically located the orientations are just opposite. So, if I have theta 2 here I have minus theta 1 here, if I have theta 1 here, I have minus theta 1 here.

$$[+\theta_1|+\theta_2|-\theta_2|\theta_1]$$

So, an example is 30, 30, let us call it 30, 45, minus 45, minus 30, that is anti-symmetric.

$$[30|45|-45|-30]$$

And there can be asymmetric. So, it is neither symmetric nor anti-symmetric. So, it can be supposing theta 1, minus theta 2, minus theta 1, minus theta 2. So, it is neither symmetric nor anti symmetric. if we look at it with respect to the mid plane, we can see that.

$$[+\theta_1|-\theta_2|-\theta_1|\theta_2]$$

So, maybe 30, it is plus. I cannot have two minus. So, it is plus. 30, 45, minus 45, minus 30. So, it is asymmetric.

$$[30|45|-45|-30]$$

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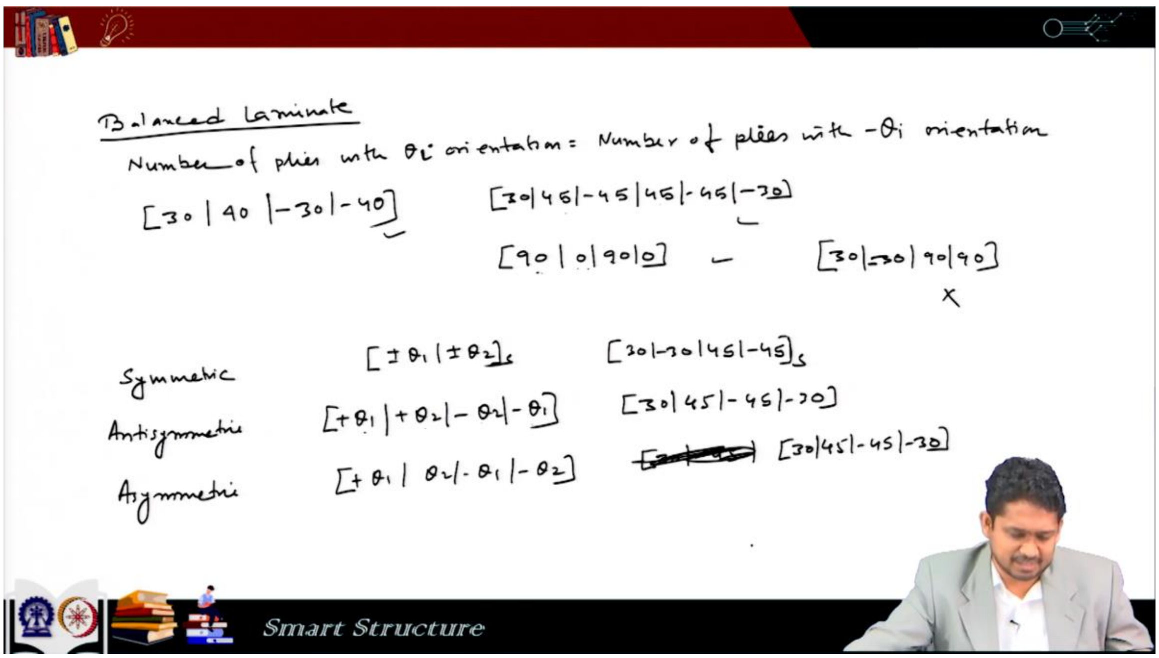

Now, for any balanced laminates: Now, A_{xs} is equal to A_{ys} is equal to 0 for any balanced laminates, it can be proved. And then, if the laminate is antisymmetric, then we have D_{xy} is equal to D_{ys} is equal to 0, for any antisymmetric laminates.

$$A_{xs} = A_{ys} = 0$$
$$D_{xs} = D_{ys} = 0$$

So, in antisymmetric laminate, there is no coupling between the bending and the twisting components. Sometimes this coupling is desirable, sometimes this coupling is non desirable and accordingly we can design our laminates. So, that is what gives us tremendous design flexibility when we use a composite material. And then, if the laminate is cross ply and antisymmetric, like a laminate 0, 90, 0, 90, then the B matrix looks like this. So, B_{xx} and B_{xy}, Bxx and B_{yy} are opposite of each other and rest of the quantities are 0, for antisymmetric cross ply laminate.

$$\begin{bmatrix} B_{xx} & 0 & 0 \\ 0 & -B_{xx} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And then we have 0 0 B_{xs}, 0 0 B_{ys} and B_{xs} B_{ys} 0 for antisymmetric angle ply laminate.

$$\begin{bmatrix} 0 & 0 & B_{xs} \\ 0 & 0 & B_{ys} \\ B_{xs} & B_{ys} & 0 \end{bmatrix}$$

So, we know the transformation relation for each plies for theta and we know that in a laminate where I have balanced. There are thetas and minus thetas. So, according to we can look at the transformation, and after looking at the lamination sequence, we can just get these quantities whether they are 0 or not it can be easily shown.

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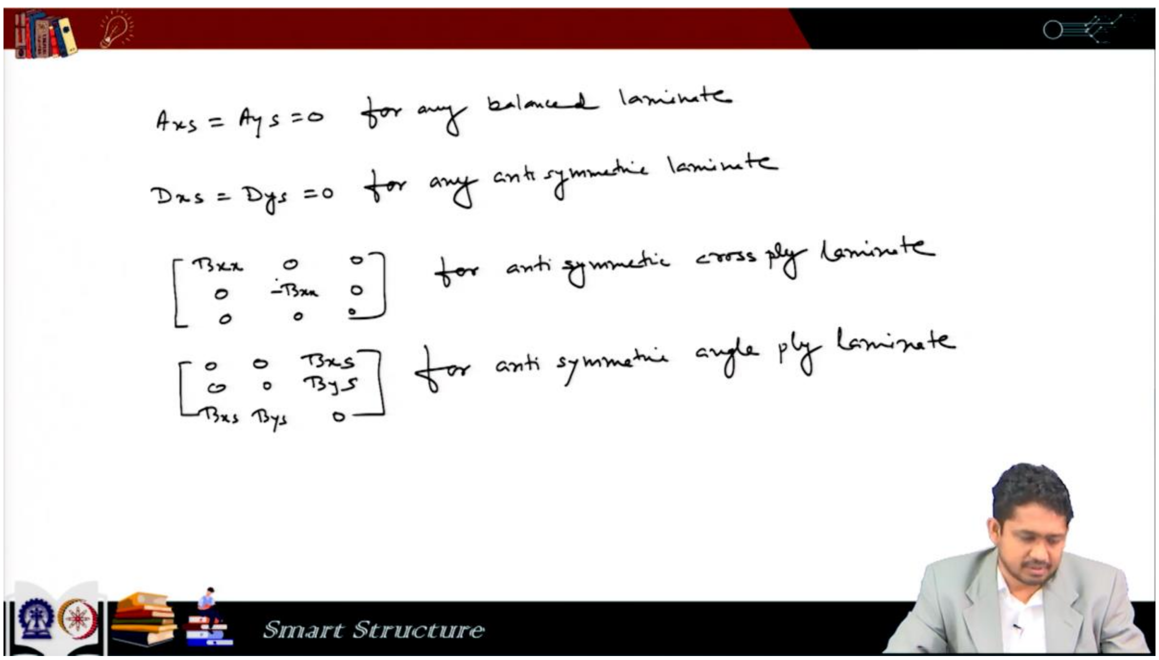

Now, we will talk about one another type of laminate, and that is called quasi isotropic laminate. So, in a quasi isotropic laminate, adjacent laminates are oriented at pi by n, where n is the number of laminates.

An example can be a laminate which has orientation 60, minus 60, 0 symmetric.

$$[60|-60|0]_s$$

Now, in these laminates, the behavior of A is of isotropic nature, but B and D matrices are not. So, if we look at the A matrix, it looks like this.

$$[A] = \begin{bmatrix} A_{xx} & A_{xy} & 0 \\ A_{xy} & A_{yy} & 0 \\ 0 & 0 & (A_{xx} - A_{yy})/2 \end{bmatrix}$$

So, this is how A of a quasi isotropic laminate looks like, and we have A_{xx} is equal to A_{yy} here. So, this is more of an isotropic behavior. So, A matrix is isotropic, but B and D are not. So, we saw how to relate the stress resultants with the strain components and then we defined matrices like A, B and D matrix. And we saw that depending on the lamination sequence, some of the couplings are there, some of the couplings are not there. And we get some special relations also in some cases like here, we have A_{xx} is equal to A_{yy} and so on. So, these are quite useful observations and they are quite useful in designing a composite laminate.

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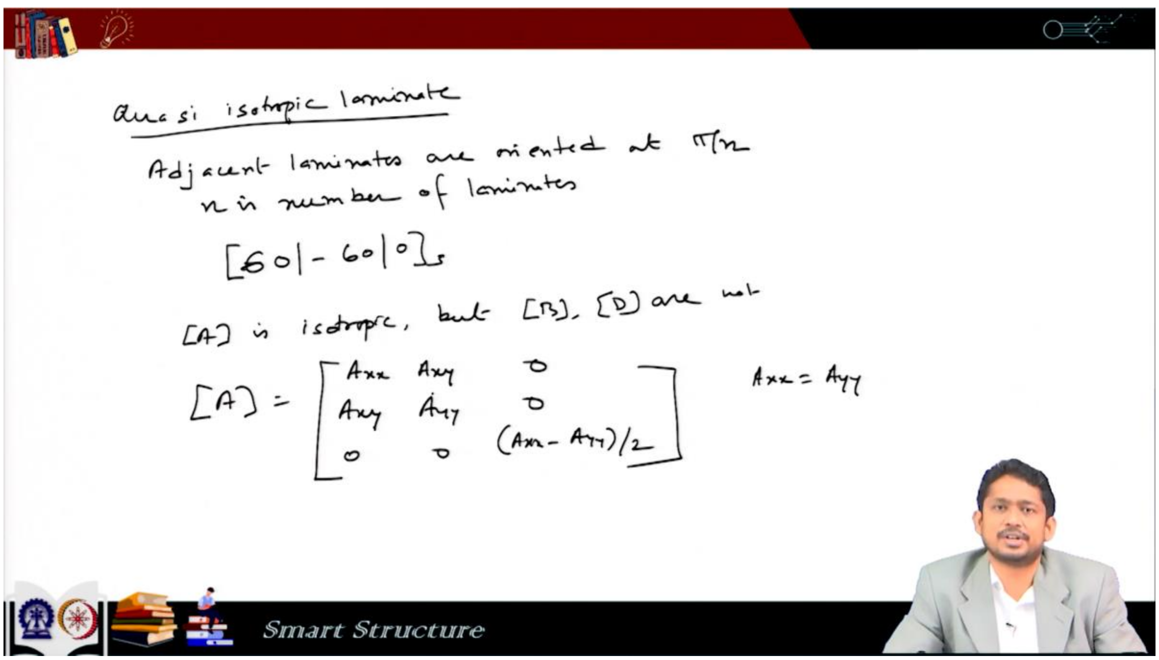

So, with that I would conclude this lecture here.

Thank you.