Smart Structures

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Week 06
Lecture No: 33
Constitutive Relation of Unidirectional FRP Composite Ply (continued)
Part 04

So, first we have seen the behavior of composite plies, behavior of individual composite plies with respect to its own principal material direction. So, we have seen ply like this and the plies are all unidirectional and they have fibers oriented in some direction and we can call this direction 1, we can call this direction 2 and we have defined properties like E_1, E_2, nu_{12}, nu_{21}, G_{12} which we also called E_6 and then we have by applying the plane stress condition we have got into the constitutive relationship, where we relate the in plane strains defined with respect to 1 2 system to the in plane stresses defined with respect to 1 2 system.

Now our problem is that our composite laminate it made of different plies and each ply has its, each ply can have different fiber orientation direction. So, I may have another ply where the fiber is oriented in an altogether different direction, maybe in this direction. So, now 1 2 changes, we can call this as 1, this as 2. So, each ply can have its own 1 2 directions. So, when we want to analyze a laminate we we need to define a global coordinate system and better that with respect to that global system we define our properties. So, if we suppose I have a laminate which is made of these plies. So, better we define a system like this xy which remains same for the all the plies. So, which is a global for the entire laminate and we defined our properties, defined our constitutive law with respect to this xy system then we can analyze the laminate. So, that will be our today's job.

So, from the constitutive relation which we already defined with respect to 1 2 systems now will be transferred to the xy system. So, we will transform the constitutive relation. Now to transform the constitutive relation, we will start with transformation of stresses at first we transform the stresses.

So, we have, we already know that we have a relation in this form our Q_1 and Q_2 are same. So we can write it like this. So, we have the relation in this form in the 1 2 system we have to transform it to xy system. So, first we transform the stresses. So, first we transform this vector and then we transform, then we transform the strains. So, we transform the strains and once we are able to transform this into x y and strain into xy, transformation of this Q is easy. So, we start with the stress transformation.

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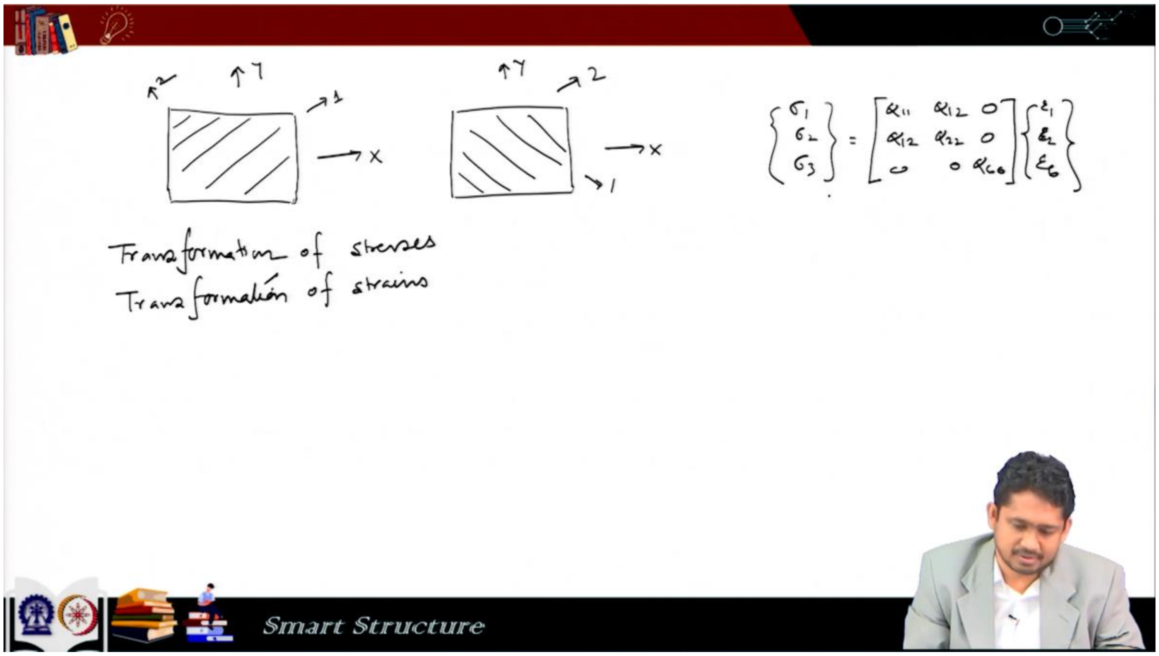

Now, let us imagine that our ply has - this as direction 1 and this is direction 2 and we need to convert it to - that convert the relation to xy. So, this we can call theta. So, the axis 1 is oriented at an angle theta with axis x and similarly axis 2 is also oriented at an angle theta with respect to axis y.

Now let us draw a line which is perpendicular to our axis, axis 1. So, this is perpendicular. And similarly, draw a line here, which is also perpendicular to axis, which is perpendicular to axis 2. And then we complete this rectangle.

So, to do this let us imagine that these are ply and this has direction 1 and this is direction 2 and we have direction x and direction y and axis 1 with respect to axis x is has an orientation of theta. Similarly, the angle between axis 2 and axis y is also theta. Now, let us draw a perpendicular to direction 1. So, to the axis 1 we are going to draw a perpendicular and to axis 2 also we are drawing another perpendicular and then we are joining them. So, let us imagine that we have a ply and the ply has direction 1, in this direction and direction 2 along this and it has x axis here and y axis here. So, the angle between direction 1 and direction x is theta and similarly angle between y and 2 is theta. Now we can draw a line which is perpendicular to axis 1 and then we can separate this part. So, if we draw the free body diagram of the separated part this would look like this.

So, if we look at this ply, then with respect to the xy system, we have sigma x along this direction. So, we have sigma x. So, let us use a different color to show the stresses. So, we have sigma x and then we have sigma y along this direction. Sigma x, sigma y and then we have sigma s over here, and sigma s over here sigma x, sigma s. So, sigma x is basically our sigma xx, sigma y is sigma yy, and sigma s is tau xy.

$$\sigma_x = \sigma_{xx}$$

$$\sigma_y = \sigma_{yy}$$

$$\sigma_s = \tau_{xy}$$

Similarly, so, if I draw or if I cut it and separate this part and draw the free body diagram here, what we see is - we will see sigma x here, we will see sigma y here and then, we see sigma s here and we will see sigma s here. Now this angle is theta and is in this inclined plane, as we know this is perpendicular to direction 1. So, perpendicular to this, we have sigma 1 and we know that, we know that this angle is theta and this is theta also. Now what we do is - we would do the force balance in this part and let us assume that this entire, this inclined edge has a length of l_1. So, if we sum up all the forces along direction 1. So, let us give it a name maybe let us call it 1. So, or better let us call it A. So, in A, considering this part A, if we balance all the forces along direction 1 and set it equal to 0 this gives me this equation. So, along x axis we have the forces due to this stress sigma x. So, sigma x multiplied by this length and this length is l_1 cosine theta.

So, sigma x multiplied by m and we are saying that our m is equal to cosine theta and n is equal to sine theta. So, sigma x multiplied by m into l_1 and then, here we have sigma s. So, sigma s multiplied by n into l_1, because this is our l_1 sine theta and then, we have minus sigma 1. So, we have the component of this along x direction.

$$\sum F_x = \sigma_x m l_1 + \sigma_s n l_1 - \sigma_1 l_1 m - \sigma_6 l_1 n = 0$$

So, the force here is sigma 1 multiplied by l and the component of along x is cosine theta with a minus. So, minus sigma 1 l_1 cosine theta means m, we have already defined l_1 m and also, along this we have a stress component whichis sigma 6. So, a stress shear strain along this inclined line which is sigma 6. So, plus sigma 6 l_1 n and this entire quantity is 0.

$$\sigma_x m + \sigma_s n - \sigma_1 m - \sigma_6 n = 0$$

Now we can say that l_1 we can just take out from all the sides. So, we get m sigma x plus n sigma s minus m sigma 1 plus n sigma 6 is equal to 0. Similarly, we can sum up all the force along y direction and set that equal to 0 and that would give us sigma y multiplied by n into l_1 plus sigma s, we have here sigma s multiplied by m into l_1 and then we have minus sigma 1, it is a component along the direction 1. So, sigma 1 multiplied by l_1 into sine theta with a negative sign. So, sigma 1 l_1 n and we have sigma 6 l_1 m and that is equal to 0.

$$\sum F_y = \sigma_y n l_1 + \sigma_s m l_1 - \sigma_1 l_1 n - \sigma_6 l_1 m = 0$$

So, that gives us n sigma y plus m sigma s minus n sigma 1 minus m sigma 6 is equal to 0.

$$\sigma_y n + \sigma_s m - \sigma_1 n - \sigma_6 m = 0$$

So, let us call this equation 1 and let us call this equation 2. Now we can solve this equation 1 and equation 2 and by solving we have to express sigma 1 and sigma 6 in terms of sigma x and sigma y and sigma s. So, first if you want to get our sigma 1, what we can do is we multiply sigma 1 with m and sigma 2 with n and sum them and that would give us sigma 1 is equal to m squared sigma x plus n squared sigma y plus 2 m n sigma s. Now, our m is equal to cosine theta and n is equal to sine theta. So, m squared plus n squared is equal to 1 and using this, we get it.

$$\sigma_1 = m^2\sigma_x + n^2\sigma_y + 2mn\sigma_s$$

Now our job is to get sigma 6 and to get sigma 6 we can have multiplied 1 with n and we can multiply 2 with m and subtract and that would give us sigma 6 is equal to minus of m n sigma x plus m n sigma y plus m squared minus n squared sigma s. So, we could express sigma 1 and sigma 6 in terms of sigma x, sigma y, sigma s and m n.

$$\sigma_6 = -mn\sigma_x + mn\sigma_y + (m^2 - n^2)\sigma_s$$

Now, we can draw another perpendicular here with to the line 2. So, this we can draw another perpendicular here. So, this is also perpendicular or previous line. Now we will take out this part and look at the force balance there to get other required relations.

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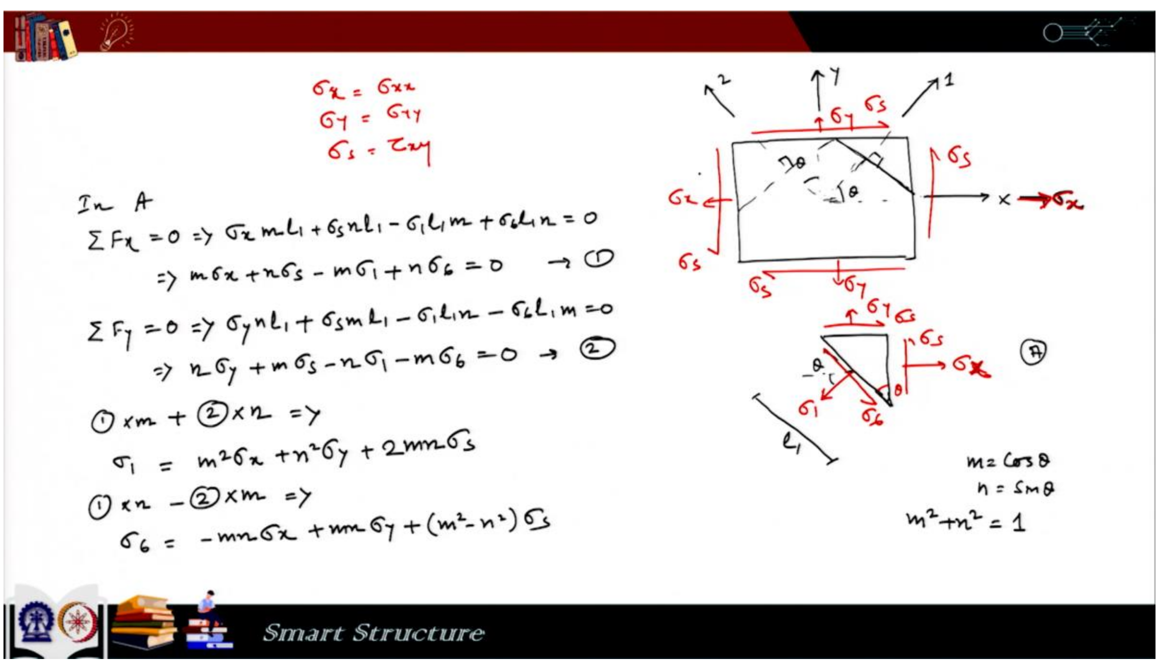

So, for that part, if we draw that separately, this gives us sigma y and we have sigma s in this direction. We have sigma x, sigma s in this direction and perpendicular to this, we have sigma 2 and then, we have sigma 6. And this angle is theta. Now, we have to balance the forces and let us say that this length is suppose l_2 and let us call this part as part B. So, if we again, if we apply summation of all the forces along x is equal to 0 that gives us this equation sigma s multiplied by l_2 m minus sigma x l_2 n plus sigma 2 l_2 n minus sigma 6 l_2 m is equal to 0 and then by writing the force equilibrium along y direction we get another equation which is sigma y l_2 m minus sigma s l_2 n minus 2 l_2 m minus sigma 6 l_2 n is equal to 0 and let us call it equations 3 and 4.

$$\sum F_x = \sigma_s l_2 m - \sigma_x l_2 n + \sigma_2 l_2 n - \sigma_6 l_2 m = 0$$

$$\sum F_y = \sigma_s l_2 n - \sigma_2 l_2 m + \sigma_2 l_2 m - \sigma_6 l_2 n = 0$$

And again we can do the same thing. We can just solve this two equation and we can find out our expressions for sigma 2 and sigma 6 in terms of in terms of sigma y, sigma x, and sigma s. So, doing the same similar thing, we get from equation 3 and 4 we get sigma 2 is equal to sigma x multiplied by n squared plus sigma y multiplied by m squared plus 2 m n sigma s.

$$\sigma_2 = \sigma_x n^2 + \sigma_y m^2 + 2mn\sigma_s$$

And from this by solving these two we get sigma 6 once again and this sigma 6 should be what we got from here. So, from this part from equation 1 and 2 by solving whatever sigma 6 we got same thing we should get here also.

$$\sigma_6 = -mn\sigma_x + mn\sigma_y + (m^2 - n^2)\sigma_s$$

So, if the sigma 6 that we get from these two equations is different from what we get here that means, that we have some error in the formulation, otherwise they should give the same sigma 6.

So, now we have got sigma 1, sigma 2, sigma 6 all in terms of sigma x, sigma y, and sigma s and m n. So, by solving these two what we get if we write this in a matrix form it looks like this.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}$$

So, this is our stress transformation. Now, we have to define the strain transformation.

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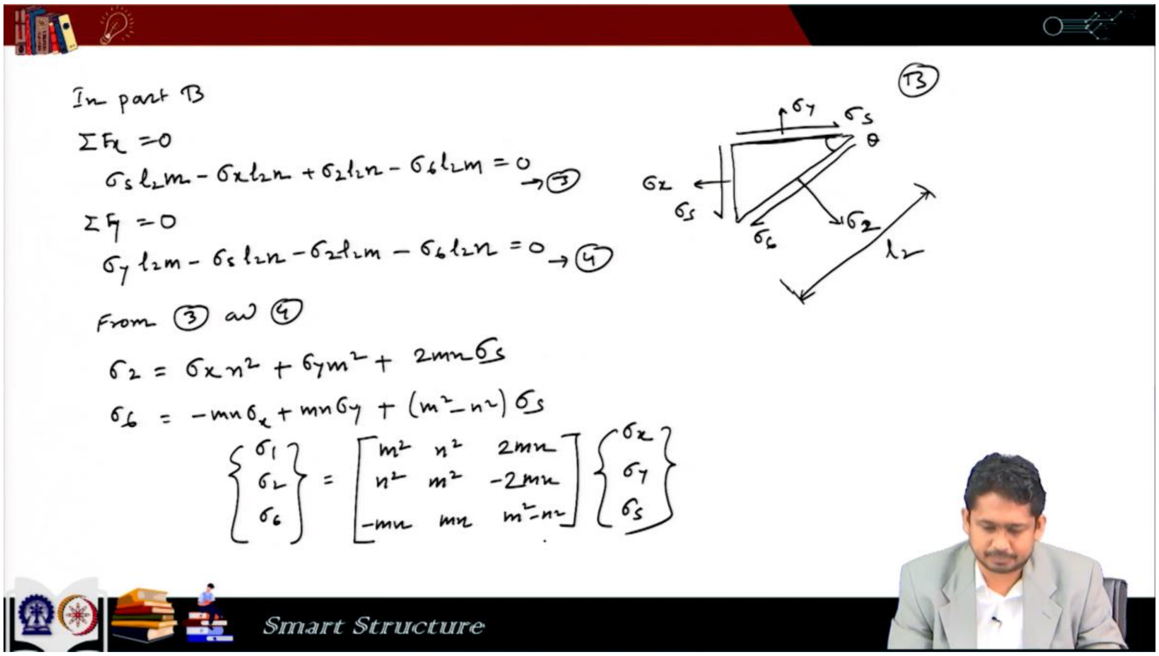

To define strain transformation, let us imagine that we have an axis system x y and we have an another coordinate system 1 2 and let us imagine that we have a point here.

So, with respect to this, if I draw a vertical line to the xy, what we get here is x. And if I draw a vertical line here, what we get here is y. And similarly, if I draw a vertical line on the axis 1, so, we get x_1 here. And if I draw a vertical line on axis 2, we get y_1 here and this is our theta and let us call it a point p. So, from this diagram, we can say that x is equal to our x_1 m minus y_1 n and similarly, y is equal to x_1 n plus y_1 m.

$$x = x_1 m - y_1 n$$

$$y = x_1 n + y_1 m$$

So, we can write x y is equal to m minus n, n m; x_1 y_1.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} m & -n \\ n & m \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}$$

And we can invert this relation and also can write x_1 y_1 is equal to m n, minus n m; x y. It is just inversion of this equation and which gives us this.

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Now, we have already known that we have three strain components epsilon x, epsilon y, and epsilon xy or in the 12 system epsilon 1, epsilon 2 and epsilon 6. So, with these findings, we have to now do a transformation between these two stresses.

Now, before that because we know that our x in terms of x_1 and y_1 is this. So, if we define a movement along x and if we call it u. So, u which is a displacement along x can also be written as or if you write other way if we define a movement along one direction and which we call it u_1. So, u_1 with respect to a movement along x and y can also be written as m u plus n v. Similarly, v_1 which is a movement along the direction 2 can also be written as minus of m u plus n v.

$$u_1 = mu + nv$$

$$v_1 = -mu + nv$$

So, u_1 v_1 are movement along x_1 y_1 and u v are movement or displacement along x and y. So, they also follow the same relation.

Now, let us define a strain epsilon 1. So, epsilon 1 as per definition should be derivative of displacement along x_1 with respect to x_1. So, we can write using the chain rule of differentiation as del u by del x multiplied by del x by del x_1 plus del u_1 by del y multiplied by del y by del x_1. And now we can write u_1 in terms of u and v. So, m u plus n v and del x del x_1, if I look at this relation, from this del x by del x_1 becomes 1 because our x is equal to, because our x is equal to x_1 m minus y_1 n. So, del u by del x_1 is equal to m, plus again we have del by del y and u_1 is equal to m u plus n v, and then del y by del x_1, which we have here del y by del x_1 is n.

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial x_1} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial x_1} = \frac{\partial}{\partial x}(mu + nv)m + \frac{\partial}{\partial y}(mu + nv)n$$

Now, if we differentiate this m u with respect to x, we have m multiplied by del u by del x, and we know that del u / del x is x. Similarly, here we have n multiplied by del v by del y, and we know that the del v by del y is epsilon y. So, del u by del x is equal to epsilon x and del v by del y is equal to epsilon y. And this term will give us m n multiplied by del v by del x and this will give us m n multiplied by del u by del y.

And we know that del v by del x plus del u by del y is epsilon xy or epsilon s. So, this entire quantity can be written as mn multiplied by epsilon x plus mn multiplied by epsilon y, not m n its m^2. And then we have mn epsilon s.

$$\varepsilon_1 = m^2\varepsilon_x + n^2\varepsilon_y + mn\varepsilon_s$$

So, to do this we use the relations epsilon x is equal to del u by del x, epsilon y is equal to del v by del y, and epsilon s which you also call as gamma xy is equal to del u by del y plus del v by del x.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial u}{\partial y} \qquad \varepsilon_s = \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

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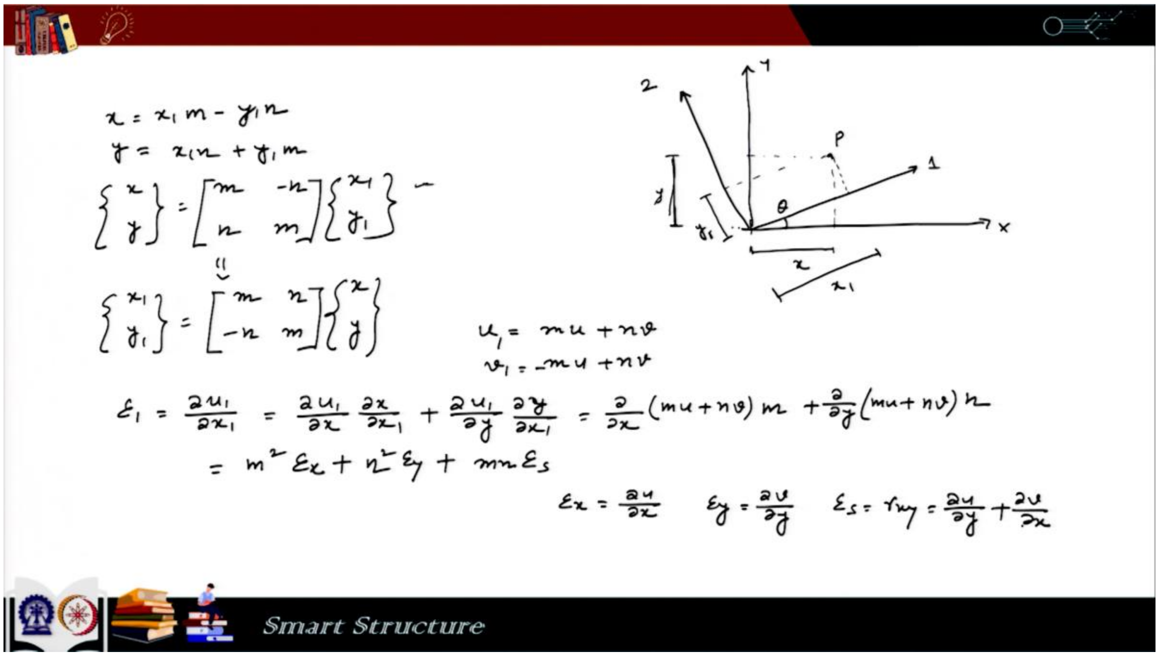

Similarly, now we have epsilon 2 which is equal to del v_1 by del y_1. And this we can write as del v_1 by del x multiplied by del x by del y_1 plus del v_1 by del y multiplied by del y by del y_1.

$$\varepsilon_2 = \frac{\partial v_1}{\partial y_1} = \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial y_1} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial y_1}$$

Now, we know that v_1 is equal to minus n u plus m v and del x by del y_1 is minus n. And similarly, we have here also minus n u plus m v and we have del y by del y_1 is equal to m.

$$\varepsilon_2 = \frac{\partial}{\partial x}(-nu + mv)(-n) + \frac{\partial}{\partial y}(-nu + mv)m$$

So, again if we write this equation finally, it would give us n squared multiplied by epsilon x plus m squared multiplied by epsilon y minus mn epsilon s.

$$\varepsilon_2 = n^2\varepsilon_x + m^2\varepsilon_y - mn\varepsilon_s$$

Now, our job is to transform epsilon 6. So, epsilon 6 by this definition would be del u_1 by del y_1 plus del v_1 by del x_1.

$$\varepsilon_6 = \frac{\partial u_1}{\partial y_1} + \frac{\partial v_1}{\partial x_1}$$

And we have del u_1 by del x multiplied by del x by del y_1 plus del u_1 by del y multiplied by del y by del y_1 plus we have, del v_1 by del x multiplied by del x by del x_1 plus we have del v_1 by del y multiplied by del y by del y_1.

$$\varepsilon_6 = \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial y_1} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial y_1} + \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial x_1} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial x_1}$$

Now, we can write del by del x and this is equal to m u plus n v, and del x by del y_1 is equal to minus n. And this we can again write m u plus n v and del y by del y_1 is equal to m, del y by del y_1 is equal to m. And here again we have del del x of v_1 and v_1 is equal to minus n u plus m v and del x by del x_1 is m plus we have del del y multiplied by v_1 and v_1 is equal to minus n u plus m v and del y by del y_1 is - So, this should be x not y, excuse me. So, this should this is del y by del x_1, and del y by del x_1 is n as per this.

$$\varepsilon_6 = \frac{\partial}{\partial x}(mu + nv)(-n) + \frac{\partial}{\partial y}(mu + nv)m + \frac{\partial}{\partial x}(-nu + mv)m + \frac{\partial}{\partial y}(-nu + mv)n$$

Now, we can again simplify the entire relation and that gives us the final relation and the relation becomes minus of 2mn epsilon x plus 2mn epsilon y plus m squared minus n squared epsilon s or epsilon x y. So, epsilon s is equal to gamma xy, not epsilon gamma xy.

$$\varepsilon_6 = -2mn\varepsilon_x + 2mn\varepsilon_y + (m^2 - n^2)\varepsilon_s \qquad\qquad \varepsilon_s = \gamma_{xy}$$

Now, we can write this relation in this form epsilon 1 epsilon 2 epsilon 6 as m squared n squared mn, n squared m squared minus mn, minus of 2mn plus 2mn m squared minus n squared and that is multiplied by epsilon x epsilon y epsilon s.

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

Now, if we compare this relation with the stress-strain relation which we got before, if we look at this matrix which transforms the strains with respect to xy to the strain with respect to 1 2. If I compare this matrix with the previous matrix which does the stress transformation, we can see some difference. We have 2s here and we do not have 2s here whereas, here we do not have 2s, but here we have 2s. Now, if we want to make the previous matrix applicable here what we can do is we can define the shear strain relation little bit in a different way. So, instead of writing epsilon 6, if I write epsilon 6 by 2, then this relation changes and it looks like this.

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

So, we define a new shear strain here and this we call tensorial shear strain. This we call tensorial shear strain. And this matrix this transformation matrix we call T.

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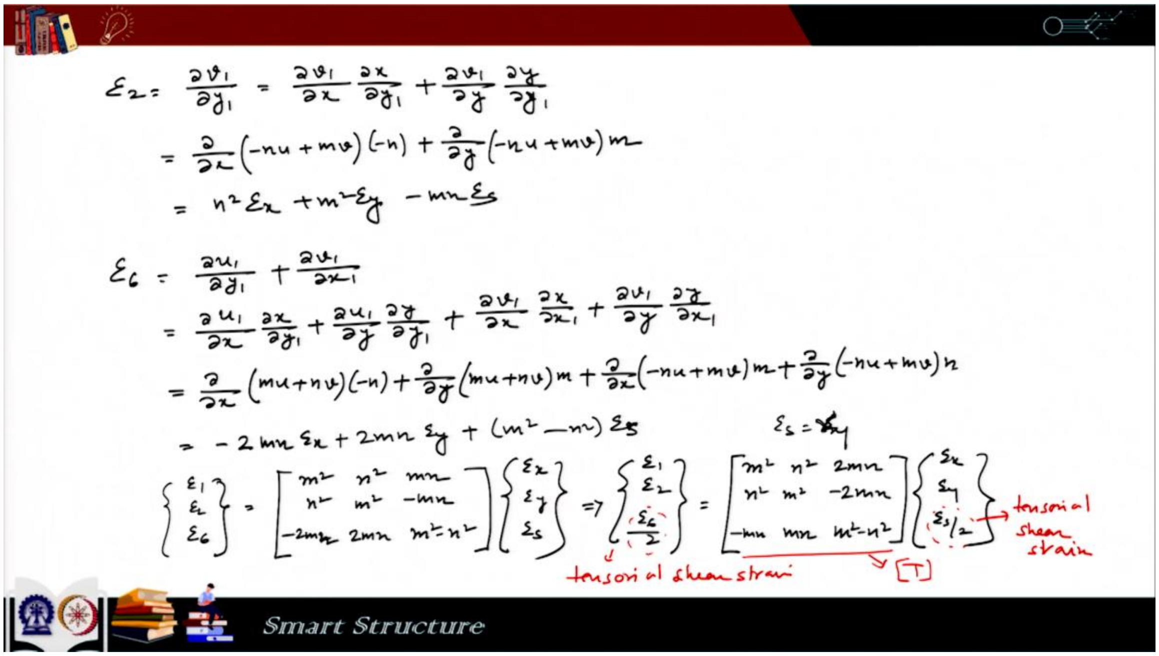

So, now with the help of this, we can with the help of this, we can rewrite epsilon 1, epsilon 2, epsilon 6, the relation that we got Q_{11}, Q_{12}, 0, Q_{12}, Q_{22}, 0, 0, 0, and here in the shear part, shear strain part instead of using epsilon 6, let us use epsilon 6 by 2. Now, accordingly to keep the value of sigma 6 remain same, we multiply Q_{66} by 2 here.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6/2 \end{Bmatrix}$$

Now, we already know that this sigmas are in terms of sigma x, sigma y, sigma s are this sigma 1, sigma 2, sigma x equal to this transformation matrix multiplied by sigma x, sigma y, sigma s, sigma 1, sigma 2, sigma 6 is equal to this in terms of the transformation matrix and similarly, we will transform our, I mean, write our epsilons in also the transform form and this we know epsilon x, epsilon y, and epsilon s.

$$[T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s/2 \end{Bmatrix}$$

Now we can bring it this side and now we can write T inverse multiplied by Q_{11}, Q_{12}, 0, Q_{12}, Q_{22}, 0, 0, 0, $2Q_{66}$ and then, now this T matrix has a property that T inverse is equal to, T inverse is equal to T of minus theta.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s/2 \end{Bmatrix} \qquad [T]^{-1} = [T(-\theta)]$$

So, now, this relation can be written as sigma x, sigma y, sigma 6 is equal to whatever we get after this multiplying this with Q matrix with T, we write this as Q_{xy} and again we adjust here epsilon s. So, after doing all the adjustment, whatever we get here, we can write it in this form epsilon x, epsilon y, epsilon s. So, this becomes our new constitutive relation with respect to the xy system.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix}$$

So, this was the constitutive relation with respect to 1 2 systems, and this is with respect to the xy system. So, to keep things shortened, we write Q with respect to the 1 2 system as Q_{11}, Q_{12}, 0, Q_{12}, Q_{22}, 0, 0, 0, Q_{66} and when we write Q with respect to x y system this is the expanded form.

$$[Q]_{1,2} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \qquad [Q]_{x,y} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix}$$

So, this is our transformed constitutive relation where Q is the stiffness matrix with respect to the x y system and these are the stress and strains with respect to the x y system.

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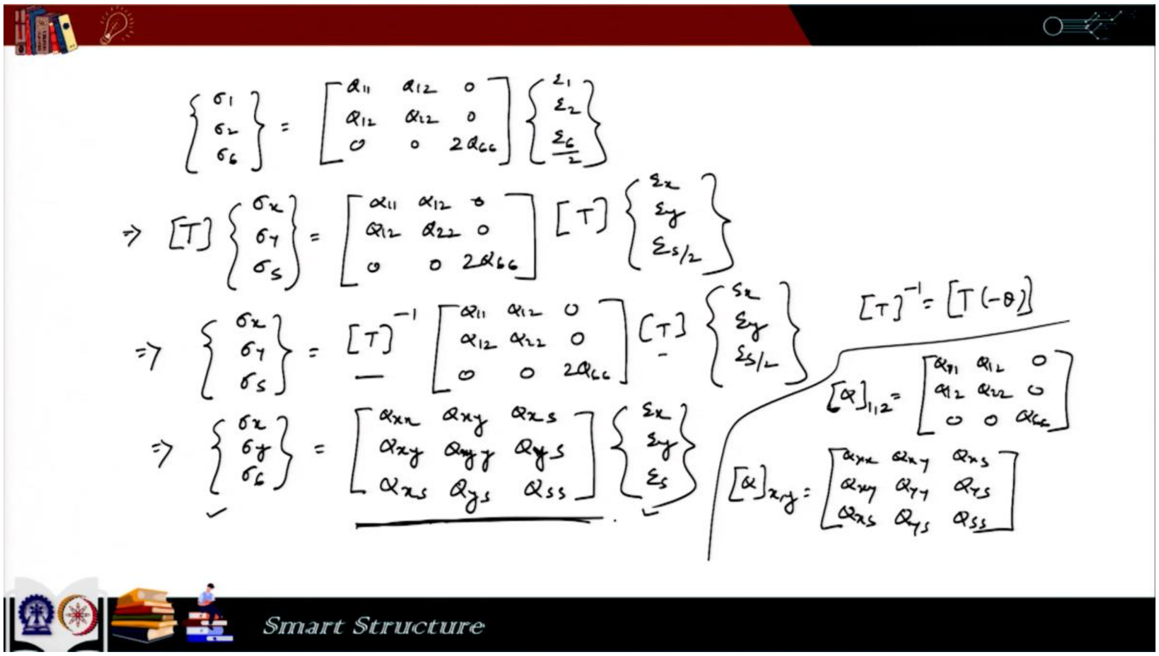

So, with that we would conclude this lecture today.

Thank you.