Smart Structures
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Week - 06
Lecture No - 32
Constitutive Relation of Unidirectional FRP Composite Ply (continued)

Welcome to the third lecture.

In the previous lecture, we looked into the equivalent elastic modulus along direction 1, 2 and also the shear modulus in 1, 2 plane of a composite ply in terms of its same properties of the constituents. We will continue from there today. Now we look into the Poisson's ratio.

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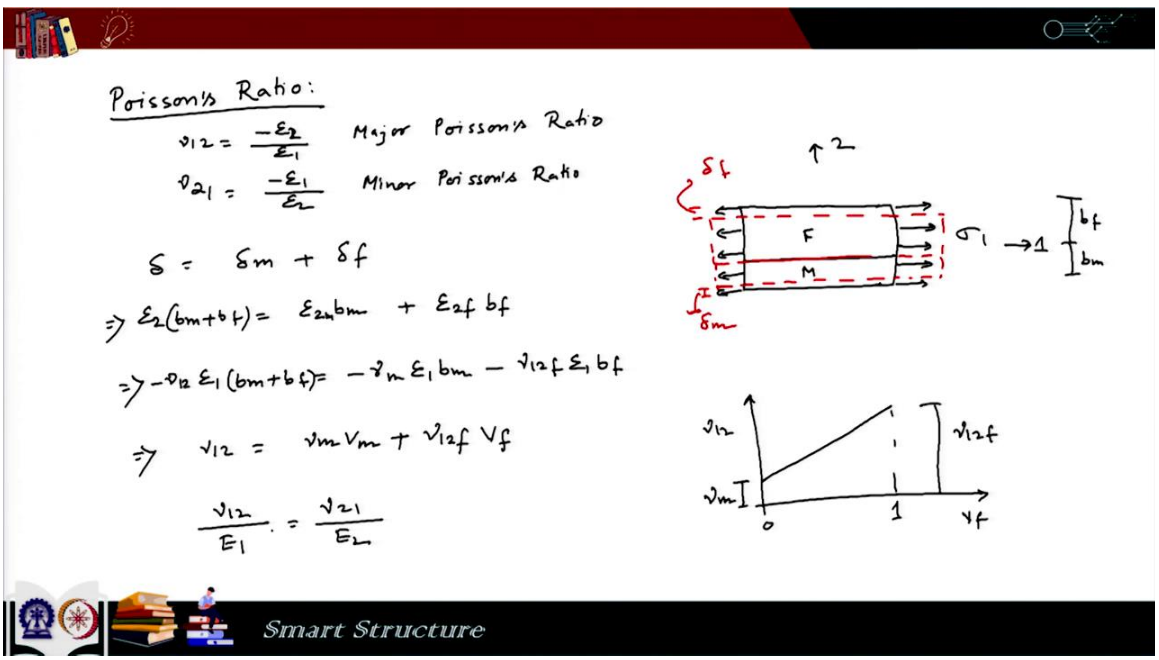

So we have two Poisson's ratio one nu 12 which we call minus E2 by E1. So Poisson's ratio as we know that when it stretches at one side it compresses at the other side or vice versa. Now the direction of those two perpendicular stress with a negative sign is Poisson's ratio. Now nu 12 is minus epsilon 2 by epsilon 1 which we call major Poisson's ratio and nu 21 is minus epsilon 1 by epsilon 2 which we call minor Poisson's ratio. Now again if we draw the idealized diagram we have F at fiber at one side, M at other side and let us say it is loaded in direction 1. So when it deforms it would deform in this way.

So it will expand in 1 direction contract in other direction. Now the deformation along the fiber let us call it delta F and contraction along direction 2 of the matrix part let us call it delta M and then we have this as our axis one and this as our axis two and again we can write delta is equal to delta M plus delta F and delta M is equal to the epsilon 2 of the matrix multiplied by bm. So let us write this dimension as bm and this as bf and this is epsilon 2f of the fiber multiplied by bf and then here we have epsilon 2 multiplied by bm plus bf. Now these strains can be written in terms of the strains in the longitudinal direction. So epsilon 2 is nothing but minus nu 12 multiplied by epsilon 1 to bm and this is minus nu 12 for the matrix and for matrix, actually we can write nu 12 is just nu m and nu 12f multiplied by epsilon 1 multiplied by bf and here it is an equivalent quantity so equivalent nu 12 of the composite multiplied by epsilon multiplied by bm plus bf.

Then again we have epsilon 1 same here. We saw that when it is loaded in the longitudinal direction, the strain along the longitudinal direction is same for matrix and fiber and that is epsilon 1. So if we cancel epsilon 1 and bring bm plus bf in the denominator at the right hand side we can write nu 12 is equal to nu m multiplied by Vm plus nu 12f multiplied by Vf and again if we look at the variation of this quantity nu 12 with Vf it gives a linear variation as the equation is. So when Vf equal to 0 it is just nu m and when Vf is equal to 1 it is nu 12f. So this is Vf equal to 1 here and we have 2 nu 12 for composites so nu 12 divided by E 1 that is equal to nu 21 divided by E 2. So E 1 and E 2 are different so nu 21 and nu 12 are also different and they follow this relation. So, if I know one of the Poisson's ratios and if I know the elastic modulus along direction 1 and direction 2, we can find out the other Poisson's ratio as well.

$$\nu_{12} = \frac{-\varepsilon_2}{\varepsilon_1}$$

$$\nu_{21} = \frac{-\varepsilon_1}{\varepsilon_2}$$

$$\delta = \delta_m + \delta_f$$

$$\varepsilon_2(b_m + b_f) = \varepsilon_{2m} b_m + \varepsilon_{2f} b_f$$

$$-\nu_{12}\varepsilon_1(b_m + b_f) = -\gamma_m \varepsilon_1 b_m - \nu_{12f}\varepsilon_1 b_f$$

$$\nu_{12} = \nu_m V_m + \nu_{12f} V_f$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

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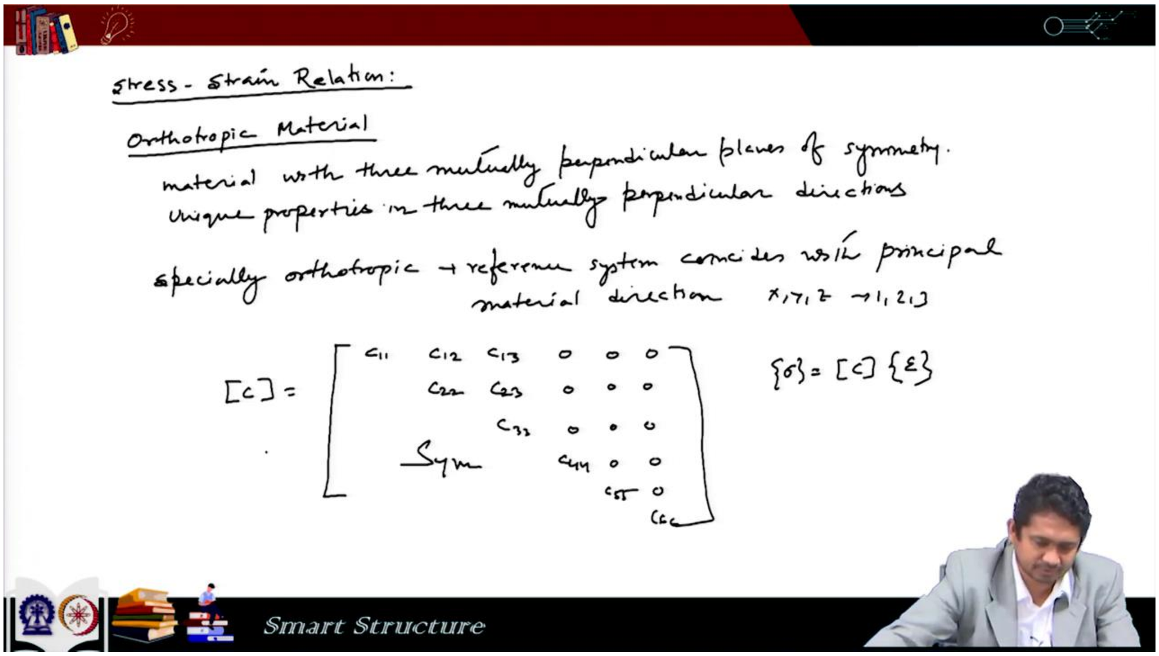

Now we will talk about the stress strain relation and we will go towards the stress strain relation in a thin ply. For that we need to define some quantities. We need to define orthotropic material. We need to define the elastic stress strain relation for orthotropic material in a 3D and then from there we will look into the stress strain relation in a ply in 2D. Now orthotropic material is material with 3 mutually perpendicular planes of symmetry. So it has unique properties unique and independent also. We can just see unique properties in 3 mutually perpendicular directions and then there is something called spatially orthotropic where reference system which means the axis system, reference system coincides with principal material direction. So x, y, z coincides with 1, 2, 3 and we will see that when it is spatially orthotropic, some of the coupling does not exist.

We will see that later on. It is symmetric, so we can populate this side. So stress is equal to C matrix multiplied by strain and C matrix is this. This is for an orthotropic material and then comes another material which is transversely isotropic.

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

$$\{\sigma\} = [c]\{\varepsilon\}$$

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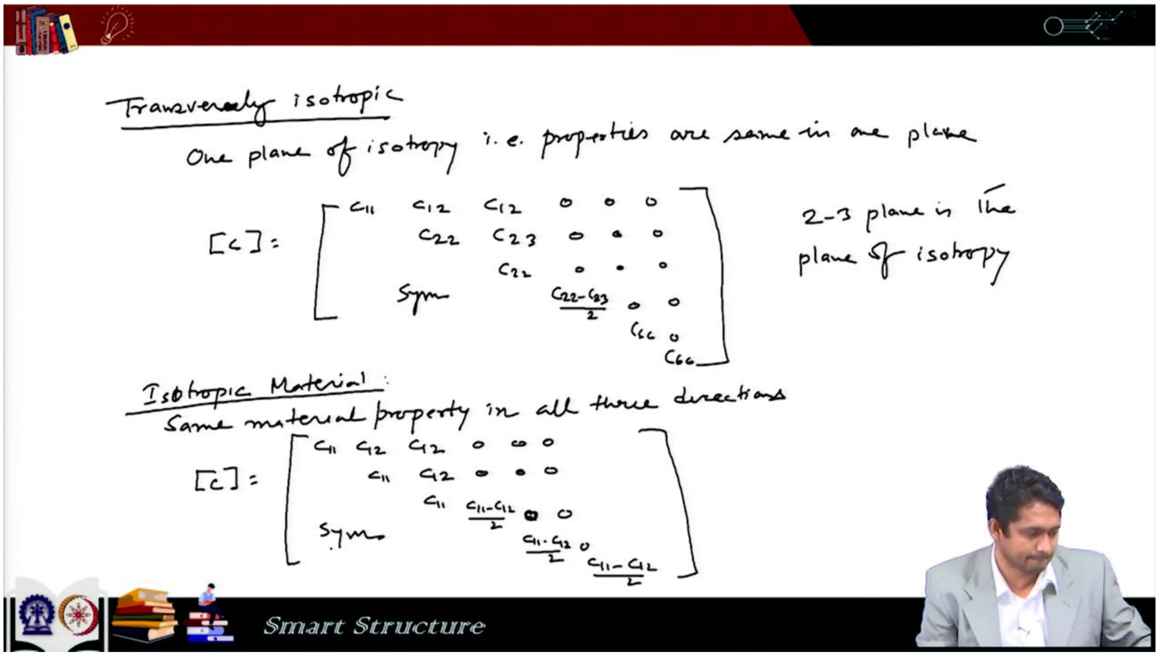

So it has one plane of isotropy which means properties are same in one plane. So for example, in case of piezoelectric sheets, we saw that the properties are same in plane direction but in the out of plane direction, properties are different. In this case the C matrix becomes C 11, C 12, C 12 and then we have C 22, C 23, C 22. C 23 and this is symmetric. So in this case 2, 3 plane is the plane of isotropy. So that is why we can see that C 22 and C 23 are same and similarly shear modulus here and here are also same.

And isotropic materials are same material property in all three directions. In this case the C matrix turns out to be C 11, C 12, C 12, C 23 and then we have C 11, C 12, 0, 0, 0, C 11, 0, 0. Here we have 0, C 11, C 12 by 2 and here we have C 11 minus C 12 by 2. This 0 we do not need to write because it is symmetric and this is symmetric. So, this is our isotropic material.

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & \frac{c_{22} - c_{33}}{2} & 0 & 0 \\ & & & & c_{66} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{11} & c_{12} & 0 & 0 & 0 \\ & & c_{11} & 0 & 0 & 0 \\ & & & \frac{c_{11} - c_{12}}{2} & 0 & 0 \\ & & & & \frac{c_{11} - c_{12}}{2} & 0 \\ & & & & & \frac{c_{11} - c_{12}}{2} \end{bmatrix}$$

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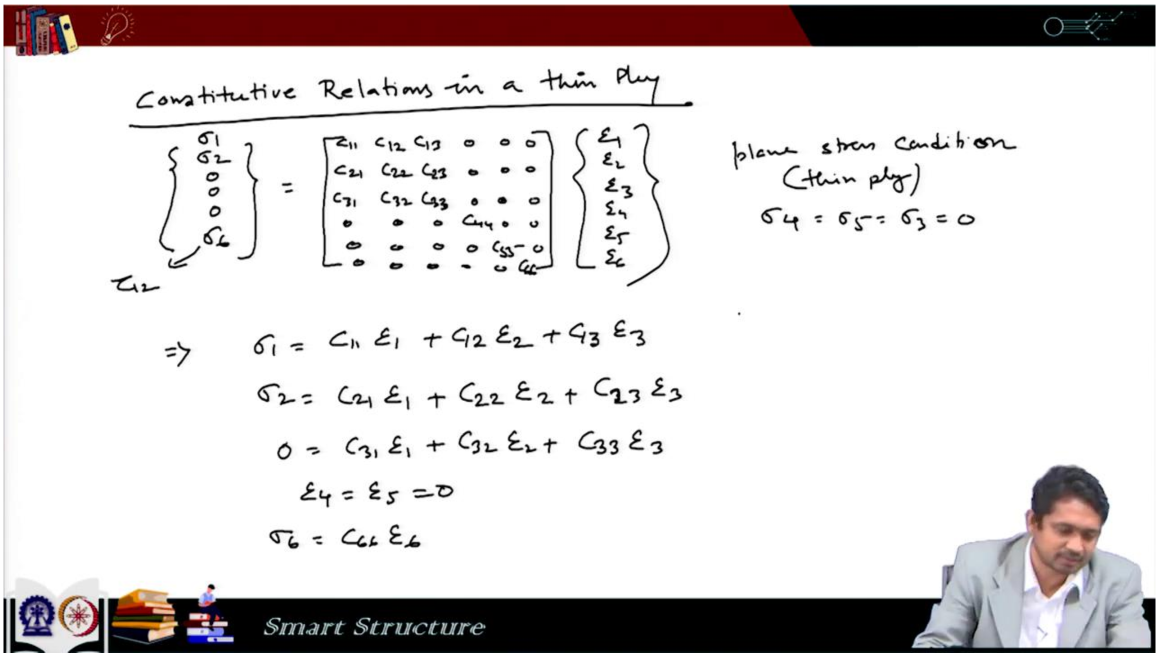

Now, we are in a position to find out the constitutive relations in a thin ply. So, we have seen that the stress is equal to the C matrix multiplied by the strain matrix for the orthotropic material and our materials are orthotropic. So, C 12, C 13, 0, 0, 0 and then we have C 21, C 22, C 23 it is symmetric. So, C 12 and C 23 are same. C 31, C 32, C 33, 0, 0, 0. 0, 0, 0, C 44, 0, 0. 0, 0, 0, 0, C 55, 0. 0, 0, 0, 0, 0, C 66. And this is multiplied with epsilon 1, 2, epsilon 3, epsilon 4, epsilon 5, epsilon 6 and here we have stress. Now, in this case we assume that there is a plane stress condition that exist.

So, we have plane stress condition because it is thin ply. Because the plies are thin. So, when it is plane stress condition we can say that sigma 4 is equal to sigma 5 is equal to sigma 3 they are all 0. So, any stress in the direction 3 is 0. So, we have sigma 1, sigma 2 then 0, 0, 0 and sigma 6 and sigma 6 as we know it is nothing but tau 12.

So, if we follow this equation what we get here is sigma 1 is equal to C 11 multiplied by epsilon 1 plus C 12 multiplied by epsilon 2 plus C 1 plus C 13 multiplied by epsilon 3. And then we have sigma 2 is equal to C 21 multiplied by epsilon 1 plus C 22 multiplied by epsilon 2 plus C 23 multiplied by epsilon 3. And here we can see that there are lot of 0s and that gives me that 0 is equal to C 31 into epsilon 1 plus C 32 into epsilon 2 plus C 33 into epsilon 3 and from here we get epsilon 4 equal to 0, epsilon 5 equal to 0 and finally, have sigma 6 is equal to C 66 into epsilon 6. Now we will keep this equation. This is finally, going to help us derive the stress strain relation for a thin ply. Now before that we have to know what this C's are.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

$$\sigma_4 = \sigma_5 = \sigma_3 = 0$$

$$\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3$$

$$\sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3$$

$$0 = c_{31}\varepsilon_1 + c_{32}\varepsilon_2 + c_{33}\varepsilon_3$$

$$\varepsilon_4 = \varepsilon_5 = 0$$

$$\sigma_6 = c_{66}\varepsilon_6$$

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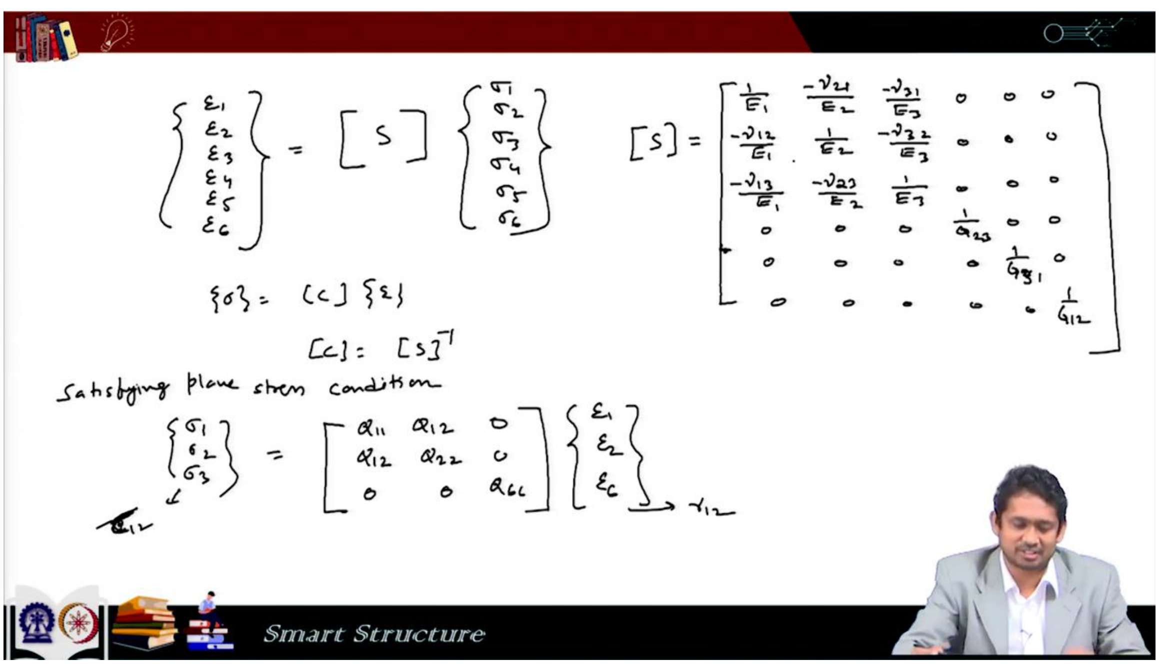

Now if you want to know the C, first we would write the relation in the inverted form. So, epsilon 1, epsilon 2, epsilon 3, epsilon 4, epsilon 5, epsilon 6 is equal to a S matrix multiplied by sigma 1, sigma 2, sigma 3, sigma 4, sigma 5, sigma 6 where S matrix is written as 1 by E 1 minus nu 21 by E 1 minus nu 31 by E 1. And we have nu 12 by E 1. So, we have minus nu 21 by E 2 and we have minus nu 31 by E 3 and here we have minus nu 12 by E 1, here we have 1 by E 2 and here we have nu 32 by E 3 and then here we have nu 13 by E 3 minus nu 23 by so, this is E 1, E 2 and here we have 1 by E 3.

So, 0 0 0 we have 0 0 0, 0 0 0 and then 0 0 0, here we have 1 by G 12 0 0, 0 0 0, 1 by so, this is G 23 and here we have 1 by G 31, 0 and here we have 0 0 0 0 0, 1 by G 12. Now, that is our S matrix. So, if we compare the relation that we had before, there we had sigma is equal to C multiplied by epsilon. So, that gives me that C is just inverted form of S matrix. So, we need so, with that we can we can know C in terms of this material properties E 1, E 2, E 3, G 12, G 23, G 31 and nu 12, nu 23 and nu 31.

So, then if we combine everything so, we know the C matrix in terms of those E's, nu's and G's and then we have this relation. So, using these relations and these C matrix in the form of those E's finally, we get. So, we can say that satisfying the plane stress condition, we can write this as Q 11, Q 12 0, Q 12 Q 22 0 and we have 0 0, Q 66 which is equal to epsilon 1, epsilon 2, epsilon 6 and epsilon 2 we can write gamma 12 also, this we can write tau 12 also. So, these Q's come in terms of the C's and we know the C's in terms of the E's, nu's and G's.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

$$[S] = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

$$\{\sigma\} = [c]\{\varepsilon\}$$

$$[c] = [S]^{-1}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

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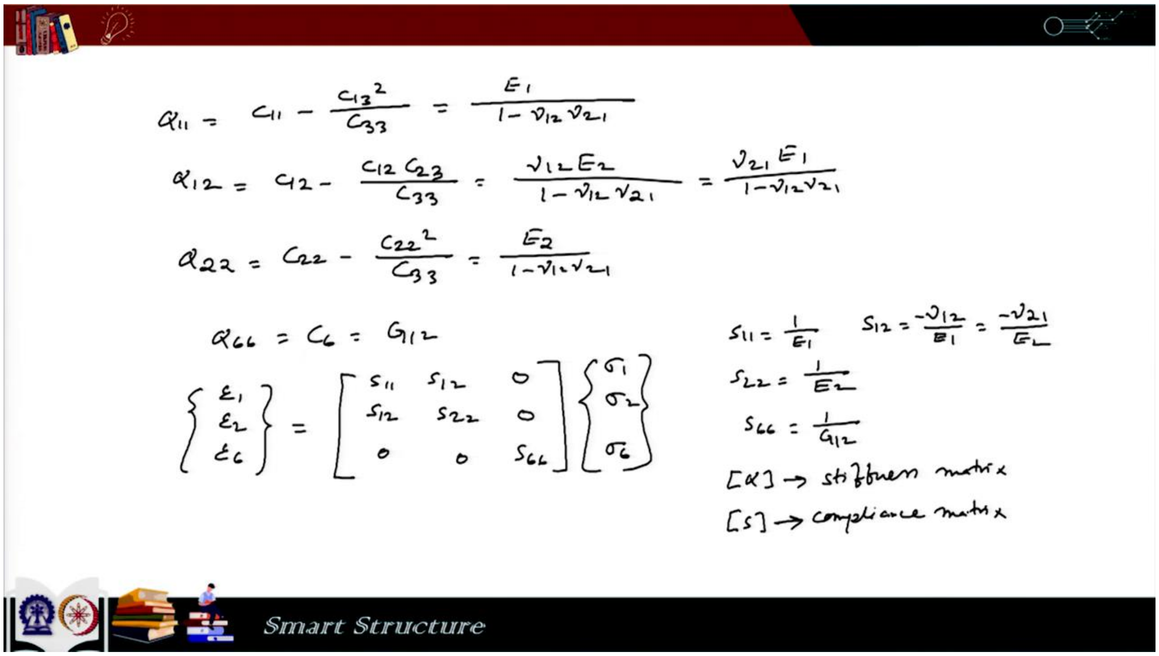

So, finally, that expression looks like the expression for the individual components of that Q matrix looks like this Q 11 is equal to C 11 minus 13 square divided by C 33. So, we have E 1 divided by 1 minus nu 12 nu 21, Q 12 is C 12 minus C 12 C 23 divided by C 33 and that comes as nu 12 E 2 by 1 minus nu 12 nu 21 or we can write nu 21 E 1 by 1 minus nu 12 nu 21 and then we have Q 22 is equal to C 22 minus C 22 square by C 33 which looks like E 2 by 1 minus nu 12 nu 21 and then we have Q 66 which is just C 6 and which is just equal to G 12. Now this generally is called stiffness matrix Q and this can be written in the inverted form as these also epsilon 1 epsilon 2 epsilon 6 as S 11 S 12 0 S 12 S 22 0 0 0 S 66 and here we have sigma 1 sigma 2 sigma 6. S 11 is 1 by E 1, S 12 is minus nu 12 by E 1 which can also be written as minus nu 21 by E 2, S 22 is 1 by E 2 and S 66 is 1 by G 12. So, Q is often called stiffness matrix, but do not confuse it with stiffness term which we define as load by displacement and this is often called compliance matrix. So, now we have defined the stress strain relation in a thin ply in the principle material system.

$$Q_{11} = c_{11} - \frac{{c_{13}}^2}{c_{33}} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = c_{12} - \frac{c_{12}c_{23}}{c_{33}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}$$

$$Q_{22} = c_{22} - \frac{{c_{22}}^2}{c_{33}} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

$$Q_{66} = c_6 = G_{12}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$S_{11} = \frac{1}{E_1}$$

$$S_{12} = \frac{-\nu_{12}}{E_1} = \frac{-\nu_{21}}{E_2}$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{66} = \frac{1}{G_{12}}$$

Now, later on we have to convert it to a global x y system. So, that we will do in the next lecture. So, let us end this lecture here.

Thank you.