Smart Structures

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Week - 06
Lecture No - 31

Constitutive Relation of Unidirectional FRP Composite Ply

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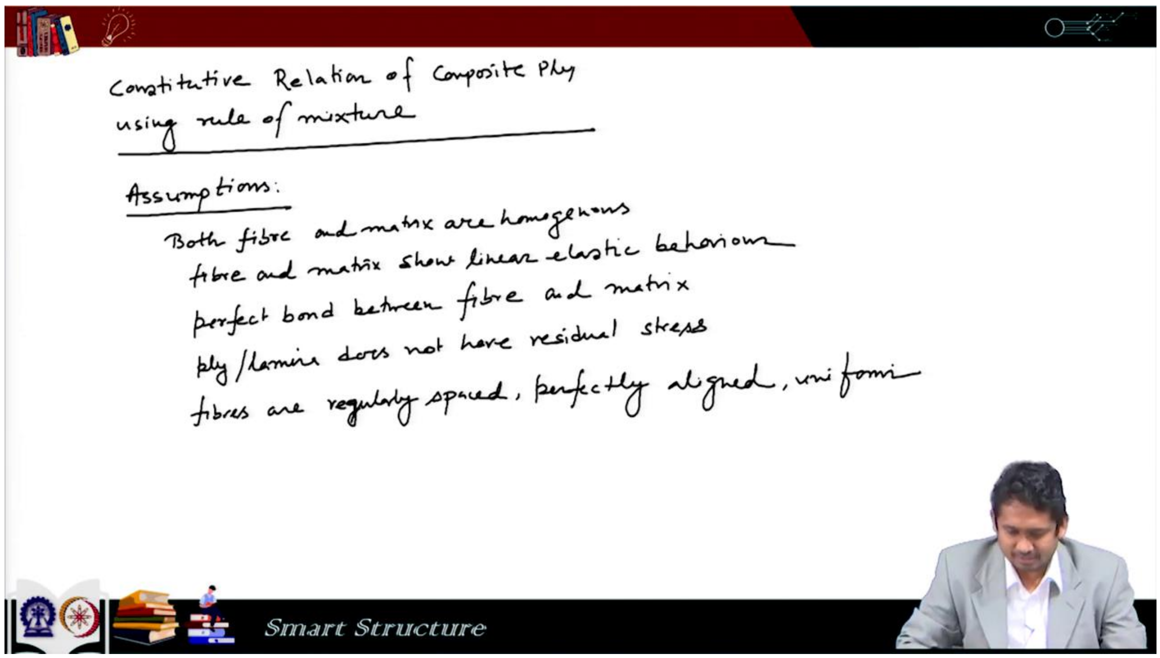

Welcome to the second lecture of week 6.

Today we will discuss about deriving the constitutive relations of composite ply with the help of rule of mixture-based homogenization. So we will use rule of homogenization for constitutive relation of composite ply using rule of mixture. Now if you want to do it, we have to make some assumptions. So there are some assumptions. These assumptions are (i) both fiber matrix are homogeneous and (ii) fiber matrix show linear elastic behavior and then we also assume that (iii) there is perfect bond between fiber matrix and (iv) ply which will also called lamina does not have residual stress. (v) We also assume that fibers are regularly spaced and perfectly aligned and also they are uniform. Now with these assumptions that make the situation somewhat ideal, we will derive all the relations.

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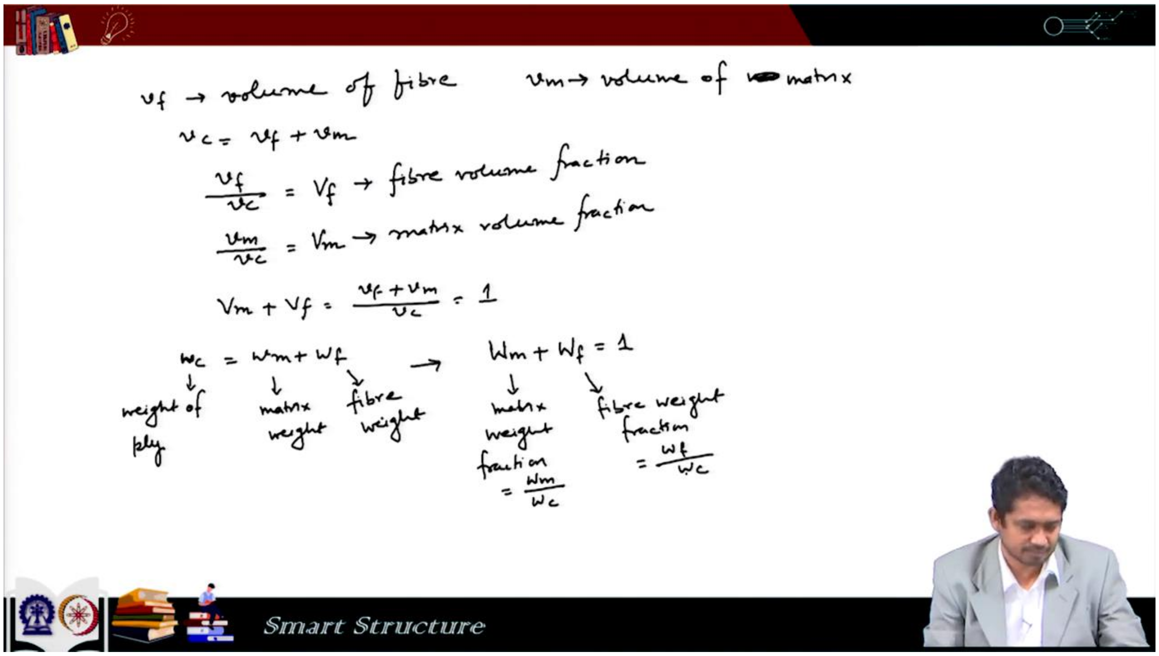

So, our ply has both matrix and laminate. So if we assume that vf is the volume, sorry our ply have both matrix and fiber and if we assume that vf is the volume of fiber and vm is the volume of matrix volume of matrix and then we say that vc is vf plus vm which is the total volume of ply then we can define something like vf by vc which we call capital Vf which means this is fiber volume fraction and then similarly we can write vm by vc which we call Vm is matrix volume fraction and if we add Vm plus Vf this is vf plus vm by vc and we also we know that vc is also vf plus vm so it is 1.

Now it is assumed that there is no void. If there is any void so accordingly there can be a void volume fraction also and then void volume fraction plus volume fraction of matrix and fiber that becomes 1. Similarly we can define weight wc as wm plus wf where this is wc is weight of ply and this is matrix weight and this is fiber weight and then using the same logic we can write the matrix weight fraction plus fiber weight fraction is 1. This is matrix weight fraction which is weight of matrix by weight of composite and this is fiber weight fraction which is weight of fiber by weight of composite.

$$v_c = v_f + v_m$$

$$\frac{v_f}{v_c} = V_f$$

$$\frac{v_m}{v_c} = V_m$$

$$V_m + V_f = \frac{v_f + v_m}{v_c} = 1$$

$$w_c = w_m + w_f$$

$$W_m + W_f = 1$$

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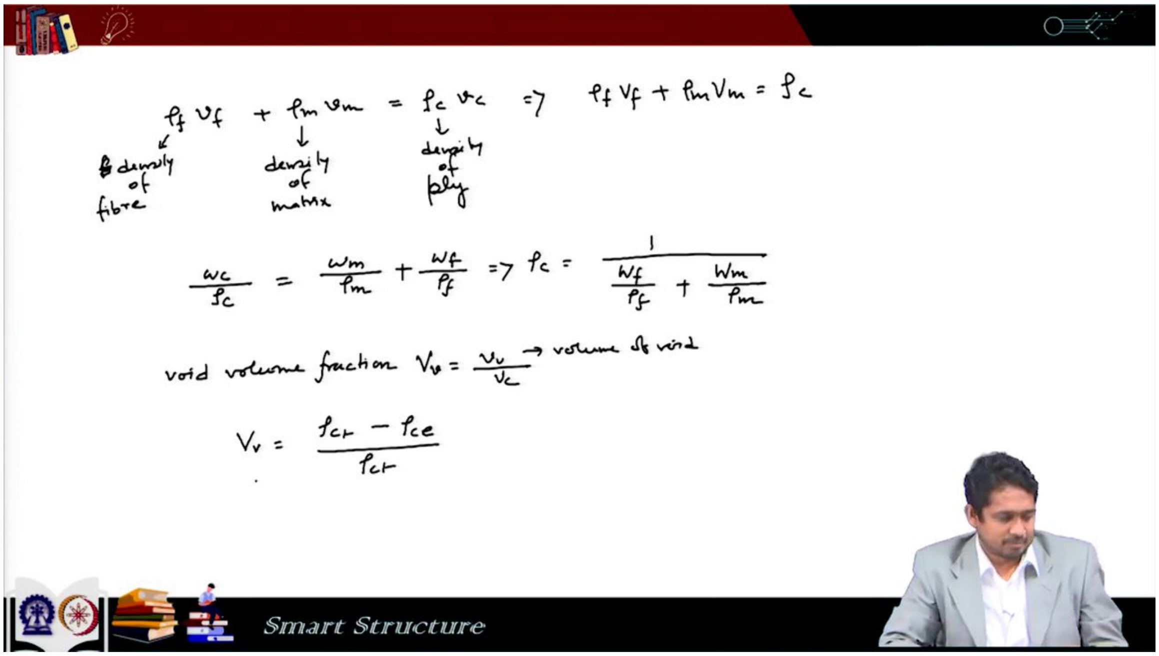

Now we can write that rho f into vf where rho f is the density of fiber plus rho m into vm which is density of matrix. We can write this as rho c into vc. Here rho c is density of ply and again this can be written as rho f vf plus rho m vm is equal to rho c. So the density of the ply is weighted combination of the density of matrix and density of fiber where the weights are the volume fractions. We can also write weight of composite by the density of composite which is the volume of the composite as weight of the matrix divided by the density of the matrix which is the volume of the matrix plus weight of the fiber, density of the fiber which is volume of the fiber. So from here we can directly show that the density of fiber can be written as weight fraction of fiber divided by fiber density plus weight fraction of the matrix divided by matrix density.

Now if we have something called void volume fraction as vc, so this is volume of void. So if there is void then the density thus calculated is going to be more than the actual density because here we do not have any void and voids do not have weight. So this is we know the density that is got theoretically and experimentally if we get any less density then it becomes rho ct which means theoretically obtained density which is this minus rho ce which is experimentally obtained density divided by rho ct.

$$\rho_f v_f + \rho_m v_m = \rho_c v_c$$

$$\rho_f V_f + \rho_m V_m = \rho_c$$

$$\frac{w_c}{\rho_c} = \frac{w_m}{\rho_m} + \frac{w_f}{\rho_f}$$

$$\rho_c = \frac{1}{\frac{w_f}{\rho_f} + \frac{w_m}{\rho_m}}$$

$$V_v = \frac{v_v}{v_c}$$

$$V_v = \frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}$$

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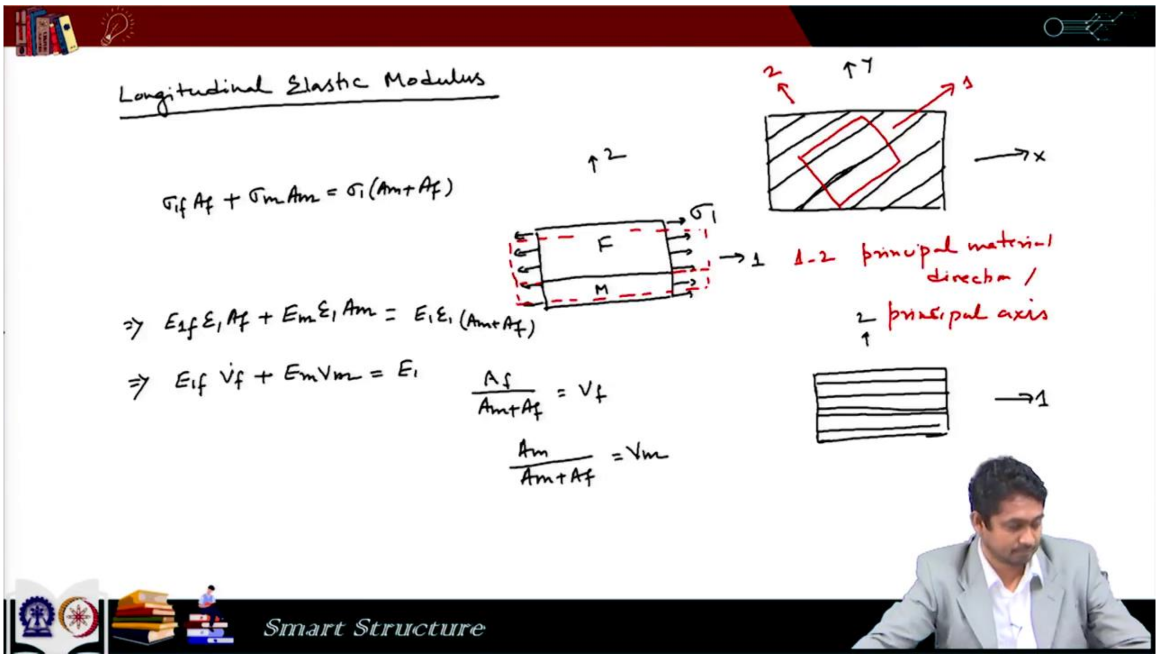

Now we will define the elastic properties. So let's talk about longitudinal elastic modulus. So our composite ply looks like this. We have plies oriented in some direction. Some theta its plies are oriented and our assumption is that they are aligned properly and spacing is uniform. So here we can talk about two axis one is let's define as x and y. So this is we can say it is a global axis and we can align one another axis.

We can define another axis which we assume to be oriented along the fiber direction and let's call it axis one and let's call it axis two. Now one two is principle material direction or we call it principle axis. Now we will find out the properties of the plies in the principle axis. So along the principle axis we can make the same drawing as this. If we draw it along principle axis it will look like this.

Now our goal is to find out the elastic modulus along direction one. So let's idealize and assume that all the fiber volume is concentrated on one side and the matrix is at the other side. This is just an assumption, idealization I would say for the sake of illustration. It has nothing to do with the physical picture it is just for the sake of illustration. We are showing that fiber is at the one side and matrix is at the other side.

And let's assume that this entire lamina in the direction one is under a stress sigma one. So sigma one is applied along direction one and then under the action of this, this deforms and it deforms in this way. The deformed shape can look like this. The red dashed line shows the deformed shape. So what's happening here is because there is a perfect bond between matrix and fiber, so one is not going to slip over other.

So they are always in contact which means that the longitudinal strain experienced by the fiber and matrix are same. So strains are same. So the total load carried out by the matrix and total load carried out by the fiber on being added would give me the total load carried by the ply. So we can write sigma 1f which means the sigma 1 in the fiber is equal to multiplied by the area of the fiber plus the area of this face, the area is normal to the one axis plus sigma m multiplied by area of the matrix is equal to sigma one multiplied by the total area of the matrix and fiber. So here sigma 1f means the stress at the fiber, sigma m means stress at the matrix and sigma 1 means the sigma 1 over the composite.

Now stress are not different whatever the stress here whatever the stress here and they are just sigma 1. So sigma 1 is equal to sigma f is equal to sigma m. Now sigma 1f is E 1f multiplied by epsilon 1 and we are multiplying that with Af, sigma m means E m and that is multiplied with epsilon 1 and we are multiplying that with Am and then we have sigma 1 which is E one, E1 is equivalent elastic modulus of the ply that is multiplied with epsilon 1 into Am plus Af. So please understand the stress are different but the strains are same. Then E1 cancels from both side and we bring Am by Af in the denominator here and that gives us E 1f Vf plus Em Vm is equal E1.

Because if we have Af divided by Am plus Af, it is the ratio of the fiber area divided by the total area and the other dimension is same. So it is the ratio of the volumes also and that is Vf. Similarly, Am by Am plus Af is equal to Vm. So that is our elastic modulus in the longitudinal direction of the composite in terms of the individual elastic modulus in the longitudinal direction of the matrix and fiber.

$$\sigma_{1f}A_f + \sigma_m A_m = \sigma_1(A_m + A_f)$$

$$E_{1f}\varepsilon_1 A_f + E_m\varepsilon_1 A_m = E_1\varepsilon_1(A_f + A_m)$$

$$E_{1f}V_f + E_m V_m = E_1$$

$$\frac{A_f}{A_m + A_f} = V_f$$

$$\frac{A_m}{A_m + A_f} = V_m$$

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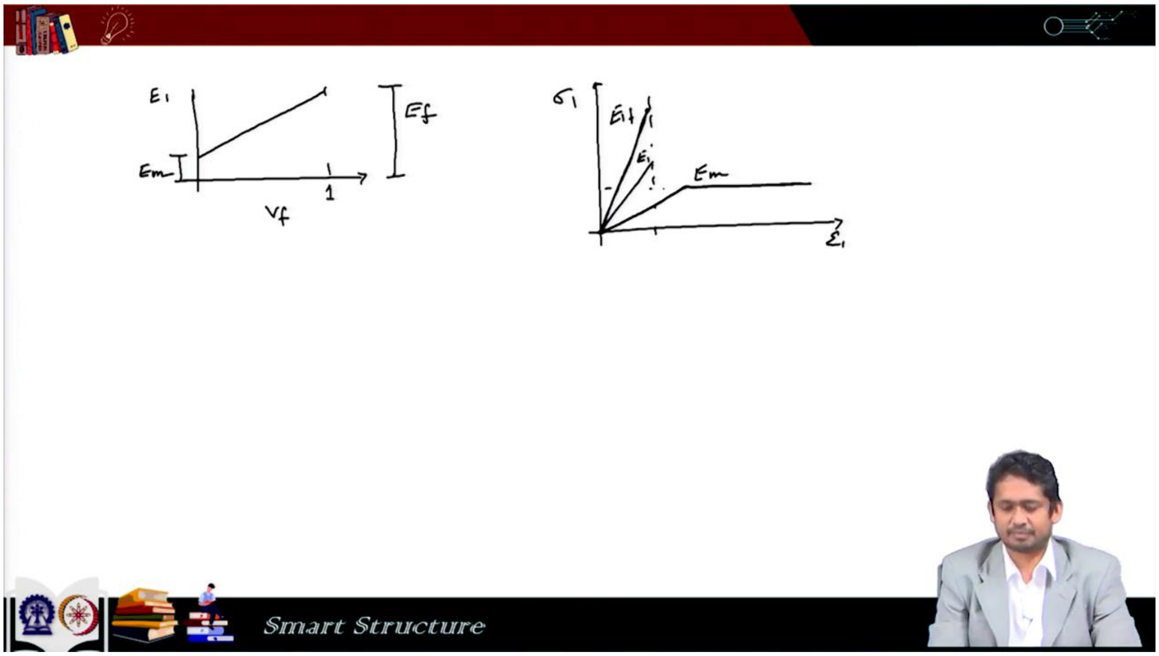

Now given this expression if we want to see the variation of E1 with change in Ef, it would look something like this. So, this is our Vf it is increasing in this direction. This is Vf equal to one. So when V f equal to zero which means there is no fiber E one is just equal to E m and when Vf is equal to 1 which means there is no matrix, it is only fiber then E1 is equal to Ef and in between them, they are varying linearly. And also, if we look at the individual stress strain behavior of the matrix and fiber, it looks like this. The fiber behaves like this.

It is a matrix, so the matrix behaves like this. The matrix does not break suddenly and then we have the fiber. Fiber has a much higher elastic modulus and it is brittle. It breaks all of a sudden. And the composite is somewhere in between, so the elastic modulus of the composite along one direction is between what we get for the fiber and for the matrix. So fiber is very stiff, matrix is relatively less stiff and the composite is in between so it is E1f, this is Em and this is E1.

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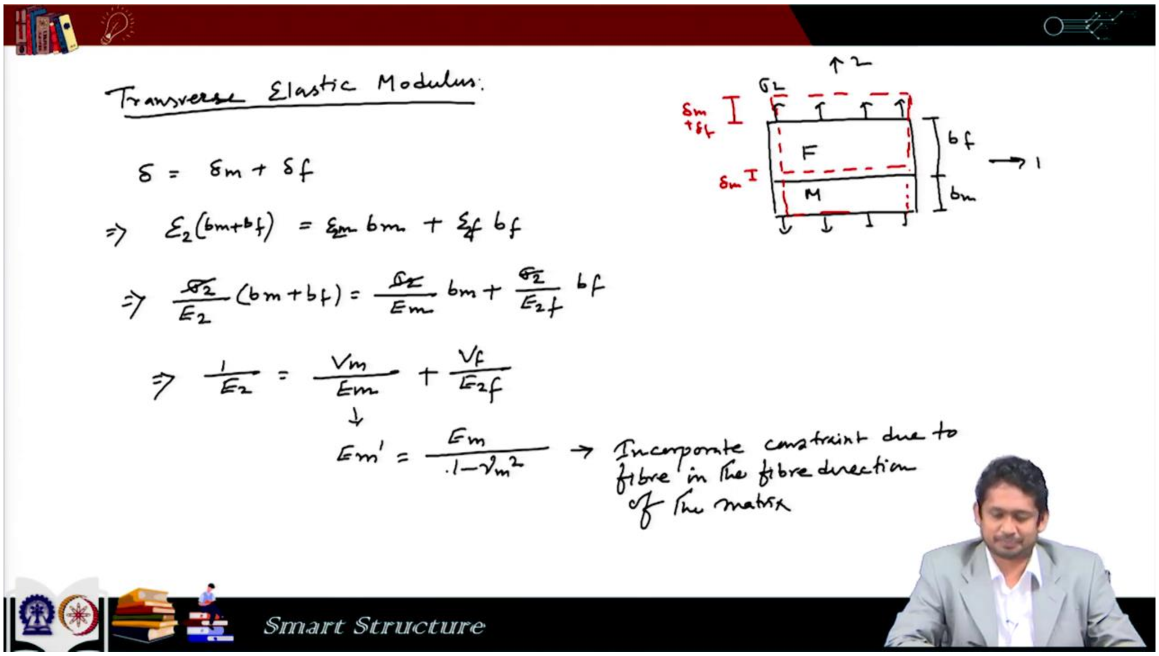

Now we will talk about the same properties in the transverse direction. In the transverse direction so we look at the same idealized diagram when it is loaded in the transverse direction. We have fiber here, we have matrix here. It is loaded in sigma 2 direction.

Stress is sigma 2. This is direction 1, this is direction 2. Now after deformation, this can look like this. So there is delta m which is displacement, I mean the elongation of the fiber matrix and then there is delta f and combining delta m and combining delta f, we get the total deformation. Now here the stress is same and strain is different. So, the total delta which is delta m plus delta f.

Now delta m is the elongation in the matrix part in the transverse direction which can be written as strain in the matrix multiplied by the initial width so if we say bm is the dimension of the matrix part along 2 direction and bf if we say matrix of the dimension of the matrix along the 2 direction, so it is epsilon m multiplied by bm. It is epsilon f multiplied by bf and then we have delta is equal to epsilon 2 multiplied by the bm plus bf. Now epsilon 2 can be written as sigma 2 divided by E2. Sigma 2 is the stress along the second direction 2, E2 is the equivalent elastic modulus along direction 2, so bm plus bf epsilon 2m is again the strain stress along direction 2 which is same so epsilon 2 by Em and then we have bm and then epsilon 2 divided by E2f multiplied by bf. Now epsilon 2 is same everywhere so it cancels out.

We have 1 by E2 is equal to Vm by Em plus Vf by E2f. So this is our relation for the elastic modulus along direction two in terms of the elastic modulus of the fiber along direction 2 and matrix elastic modulus. Now Em is generally used as Em prime where Em prime is Em by 1 minus nu m square and it is for the incorporation due to fiber in the fiber direction of the matrix. So to incorporate the constraint due to the fiber in the fiber direction of the matrix. Em is often modified to be Em prime and used in the formula. So this is what we get when we use this kind of strength of material based approach using rule of mixture to get the relation.

$$\delta = \delta_m + \delta_f$$

$$\varepsilon_2(b_m + b_f) = \varepsilon_{2m} b_m + \varepsilon_{2f} b_f$$

$$\frac{\sigma_2}{E_2}(b_m + b_f) = \frac{\sigma_2}{E_m} b_m + \frac{\sigma_2}{E_{2f}} b_f$$

$$\frac{1}{E_2} = \frac{V_m}{E_m} + \frac{V_f}{E_{2f}}$$

$$E_{m'} = \frac{E_m}{1 - {\gamma_m}^2}$$

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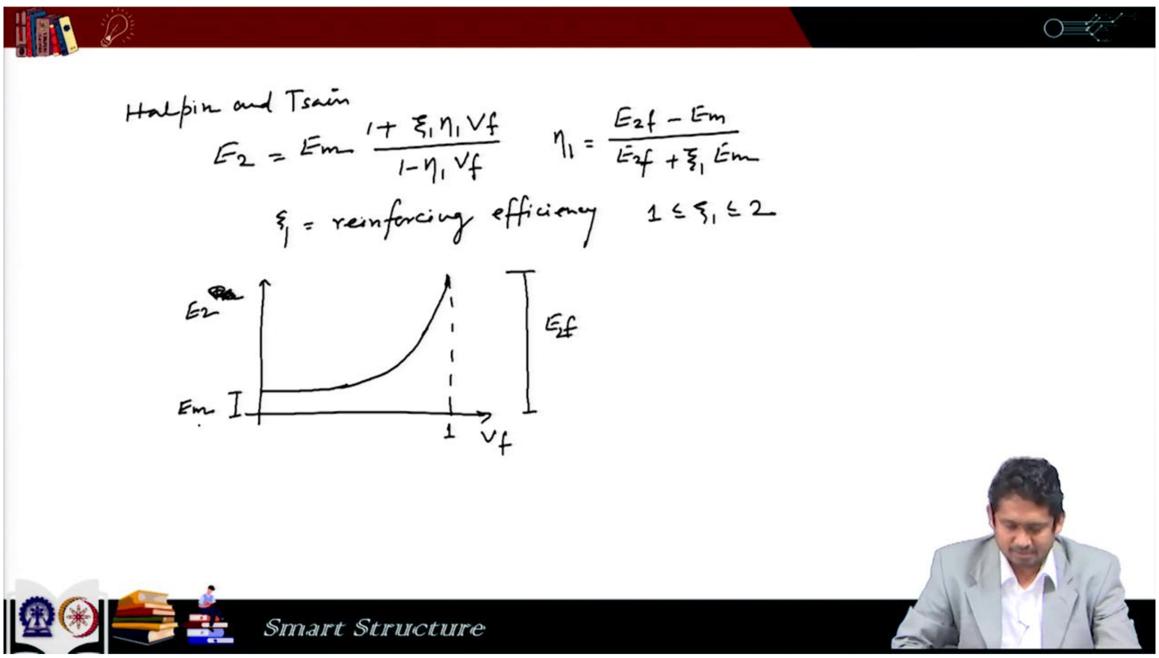

Now Halpin and Tsai they proposed a formula E2 is equal to Em multiplied by 1 plus xi1 eta 1 Vf by 1 minus eta 1 into Vf where eta 1 is equal to E2f minus Em divided by E2f plus xi 1 plus Em. So xi 1 is called reinforcing efficiency which is generally xi 1 ranges from 1 to 2. So, through experiment, we can try to fit the formula and find out what is our xi 1 and then whatever the xi one is used we can use the same xi 1 for this material for this ply material. Now if we look at the variation of the transverse elastic modulus of the ply with Vf the plot would look like this.

So here the variation is not linear again the same thing. This is Vf equal to 1. So, when Vf equal to 1, the entire ply has only fiber. So, elastic modulus is just E2f and when Vf equal to only matrix, the elastic modulus is just Em.

$$E_2 = E_m \frac{1 + \xi_1 \eta_1 V_f}{1 - \eta_1 V_f}$$

$$\eta_1 = \frac{E_{2f} - E_m}{E_{2f} + \xi_1 E_m}$$

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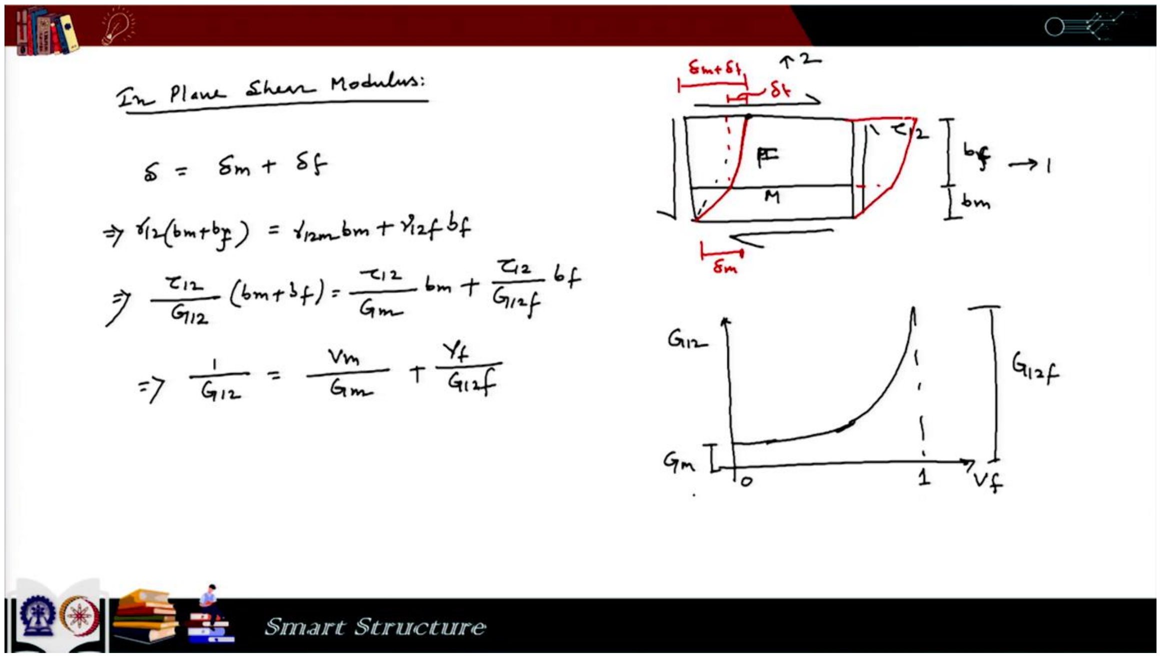

Now we will find out in-plane shear modulus. Now to find out in-plane shear modulus again, let us assume our ply is loaded in shear. So we have shear tau 12 and we have the fiber part here and the matrix part here. Now due to this kind of shear the deformation would look like this. Now this is delta m plus delta f due to shear and this is our delta m and this extra portion is delta f and that is the result of shear and also we have this as our dimension of the fiber part along the second direction. This is the dimension of the matrix along second direction.

This is direction 1. This is direction 2. So again we can write that delta is equal to delta m plus delta f. Now this delta m is nothing but the shear strain in the matrix part multiplied by this dimension bf. Similarly delta f is the shear strain in the fiber part multiplied by this dimension bf. So, delta m is shear-strain in the matrix which we can write as gamma 12m and then that is multiplied with bm.

Similarly this is gamma 12f and this is multiplied with bf and delta is the equivalent shear if I just join this point to this point, the line that I get that shows some angle here and that is going to give me. So, this angle is our equivalent shear gamma 12 and that multiplied by bm plus bf. Now equivalent shear gamma 12 can be written as the applied tau 12 divided by equivalent shear modulus G 12 and then we have bm plus bf and gamma 12 is again tau 12 that is applied. So stress is same throughout and that is divided by Gm multiplied by bm then we have tau 12 by G 12f multiplied by bf and finally after cancelling out tau 12 and bringing bm plus bf in the denominator here, we write Vm by Gm plus Vf by G 12f and again the relation would also look same as that that we saw for the transverse modulus case. So, if we plot Vf here and if we plot

G 12 here, the relation would look like this. So here we have G m. In this side, we have G 12f and this is Vf equal to one. This is zero.

$$\delta = \delta_m + \delta_f$$

$$\gamma_{12}(b_m + b_f) = \gamma_{12m} b_m + \gamma_{12f} b_f$$

$$\frac{\tau_{12}}{G_{12}}(b_m + b_f) = \frac{\tau_{12}}{G_m} b_m + \frac{\tau_{12}}{G_{12f}} b_f$$

$$\frac{1}{G_{12}} = \frac{V_m}{G_m} + \frac{V_f}{G_{12f}}$$

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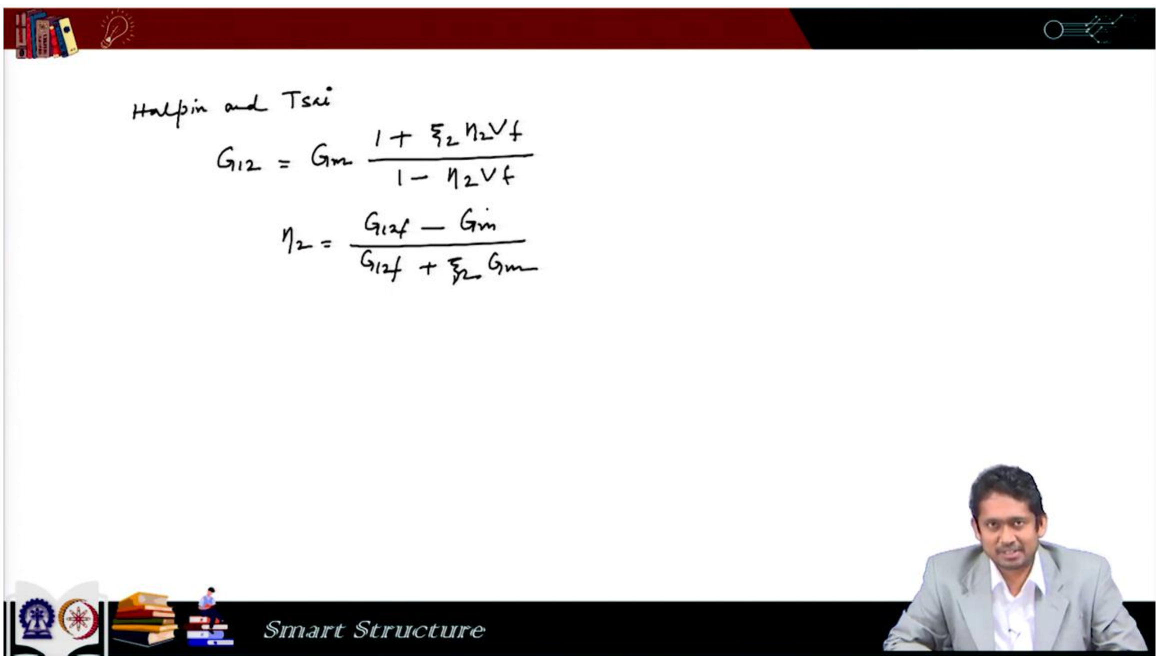

Now there is a equation by Halpin and Tsai for this case also and as per that equation it is Gm multiplied by in the denominator 1 plus Tsai two eta 2 Vf then divided by 1 minus eta 2 Vf where eta 2 is G 12f minus Gm divided by G 12f plus Tsai 2 Gm. So again we can experimentally find out Tsai 2 and we can fit the equation and we can use the same equation. Now this brings us to the end of this lecture.

$$G_{12} = G_m \frac{1 + \xi_2 \eta_2 V_f}{1 - \eta_2 V_f}$$

$$\eta_2 = \frac{G_{12f} - G_m}{G_{12f} + \xi_2 G_m}$$

We will continue from here in the next lecture.

Thank you.