Smart Structures
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Week 05
Lecture No: 29
Solution of Coupled Linear Ordinary Differential Equations
Part 05

So, far we discussed various applications like: actuation, energy harvesting, control and their mathematical modeling. Now in every exercise we saw that, at the end we get a set of coupled ordinary differential equations. So, in today's lecture, we will see how we can solve those ordinary differential equations. So, the generic form that we get is M X double dot, plus C X dot, plus K X, is equal to F. So, here F is a function of time. So, F changes with time and accordingly, our X, X dot and everything change. So, this is X.

$$[M]_{nxn}\{\ddot{X}\}_{nx1} + [C]_{nxn}\{\dot{X}\}_{nx1} + [K]_{nxn}\{X\}_{nx1} = \{F\}_{nx1}$$

Now, we will discuss numerical techniques to solve it. So, we will discuss two techniques. In the first technique, we will convert this to a first order equation and then we will solve.

So, convert to first order equation and solve. So, it is a second order differential equation, we will convert this to a first order differential equation. And let us assume that their size is n. So, these are all n by n matrices and all the vectors are n by 1. Now, if I want to convert to a first order equation, then assume another vector Y with Y_1 to Y_n is equal to X, which means x_1 to x_n, and also assume that, $Y_{n \text{ plus } 1}$ to Y_{2n} is x_1 dot, x_2 dot, all the way up to x_n dot.

So, we have a new vector Y, the first n components are just X and the rest of the n components are the derivatives of X, with respect to time. So, we can write Y_1 to Y_n as. So, we can write Y_1 dot, Y_2 dot all the way up to Y_n dot, as x_1 dot x_2 dot all the way up to x_n dot which is equal to $Y_{n \text{ plus } 1}$, $Y_{n \text{ plus } 2}$ all the way up to Y_{2n} from this equation. So, this is one set of equation that we get. Now, next what we do is - we write this in terms of the Y's.

$$\begin{Bmatrix} Y_1 \\ \vdots \\ Y_n \end{Bmatrix} = \{X\} = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} \qquad \begin{Bmatrix} Y_{n+1} \\ \vdots \\ Y_{2n} \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \vdots \\ \dot{Y}_n \end{Bmatrix} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix} = \begin{Bmatrix} Y_{n+1} \\ Y_{n+2} \\ \vdots \\ \dot{Y}_{2n} \end{Bmatrix}$$

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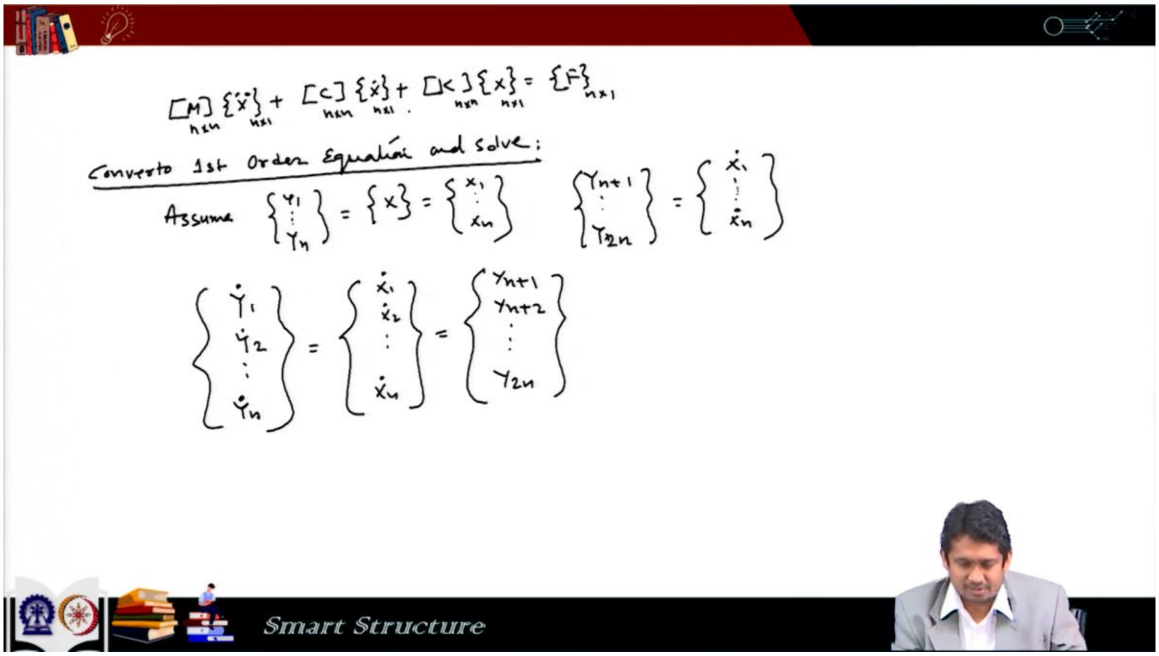

If we do that, we can write M, M is multiplied with X_1 double dot. So, X_1 double dot means dot of $Y_{n \text{ plus } 1}$ to Y_{2n} and then, we have C X dot, X dot means $Y_{n \text{ plus } 1}$ up to Y_{2n} plus we have K X which means Y_1 up to Y_n and then at the right hand side, we have F.

$$[M]\begin{Bmatrix}\dot{Y}_{n+1}\\ \vdots\\ \dot{Y}_{2n}\end{Bmatrix}+[C]\begin{Bmatrix}Y_{n+1}\\ \vdots\\ Y_{2n}\end{Bmatrix}+[K]\begin{Bmatrix}Y_1\\ \vdots\\ Y_n\end{Bmatrix}=\{F\}$$

So, from this, we can say that, $Y_{n \text{ plus } 1}$ dot up to Y_{2n} dot is equal to F vector minus C $Y_{n \text{ plus } 1}$ all the way up to Y_{2n} minus K Y_1 up to Y_n and this entire thing is multiplied with M inverse.

$$\begin{Bmatrix}\dot{Y}_{n+1}\\ \vdots\\ \dot{Y}_{2n}\end{Bmatrix}=[M]^{-1}\left\{\{F\}-[C]\begin{Bmatrix}Y_{n+1}\\ \vdots\\ Y_{2n}\end{Bmatrix}-[K]\begin{Bmatrix}Y_1\\ \vdots\\ Y_n\end{Bmatrix}\right\}$$

So, we can write in a more compact way as Y_1 dot up to Y_n dot, $Y_{n \text{ plus } 1}$ dot up to Y_{2n} dot as a matrix 0 n by n, identity matrix and then we have M inverse K and then, we have M inverse C and this entire thing is multiplied by Y_1 up to Y_n, $Y_{n \text{ plus } 1}$ up to Y_{2n} and then, plus we have 0, n by 1 plus M inverse F.

$$\begin{Bmatrix} \dot{Y}_1 \\ \vdots \\ \dot{Y}_n \\ \dot{Y}_{n+1} \\ \vdots \\ \dot{Y}_{2n} \end{Bmatrix} = \begin{bmatrix} [0]_{nxn} & [I]_{nxn} \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} Y_1 \\ \vdots \\ Y_n \\ Y_{n+1} \\ \vdots \\ Y_{2n} \end{Bmatrix} + \begin{Bmatrix} \{0\}_{nx1} \\ [M]^{-1}[F] \end{Bmatrix}$$

So, this can be written as Y vector which is a 2n by 1 vector is equal to a matrix which is a 2n by 2n matrix. So, this matrix we call A matrix multiplied by Y matrix which is a 2n by 1 vector, plus let us call it B vector, which is a 2n by 1 vector. So, and this dot. So, this is a first order system of - it is a system of first order differential equations.

$$\{\dot{Y}\}_{2nx1} = [A]_{2nx2n}\{Y\}_{2nx1} + \{B\}_{2nx1}$$

So, we started with a system of second order differential equations, but the system size was n there, there were n equations. After conversion, we are getting a system of first order differential equations, but the size has doubled.

Now, the question is how we can solve this system of equations.

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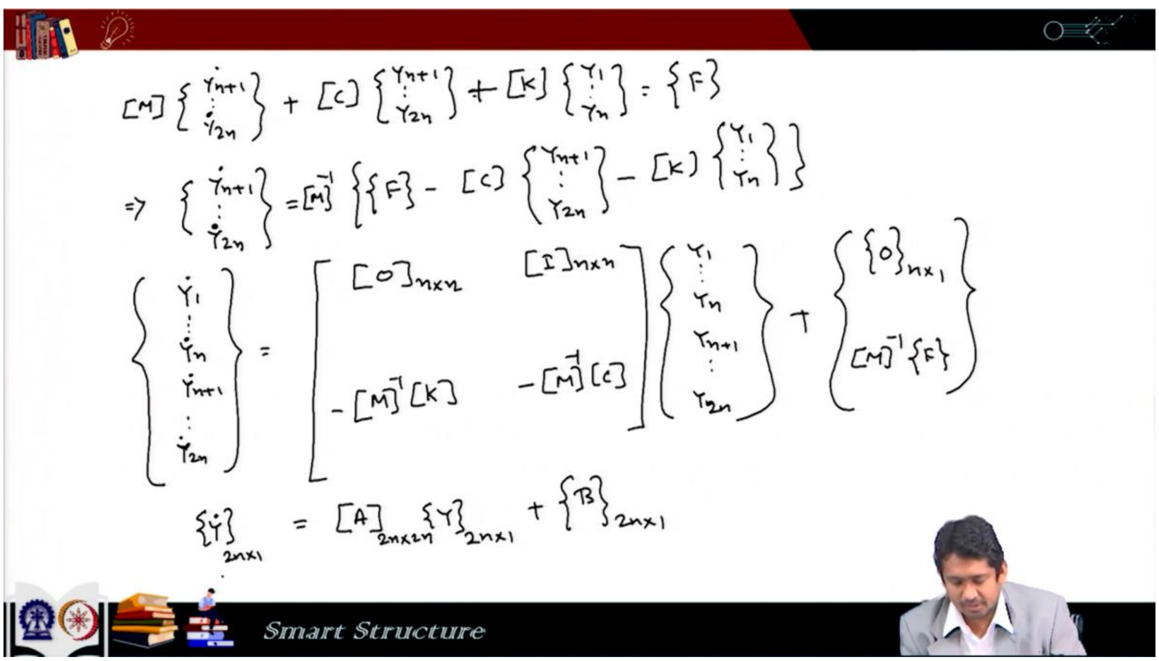

So, for that we will use - let's see a finite difference based technique. As for the finite difference scheme. So, before that let's just explain that we are going to do time marching and solve it. So, we are going to solve this by time marching.

So, at the time axis we have t_0, t_1, t_n, $t_{n \text{ plus } 1}$ and so on. So, we have to solve it at each and every discrete time point. Now, if we use finite difference scheme, we know that the derivative at the nth time step t_n, if I write that as Y dot n is equal to $Y_{n \text{ plus } 1}$ minus Y_n by delta t. So, delta t is a time interval. So, delta t is time interval.

$$\{\dot{Y}\}_n = \frac{\{Y\}_{n+1} - \{Y\}_n}{\Delta t}$$

From this, we can write $Y_{n \text{ plus } 1}$ is equal to Y_n plus delta t, plus $Y_{n \text{ plus } 1}$ is equal to Y_n plus delta t Y_n dot. So, this is our forward difference scheme.

$$\{Y\}_{n+1} = \{Y\}_n + \Delta t\{\dot{Y}\}_n$$

Similarly, for backward difference, we can write the derivative at $Y_{n \text{ plus } 1}$ of n plus 1time step is equal to $Y_{n \text{ plus } 1}$ minus Y_n by delta t and that is equal to $Y_{n \text{ plus } 1}$ is equal to Y_n plus delta t $Y_{n \text{ plus } 1}$ dot and this is backward difference scheme.

$$\{\dot{Y}\}_{n+1} = \frac{\{Y\}_{n+1} - \{Y\}_n}{\Delta t}$$

$$\{Y\}_{n+1} = \{Y\}_n + \Delta t\{\dot{Y}\}_{n+1}$$

Now, we can generalize it by a parameter alpha, and we can write it $Y_{n \text{ plus } 1}$ is equal to Y_n plus, Y_n plus 1 minus alpha, Y_n dot, plus alpha Y dot at n plus 1.

$$\{Y\}_{n+1} = \{Y\}_n + (1 - \alpha)\{\dot{Y}\}_{n+1} + \alpha\{\dot{Y}\}_{n+1}$$

So, when alpha is equal to 0, this corresponds to this scheme the forward difference scheme, when alpha is equal to 1, this becomes 0. So, this corresponds to the backward difference scheme and if alpha is between 0 and 1, it is kind of weighted combination between of the forward difference and the backward difference scheme.

Now, we have to do the time marching from here. So, if we know the history so, we suppose we want to find out the solution at $t_{n \text{ plus } 1}$ then, if we know the solutions before that then we can make use of this expressions and then we can find out the solution.

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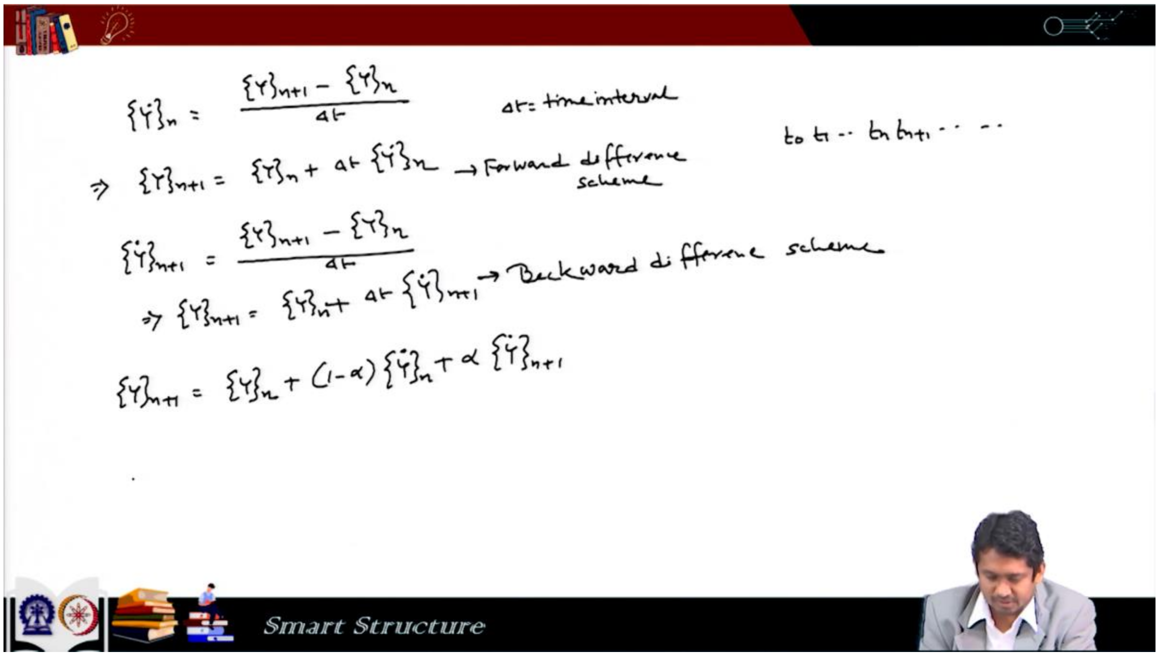

Suppose, we are doing it using the forward difference scheme. So, by the forward difference scheme, we know that $Y_{n \text{ plus } 1}$ is equal to Y_n plus delta t.

$$\{Y\}_{nx1} = \{Y\}_n + \Delta t\{\dot{Y}\}_n$$

So, Y at n plus 1 is equal to Y at n plus delta t multiplied by Y dot n. Now, we know everything before the time step at which are getting the solution. So, we know Y_n and we know Y_n dot. So, we can find out $Y_{n \text{ plus } 1}$ and then, we can from this equation, we can find out the $Y_{n \text{ plus } 1}$ dot also. And once we know everything at the n plus 1 time step we can go to the n plus second time step.

$$\{\dot{Y}\}_{n+1} = [A]\{Y\}_{nx1} + \{B\}$$

So, at time t is equal to 0, we know only the initial condition. So, at time t is equal to 0, we know the value of Y and Y dot both are known to me, then we go to time t_1. So, at time t_1, this becomes Y_1. So, it is Y_1 is equal to Y_0 plus delta t multiplied by Y dot at 0.

So, we can find out this. Then we come here, we find out Y dot at first time step is equal to A multiplied by Y at 1 plus B. And then, once this Y once this Y and Y dots are known to us again we can repeat the procedure and we can keep moving forward. Now, if we use the forward difference scheme, and if we do it otherwise, then we just need to do it in this way.

So, we have $Y_{n \text{ plus } 1}$ is equal to Y_n, Y_n dot plus alpha multiplied by $Y_{n \text{ plus } 1}$ dot and here we have $Y_{n \text{ plus } 1}$ dot is equal to A multiplied by $Y_{n \text{ plus } 1}$ plus B.

$$\{Y\}_{n+1} = \{Y\}_n + (1-\alpha)\{\dot{Y}\}_{n+1} + \alpha\{\dot{Y}\}_{n+1}$$

$$\{\dot{Y}\}_{n+1} = [A]\{Y\}_{nx1} + \{B\}$$

So, again at time t is equal to 0. We know Y_0 and we know Y dot. So, we know Y and Y dot at time t is equal to 0.

Now, at time t equal to 1 in this equation, we have 2 unknowns – this is also unknown, this is also an unknown. In the previous case, this term was not there. So, there was only one unknown which is $Y_{n \text{ plus } 1}$. So, we could easily find it out. Here we have 2 unknowns, Y at n plus 1, and Y dot at n plus 1, in the second equation also, we have 2 unknowns, Y dot at n plus 1, and Y at n plus, 1 and anything at the nth time step is known to us. So, that is not a problem although it is not as straight forward as the previous one, but again it can be solved. We have 2 equations and 2 unknowns. So, we can solve it. We can express our Y at n plus 1 and Y dot at n plus 1 in terms of Y at n and Y dot at n. And accordingly, we can keep doing it and move forward.

Now, depending on the alpha these methods have different names and different criterias. So, when alpha is equal to 0. It is called forward difference and here the order of accuracy is of order delta t. When alpha is equal to half it is called Crank. It's called Crank Nicolson scheme and here the order is delta t square. And when alpha is 2 by 3 it is called Galerkin method. But please understand, this Galerkin method is not what we did before when we wanted to solve the partial differential equations. And when it is 1 it is backward difference method and here the order of accuracy is delta t.

Now, this method is conditionally stable. And these are stable.

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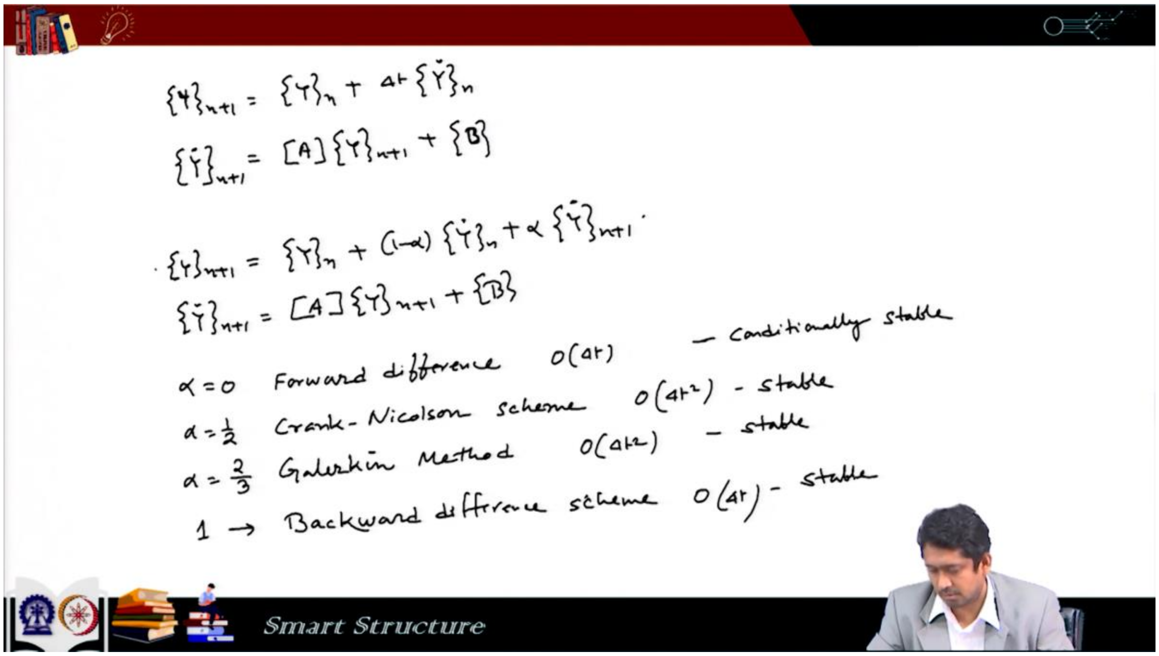

Now, we will discuss the other option, where we solve the second order differential equation.

So, solve the second order differential equations. So, here we do not convert it to first order differential equation. Now, for that what we do is - we have seen that using finite difference, a function can be written in terms of its previous value and its derivatives. So, here we will do it twice. The derivatives will also be written in terms of the present value of the derivative and the double derivatives.

So, we have X dot n plus 1. So, we write the derivative at the present time step in terms of the value at the previous time step plus 1 minus gamma. So, instead of alpha we write the parameter as gamma here, plus gamma delta t, n plus 1.

$$\{\dot{X}\}_{n+1} = \{\dot{X}\}_n + (1-\gamma)\Delta t\{\ddot{X}\}_n + \gamma\Delta t\{\ddot{X}\}_{n+1}$$

And then similarly, $X_{n \text{ plus } 1}$ is written as X_n plus delta t, X_n plus. After that it should be - after that we expect a term of X dot at n plus 1. And instead of that we write X dot at n plus 2 beta plus 1.

$$\{X\}_{n+1} = \{X\}_n + \Delta t\{\dot{X}\}_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\{\ddot{X}\}_n + 2\beta\{\ddot{X}\}_{n+1}\right]$$

And again, we have the equation M X at n plus 1 double dot, plus C X dot at n. So, we have M X double dot at n plus n, plus C X dot at n plus n, plus K at X n plus 1. So, we have M multiplied by X double dot at n plus 1, plus C multiplied by X double dot at n plus 1, plus K multiplied by X at n plus 1, is equal to F at n plus 1.

$$[M]\{\ddot{X}\}_{n+1} + [C]\{\dot{X}\}_{n+1} + [K]\{X\}_{n+1} = \{F\}_{n+1}$$

Now, we can see that we have three unknowns at the present time step X, X dot and X double dot. Everything at the previous time step is known to us and we have three equations. So, we can again solve it and do the time matching. So, if we write from here $X_{n \text{ plus } 1}$ becomes M plus gamma delta t, plus C plus beta delta square K inverse multiplied by plus 1 minus.

$$\{\ddot{X}\}_{n+1} = \left[[M] + \gamma\Delta t[C] + \beta\Delta t^2[K]\right]^{-1}\Bigg\{\{F\}_{n+1} - \Delta t(1-\gamma)[C]\{\ddot{X}\}_n - \frac{\Delta t^2}{2}[K](1-2\beta)\{\ddot{X}\}_n - [C]\{\dot{X}\}_n - \Delta t[K]\{\dot{X}\}_n - [K]\{X\}_n\Bigg\}$$

So, this is how we find out X double dot at n plus 1 time step using the values at the previous time step. Force is at the present time step, but we know that that's a known quantity. And from here, we can go back to this equation that gives us - from here we can go back to this equation that gives us the X dot at the present time step and from here you can come here and that's gives us the X at present time step. And we can keep repeating and we can get the solution.

Now again, when gamma is equal to to 0.5 and beta is equal to 0, we call it Explicit Central Difference. And, when gamma is equal to 0.5 and beta is equal to 0.25, we call it Average Constant Acceleration.

So, this is about solving these two equations, there are other methods too. Now, using these two equations the ordinary differential equations that we get for our actuation energy harvesting control all the problems can be solved.

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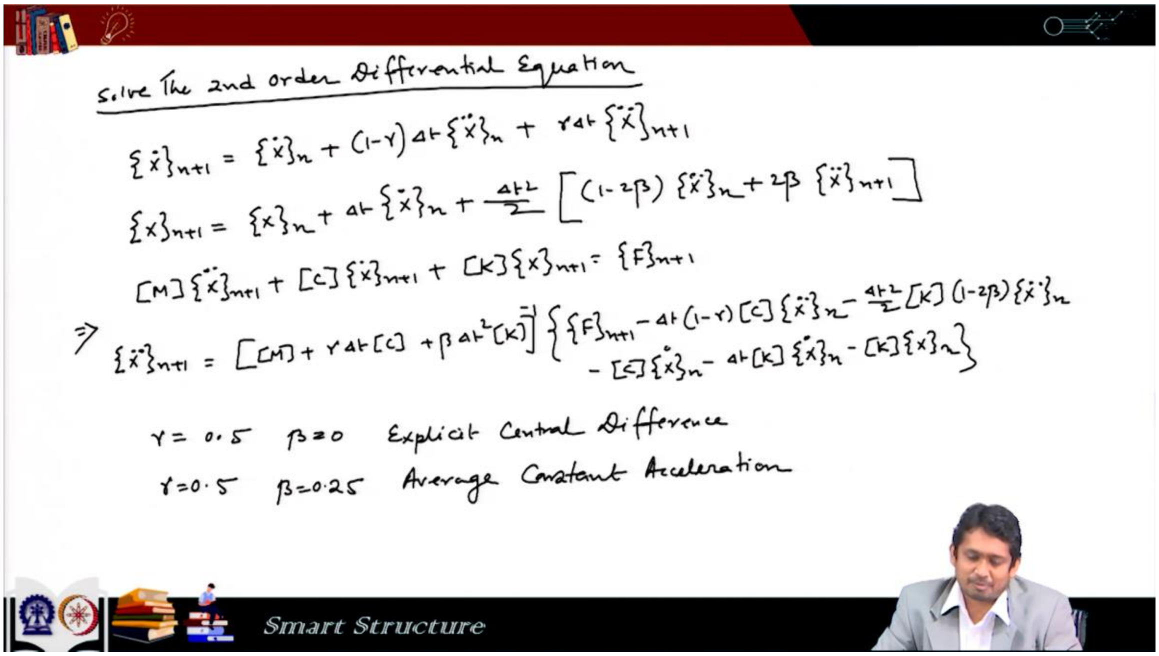

So, with that let us end this lecture.

Thank you.