Smart Structures
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Week 05
Lecture No: 28
Energy Harvesting and Vibration Control (continued)
Part 04

In the last lecture, we looked into the mathematical formulation of a beam with piezoelectric patches for energy harvesting applications. And we followed a paper for that.

Today we will look into the same formulation, but from a different perspective. We will start with the governing differential equations and we do the formulation. So, in the previous lecture, we did it by starting from the Hamilton's principle and there, we put the approximations. Here we will start with from the governing differential equation and we will do it. And after that, we look into the vibration control applications.

So, the governing equations in general for a 3D case as, we have seen that is this. Sigma i j comma i plus B_j, which is a body force is equal to rho u_j double dot and the other differential equation for the electrical domain is the Gauss law.

$$\sigma_{ij,i} + B_j = \rho \ddot{u}_j$$

$$D_{i,i} = 0$$

Now, if we again have our beam under consideration and if we have x as the axial dimension z as the vertical dimension and y is the other out of plane dimension. So, let us consider this and with Euler Bernoulli assumption and piezoelectric effect, under Euler Bernoulli assumption and considering piezoelectric effect, the linear momentum equation becomes this.

The corresponding governing differential equation is m b w double dot, plus del 2 w by del x 2 of EI_{total}, del 2 by del x 2 of EI_{total}, del 2 w by del x 2 minus p_z minus del 2 M_p by del x 2.

$$m_b \ddot{w} + \frac{\partial^2}{\partial x^2}\left(EI_{tot}\frac{\partial^2 w}{\partial x^2}\right) - p_z - \frac{\partial^2 M_p}{\partial x^2} = 0$$

So, we are very familiar with this equation. We already derived it. And for the other equation, is just this, the Gauss law is just this.

$$D_{z,z} = 0$$

Considering the fact that this piezoelectric patches are polarized in z direction, and so all the electric field voltage is in the z direction. Electrical displacement in z direction. So, these are the two equations now that we have to use to form the final energy harvesting equation. And here we will follow the notations that we were using in the strain induced actuation problems. So, in the previous lecture, we use the notations that was in the paper, but here, we have come back to the notations that we are using in the induced strain actuation problems. So, let us assume w as, as we have assumed phi w_j q w_j and the electric field was assumed to be - let us assume the electric field. So, E_z is equal to V, the voltage, which is a function of time, multiplied by psi. Psi is a function of z, and here q w j is also function of time.

So, we will make these assumptions and then we will go to the derivation. So, our approach is to multiply the first equation by phi w_j. So, if we do it.

$$w = \sum_{j=1}^{N} \phi_{w_j} q_{w_j}(t) E_z = v(t)\psi(z)$$

So, let us multiplied by phi w_i. Let us take i as the index, and then it is - we are just multiplying phi w i with the first equation and in the first equation, we put these approximations and we integrate it from 0 to L.

$$\int_0^L \phi_{w_i}\left(m_b \ddot{w} + \frac{\partial^2}{\partial x^2}\left(EI_{tot}\frac{\partial^2 w}{\partial x^2}\right) - p_z - \frac{\partial^2 M_p}{\partial x^2}\right)dx$$

And M_p is as we know as for the definition M_p is the area integral of minus z into elastic modulus multiplied by the free strain. And free strain, as we know that, it is d_{31} multiplied by electric field E_z or E_3, whatever, E_z is equal to E_3. And we have already assumed E_z as this.

$$M_p = \int_A -zE\varepsilon_p dA = \int_A -zEd_{31}E_z dA \qquad \text{and} \qquad E_z = E_3$$

So, finally, after putting this - after putting these approximations and following the procedure that we have followed; finally, this gives equation in this form. So, we have already seen how to derive the set of ordinary differential equations. From this kind of equations, it needs some shifting of derivatives also, which we have to do. And finally, if we do all these things these equations come - is equal to F.

$$[M]\{\ddot{q}\} + [K]\{q\} - [\Theta]\{v\} = \{F\}$$

So, here, M_{ij} is (can put q_w), M_{ij} is m_b, phi w_i, phi w_j dx which is equal to Ms_{ij} plus Mp_{ij}, as per the paper discussed. So, as per the paper discussed.

$$M_{ij} = \int_{0}^{L} m_b \, \phi_{w_i} \phi_{w_j} dx = M_{s_{ij}} + M_{p_{ij}}$$

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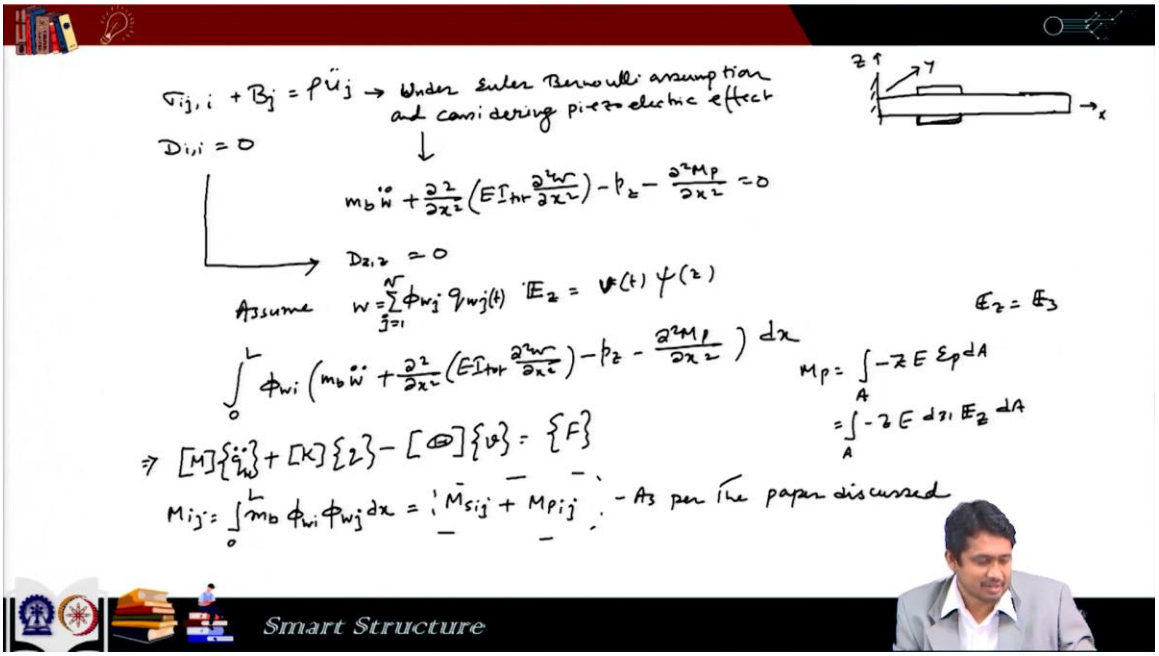

And then we have K_{ij} is equal to EI_{total}, and which is equal to Ks_{ij} plus Kp_{ij}, as per the paper. So, we are just showing the equivalence between the expressions using the two different formulations and notations.

$$K_{ij} = \int_{0}^{L} EI_{tot} \, \phi_{w_{i,xx}} \phi_{w_{j,xx}} dx = K_{s_{ij}} + K_{p_{ij}}$$

And we have theta i, which is minus 0 to L A z psi d_{31} Young's modulus multiplied by phi i comma x x dA dx. Now, please understand d_{31} multiplied by E, gives us E_{31}. So, this is phi. So, we are using phi here itself and F_i is 0 to L phi w_i p_z dx.

$$\Theta_i = -\int_{0}^{L} \int_{A} z\psi d_{31} E \phi_{i,xx} dx$$

$$F_i = -\int_{0}^{L} \phi_{w_i} p_Z dx$$

Now, here, by assuming p_z to be a distributed function of x, it is written in this way. While discussing the paper, we saw that those forces were considered to be discrete. So, that is why they were summed over.

So, that is about the first equation. In the second equation, which is D z z equal to 0, we multiply psi, and then have V and then, from there we can show that this becomes q multiplied by - So, q let us put capital Q as charge. So, capital Q is equal to charge because we are using small q to denote the coefficient associated with the displacement components. So, it is ds plus V D_z psi multiplied by v, and then dV.

So, now, this term can be written as - we can sum it over. So, we can write phi w_j comma x x q w_j plus e 3. So, electric field. So, which is psi multiplied by v and that multiplied by epsilon 3 3. So, here to multiply e here, e 3 1, and this is epsilon. This multiplied by psi dV plus Q multiplied by psi ds.

So, this gives us the equation as Theta T, multiplied by q_w plus C_p, C_p is just one value here. So, we can - we may just write it as C_p. C_p v plus Q is equal to 0. Phi T, we have already defined.

C_p comes to be volume integral of epsilon 3 3, epsilon sigma square dV. So, we have got these two equations. From the first equation, which was a version of the linear momentum equation, we got this. From the first equation which was a version of the linear momentum equation, we got this equation and the from the second equation, which is gauss law, we got this equation. And these two equations are equivalent to the equations that we got in the last lecture while discussing the paper in a different type of formulation.

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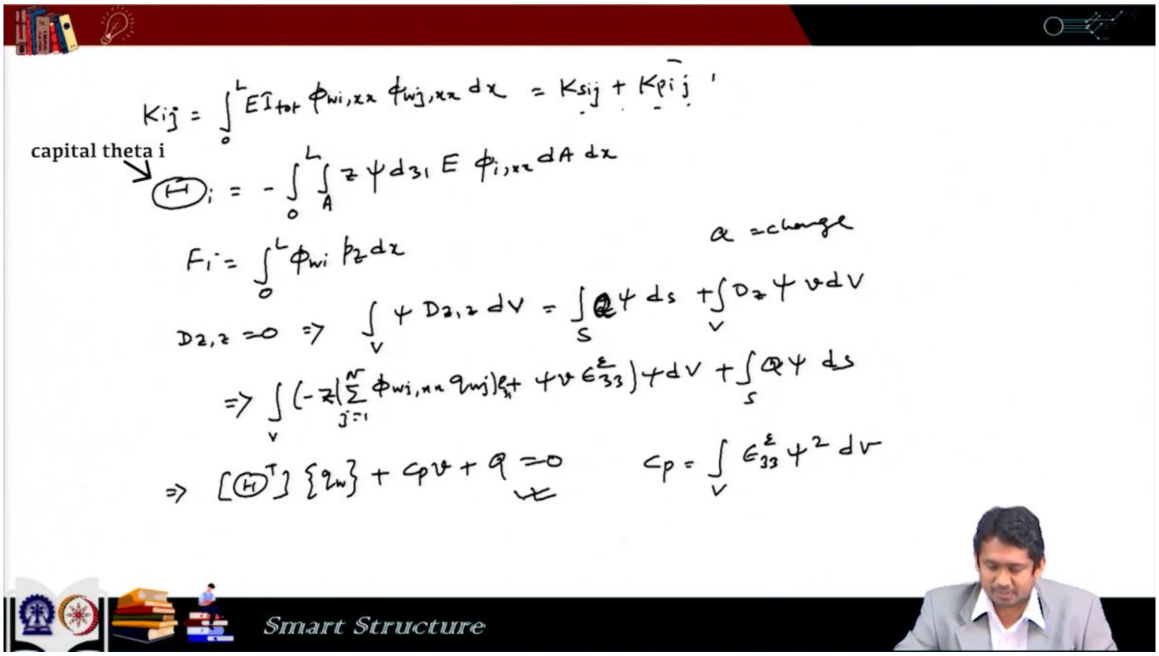

Now, we look into vibration control applications and the corresponding mathematical analysis. For that, this paper by Gordon Gee and his co-authors can be referred to. Here the idea is that we have a piezoelectric beam, sorry a beam which has piezoelectric patches for sensing and actuation. So, it has two piezoelectric patches at the top and bottom. The first top one is used as actuator and the bottom one is used as sensor.

Now, it is not needed that the actuator and sensor has to be placed back to back, they can be placed at other locations also, depending on the control strategy. Now, from the piezoelectric sensor - so this structure is under some dynamic load. So, in the experiments, the dynamic load can be given by a shaker or anything. In real life, it is due to the external factors. So, it is under some dynamic load and then - so we can just show some loads here. We can denote this as p_z. Now, from the sensor, the charge output comes, and then, there are two modes as suggested by this paper. One is the charge amplifier mode; through which we get voltage from the charge. So, voltage output from the charge. So, it can be a charge amplifier, or there can be current to voltage converter.

So, we will talk about it, what it is? Then from this, the voltage output comes and then, it is multiplied by a multiplied by a gain. We can call it controller and then finally, from there a feedback voltage is given here, which we may call v voltage. Now, the from here, the charge or current comes here. The output of this charge amplifier or current to voltage converter is a voltage. This voltage when it goes to the controller, it is multiplied by a gain and then the final voltage comes here which actuates the actuator.

Now, if you are interested in the details of this charge amplifier or current to voltage converter, this paper can be referred. The circuit diagrams are given there.

Now, if we write this, write the dynamics of this, then from there, the charge can be written as - charge output from sensor is Q which is equal to D_z integrated over the area. So, we have the electrical displacement D_z here. Let us call this as x, this dimension as z and this, let us call this as y. So, the same convention that we have been using. So, the D_z here, and it is already normal to the - it is already oriented along the z axis which means it is already normal to the xy plane. So, D_z integrated over the area that gives us the charge output Q. Now, the charge output Q is e_{31} of the piezoelectric material multiplied by the strain. So, the strain there - so, the strain here is nothing, but epsilon 0 minus z into kappa, as we know, now, epsilon 0. So, strain is equal to epsilon 0 minus z into kappa. So, epsilon 0 comes here, z for this case is this and we have to multiply kappa here and kappa comes here.

$$Q = \int_A D_z dA = \int_A \varepsilon_{31} \left(\varepsilon_0 + \left(\frac{t_b}{2} + t_{c_s} \right) \kappa \right) dA$$

$$\varepsilon = \varepsilon_0 - z\kappa$$

Now, the thickness of the host beam is t_b and we can call the thickness of the piezoelectric patch which is used as actuator to be t_A, and the thickness of the piezoelectric patch which is used as sensor to be t_{cs}, s stands for sensor, A stands for actuator. Now, with this, we can rewrite this expression as – integral from x_0 to x_0 plus l_c. And again, if you want, we can call this as x_0 as we have been calling and this dimension it is better to show in a different color just to avoid confusion. This dimension is x_0 plus l_c. So, we are assuming that the length of the sensor and actuator same. Again, that is not a requirement they can be different as well. So, we are assuming l_c, that l_{cs} is equal to l_{cA} is equal to l_c, but that is not a requirement, they can be different as well.

Now, we also know that in our analysis, we have been assumed u_0 x t to be summation of j is equal to 1 to M multiplied by phi u_j into q u_j, and w as N phi w_j q w_j.

$$u_0(x,t) = \sum_{j=1}^{M} \phi_{u_j} q_{u_j}$$

$$w(x,t) = \sum_{j=1}^{N} \phi_{w_j} q_{w_j}$$

So, with that approximation, this expression can be written as - e_{31} multiplied by b_s, b_s is the width of the sensor and then, we multiply this row vector phi u 1 x to phi u M x and then we have t_b plus t_{cs}, q w 1 x double dot, sorry, w comma x x plus t_b by 2 plus t_{cs}. and this entire row vector is multiplied by the column vector and dx.

$$Q = \int_{x_0}^{x_0+l_c} e_{31} b_s \left\{ \phi_{u_1,x} \quad \cdots \quad \phi_{u_M,x} \left(\frac{t_b}{2} + t_{c_s}\right) \phi_{w_1,xx} \quad \cdots \quad \left(\frac{t_b}{2} + \frac{t_{c_s}}{2}\right) \phi_{w_M,xx} \right\} \begin{Bmatrix} q_{w_1} \\ \vdots \\ q_{w_M} \\ q_{w_1} \\ \vdots \\ q_{w_N} \end{Bmatrix} dx$$

So, what we are doing is - if I multiply this q vector with this row vector, that gives me this expression.

Now, we can write this entire expression as A^T multiplied by q. q is this column vector and A^T is this vector multiplied by b_s into e_{31} and integrated from x_0 to x_0 plus l_c. So, here A^T is of size 1 multiplied by M plus N. And q is of size M plus N by 1.

$$Q = \{A\}^T_{1 \times (M+N)} \{q\}_{(M+N) \times 1}$$

So, again just to summarize what we have done here is - we need the expression for charge from the sensor. So, the charge is surface integral of the dz and dz is e_{31} multiplied by the strain. So, e_{31} and the strain expression is here in terms of epsilon 0 and kappa. And epsilon 0 and kappa comes in terms of the derivatives of this phi u and phi w. Here, this is not q, this is phi. And when this vector, after being multiplied with b_s and e_{31} integrated along the length of the sensor, it gives me a vector called A.

Now, we have to write the expression for the voltage that we get from the charge amplifier.

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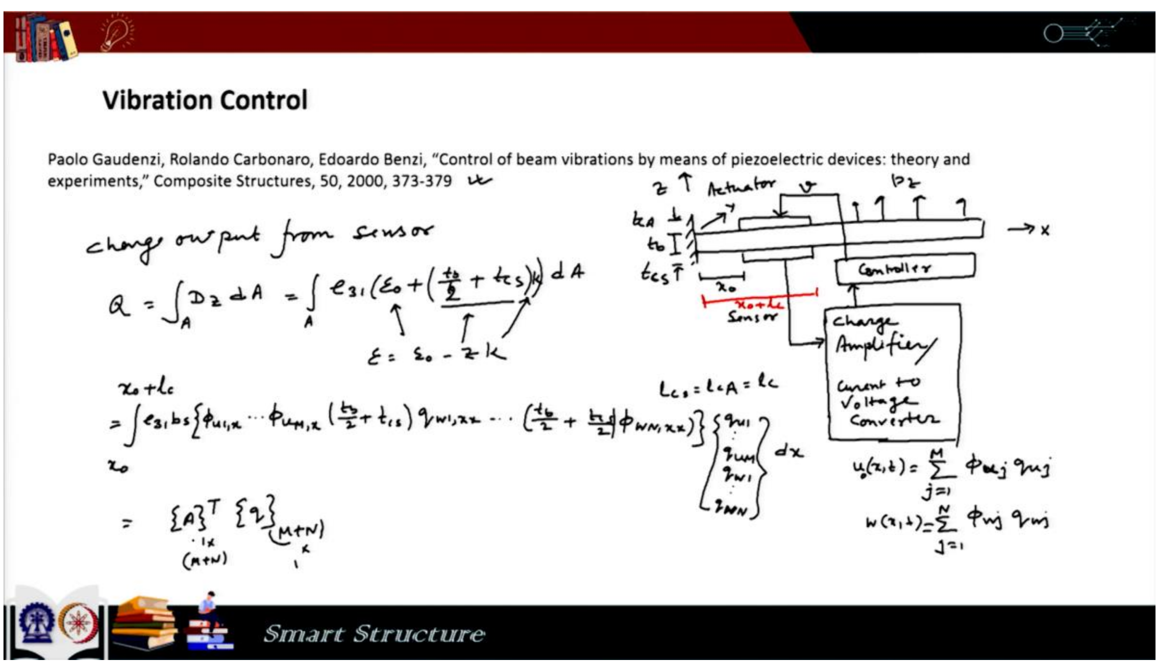

If you want the voltage from the charge amplifier, from the charge amplifier the voltage that we get is V_A is equal to 1 by C_f multiplied by Q, where C_f is capacitance of the or we can call feedback capacitance of the charge amplifier. And then, this expression can be written as 1 by C_f multiplied by the A vector multiplied by the q vector.

$$V_A = \frac{1}{C_f} Q = \frac{1}{C_f} \{A\}^T \{q\}$$

So, the input voltage to actuator can be written as - v is equal to, we multiply by some gain, G_A with the V_A. So, it is G_A by C_f A^T q.

$$v = G_A V_A = \frac{G_A}{C_f} \{A\}^T \{q\}$$

And current to voltage converter, for that the voltage: let us write it as may be V_c is R_{cv} multiplied by Q. So, R is a resistance. If we look at that structure diagram of the current to voltage converter, the position of R is there. So, R is multiplied by Q dot. Q dot is the current. So, that being multiplied with the resistance gives us the output voltage from the current to voltage converter. So, that is equal to 1 by C_f multiplied by A^T into q dot.

$$V_c = R_{cv} \dot{Q} = \frac{1}{C_f} \{A\}^T \{\dot{q}\}$$

And again, input voltage to actuator is v is equal to, let me call it - there is another gain. So, let us call it may be G_V multiplied by V_c. So, we can call it G_c also. So, it becomes Gc (it is not R_c, R_{cv}), so, G_c multiplied by R_{cv} A^T q dot.

$$v = G_c V_c = G_c R_{cv} \{A\}^T \{\dot{q}\}$$

So, what happens is - the charge amplifier gives an output which is our V_A or V_c and accordingly the controller multiplies with G_A or G_c, and the product of it is the feedback voltage V that comes to the actuator.

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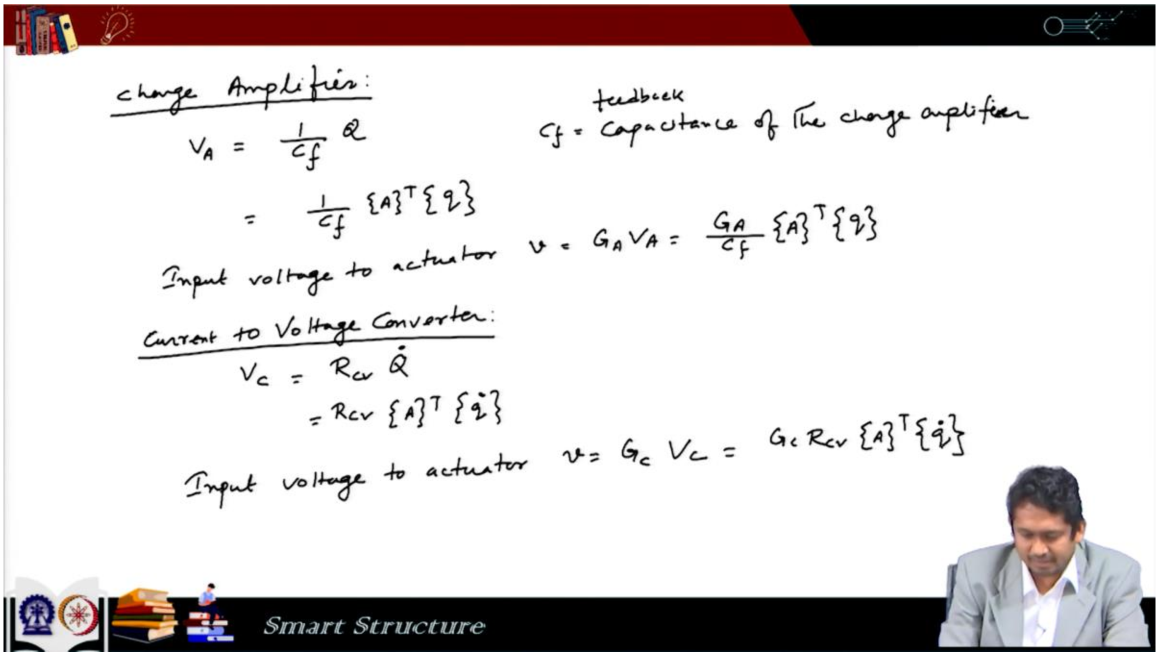

Now, we know that for the actuation case, we have already seen that the dynamics can be written as this. And that is equal to x_0, x_0 plus l_c, B^T, N_P, M_P. And here, B^T is already known to us and this is multiplied by N_P. And then it is integrated over the domain of the piezoelectric actuator, I mean, the domain along the x axis and that gives me the actuation force.

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \int_{x_0}^{x_0+l_c} [B]^T \begin{Bmatrix} N_P \\ M_P \end{Bmatrix} dx = [B_I]\{S\}v$$

And then we can write this entire thing as B_I. So, we are calling it B_I means, this B matrix integrated over x_0 to x_0 plus l_c. So, B_I multiplied by a vector S, we are introducing a new vector S into the actuation voltage v. So, S is a vector which is required to convert the voltage v to N_P and M_P. So, N_P and M_P is a vector S which, on being multiplied with v gives me the N_P and M_P. So, S can be written in terms of t_{cA}, the width of the actuator b_A and the beam thickness t_b and the elastic modulus of the piezoelectric patch and d_{31} or we can write e_{31} also. So, in terms of those, this can be written - the S vector can be found.

So, we have now this and then, we can rewrite the expression as M_q double dot plus C multiplied by q dot plus K q. And then we have, the B_I, we have S, and then, we have the gains G_A divided by C_f multiplied by A transpose, multiplied by q.

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = [B_I]_{(M+N)x2}\{S\}_{2x1} \frac{G_A}{C_f} [A]^T{}_{1+(M+N)}\{q\}_{(M+N)x1}$$

Now, this is for we will do that. Now again, this entire expression can be written as - L_A multiplied by q, because we know that, this is M plus N over q, S is 2 by 1, this is 1 by M plus N, and this is M plus N by 1. So, finally, this comes to be M plus N over M plus N and this we know it is M plus N over 1.

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = [L_A]_{(M+N)x(M+N)}\{q\}_{(M+N)x1}$$

Now, this is charge amplifier mode. Now, when for current to voltage mode, the same expression for current to voltage mode it becomes - this becomes L_C over q dot. And where L_C is B_I over S, then we have G_c R_{cv}, we already defined - G_c R_{cv}, and we have A transpose.

$$[L_c] = [B_I]\{S\}G_c R_{cv}[A]^T$$

So, for the charge amplifier mode, we have Q here and for current to voltage mode we have q dot here. So, we can see depending on the control strategy whether we used charge amplifier mode or current to voltage mode, we have q or q dot here. And the gains are decided depending on - I mean, it is a matter of control system. So, we used various techniques, various optimizations to find out the optimal gain. So that this entire system can be controlled.

Now, in real life, there can be unwanted vibration due to several reasons, aero elastic instability for aircrafts can be one of the reasons. So, if it goes to the unstable region, dynamically unstable region, flutter can occur and the vibration may look like this. In such cases if this kind of controls are applied, then the vibration can be stabilized.

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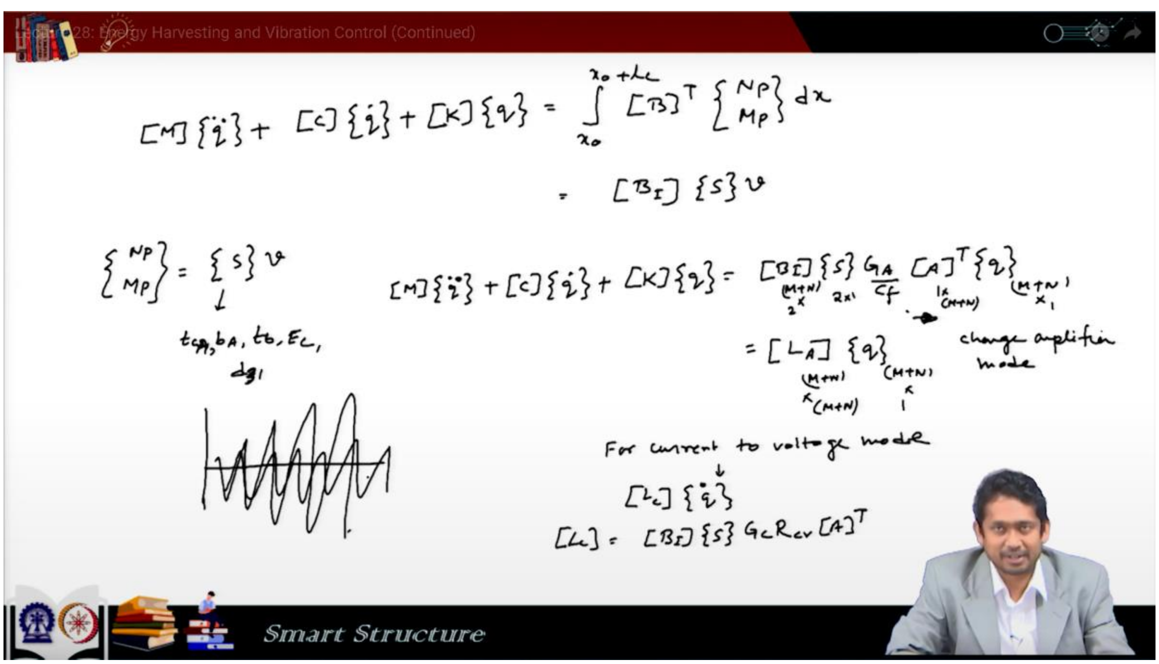

So, we have discussed about the energy harvesting and the control systems based on the formulations that we have been doing so far.

Now with that let me conclude this lecture.

Thank you.