Smart Structures
Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur
Week 12
Lecture No: 26

Dynamic Analysis of Beam for Induced Strain Actuation Using (Continued)

Welcome to the second lecture! In the last lecture we saw the Hamilton's principle and derived the governing differential equations from there. Now, today we will talk about the solutions. So, as we saw before for static cases the solution can be done directly from the variational indicators. and the energy expression or it can also be done starting from the governing differential equation by multiplying some test functions. Now, when we multiply test functions, we saw two approaches in the first approach we just multiply it and do not do any integration by parts in the second approach we do integration by parts to divide the derivatives as evenly as possible. So, we will again we look into all the three approaches for this dynamic problem. Now, our equation is so, we have written the static equation which we already derived there. So, we will just add the extra terms that comes here due to the dynamic nature. So, this equation we already knew we saw it while solving the static problem. Now, because the problem is dynamic, we are going to get one more term here which is this.

The beam equations are -

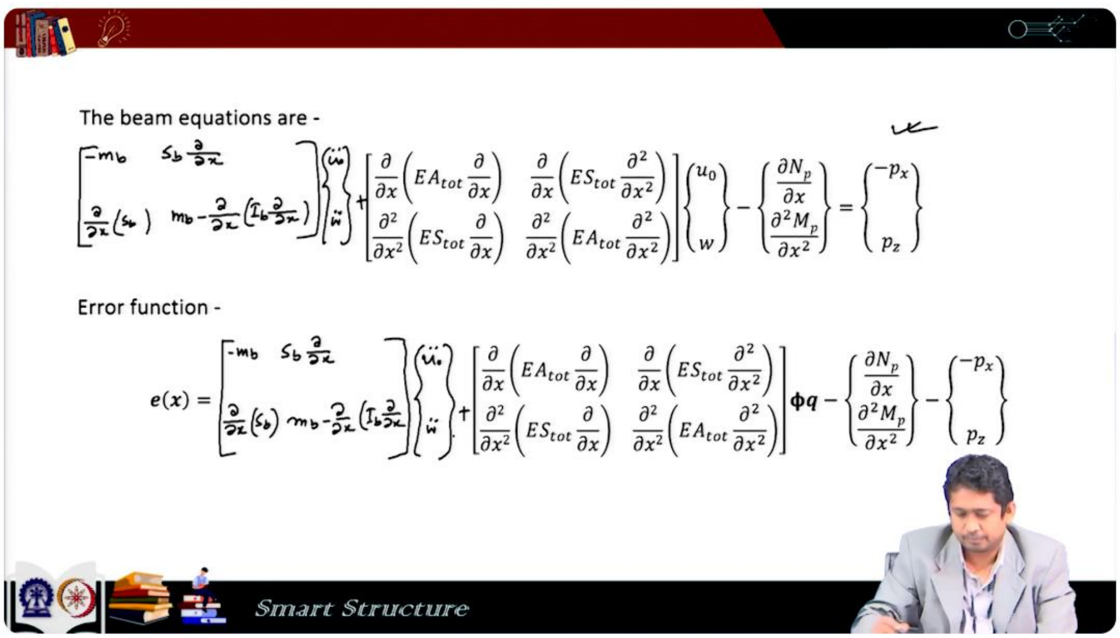
$$\begin{bmatrix} -m_b & s_b \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x}(s_b) & m_b - \dfrac{\partial}{\partial x}\left(I_b \dfrac{\partial}{\partial x}\right) \end{bmatrix} \begin{Bmatrix} \ddot{u_0} \\ \ddot{w} \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial}{\partial x}\left(EA_{tot}\dfrac{\partial}{\partial x}\right) & \dfrac{\partial}{\partial x}\left(ES_{tot}\dfrac{\partial^2}{\partial x^2}\right) \\ \dfrac{\partial^2}{\partial x^2}\left(ES_{tot}\dfrac{\partial}{\partial x}\right) & \dfrac{\partial^2}{\partial x^2}\left(EA_{tot}\dfrac{\partial^2}{\partial x^2}\right) \end{bmatrix} \begin{Bmatrix} u_0 \\ w \end{Bmatrix} - \begin{Bmatrix} \dfrac{\partial N_p}{\partial x} \\ \dfrac{\partial^2 M_p}{\partial x^2} \end{Bmatrix} = \begin{Bmatrix} -p_x \\ p_z \end{Bmatrix}$$

So, this part comes due to the inertia. Similarly, we can define the error function if you want to define the error. Again, this is the error that was there when we were solving the static problem. Now, the problem has changed. So, we have to add more term to the error. Now, the error is this.

Error function -

$$\boldsymbol{e}(\boldsymbol{x}) = \begin{bmatrix} -m_b & s_b \frac{\partial}{\partial x} \\ & \frac{\partial}{\partial x}(s_b) m_b - \frac{\partial}{\partial x}\left(I_b \frac{\partial}{\partial x}\right) \end{bmatrix} \begin{Bmatrix} \ddot{u}_0 \\ \ddot{w} \end{Bmatrix} + \begin{bmatrix} \frac{\partial}{\partial x}\left(EA_{tot} \frac{\partial}{\partial x}\right) & \frac{\partial}{\partial x}\left(ES_{tot} \frac{\partial^2}{\partial x^2}\right) \\ \frac{\partial^2}{\partial x^2}\left(ES_{tot} \frac{\partial}{\partial x}\right) & \frac{\partial^2}{\partial x^2}\left(EA_{tot} \frac{\partial^2}{\partial x^2}\right) \end{bmatrix} \boldsymbol{q} - \begin{Bmatrix} \frac{\partial N_p}{\partial x} \\ \frac{\partial^2 M_p}{\partial x^2} \end{Bmatrix} - \begin{Bmatrix} -p_x \\ p_z \end{Bmatrix}$$

(Refer Slide Time: 03:18)



Now, what we do is we multiply the Ø's to the error and make that integral 0 we have already defined the error Ø has to be assumed as we did before and after we multiply this Ø the solution comes in the form of a matrix for the static case the solution came in this form. Now, for the present case the solution would come in this form. Now, previously we assumed our Ø as so, we had u_0 as a function of x and our assumption was this is $\emptyset_{ui}$ as a function of x multiplied by q_{ui} . Now, here our u is function of both x and time.

So, it is x and t. Now, Ø remains a function of x it cannot be a function of time, but now this unknown q becomes a function of x and i goes from 1 to M the number of approximations for u_0 that is m and w was similarly we can write as a function of x and time as i is equal to 1 to M $\emptyset_{wi}(x)\, q_{wi}(t)$. So, this q's are now function of x and q we as we know q is just the vector where we have q_{u1} up to q_{uM} and q_{w1} up to q_{wM} q_{wN}. So, this is now our governing differential equation which we have to solve. So, here we have got rid of all the partial derivatives.

$$u_o(x,t) = \sum_{i=1}^{M} \emptyset_{ui}(x)\, q_{ui}(t)$$

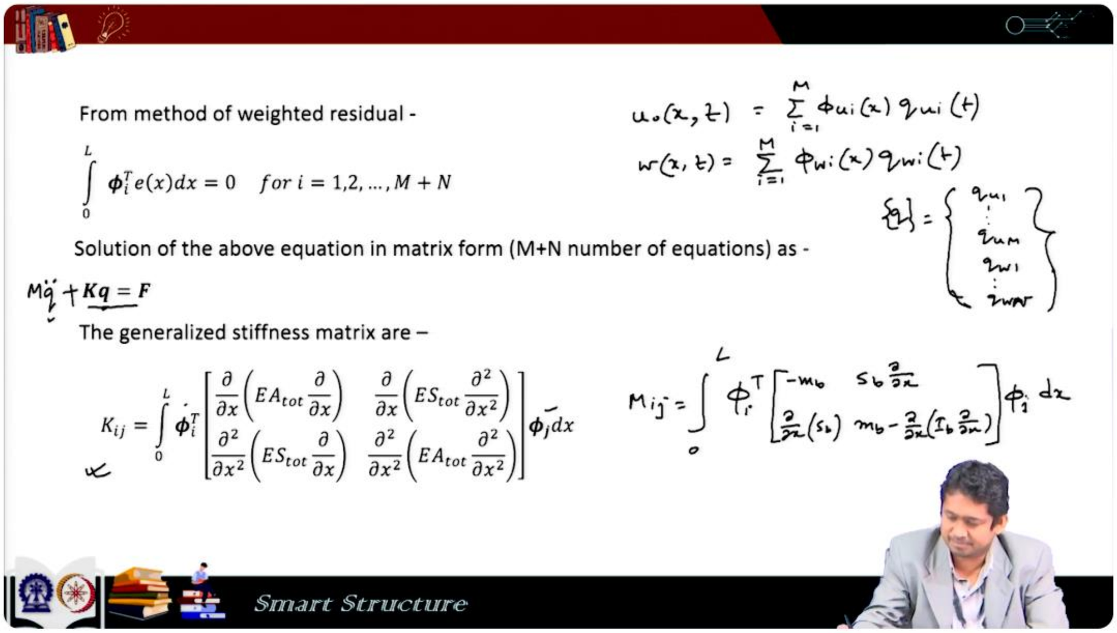
$$w(x,t) = \sum_{i=1}^{M} \emptyset_{wi}(x)\, q_{wi}(t)$$

$$\{q\} = \begin{Bmatrix} q_{u1} \\ \\ q_{uM} \\ q_{w1} \\ \\ q_{wN} \end{Bmatrix}$$

Now, when the problem was a static problem from the partial differential equation after applying the Galerkin technique we got a set of algebraic equations k*q = f was an algebraic equation but now the problem has a dynamic nature. So, instead of having an algebraic equation we are getting a system of ordinary partial differential equations. So, again for a static problem from the partial differential equations we get a set of algebraic equations I mean when we get rid of the space dependency we get a set of algebraic equations for dynamic problems after getting rid of the space dependency we get a set of ordinary differential equations. So, it becomes a initial value problem now. Here the generalized stiffness matrix K_ij we already wrote it now accordingly M_{ij} is going to be 0 to L $\emptyset_i$ transpose minus m_b S_b $del/delx(S_b\ )$ m_b minus. So, here it will be $S_b$del del x and here we have m_b minus del del x I_b del del x and this gets multiplied with $\emptyset_j$ dx and this $\emptyset_{ij}$ this $\emptyset_{ij}$they are same.

$$M_{ij}=\int_0^L \emptyset_i{}^T \begin{bmatrix} -m_b & s_b \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x}(s_b) & m_b - \frac{\partial}{\partial x}\left(I_b \frac{\partial}{\partial x}\right) \end{bmatrix} \emptyset_j \text{ dx}$$

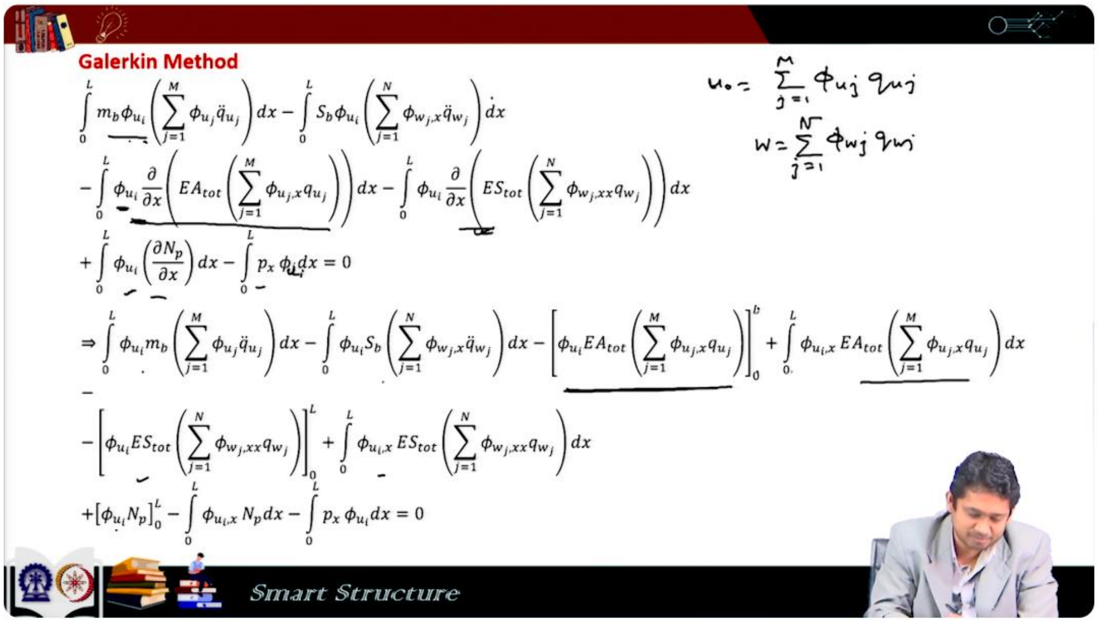
(Refer Slide Time: 07:30)



Force we already know we did it while solving the static problem. So, with so, this completes the first approach again here we did not shift the derivatives using the integration by parts. So, because of this while we approximate our Ø we have to make sure that these derivatives all exist for Ø. Now, in the second approach what we will do we will shift this we will shift this derivative. So, in that case the requirement of the differentiation order of differentiation would be less. Now, to the in the second approach when we again equation ordinary differential equation by a test function. So, the first ordinary differential equation which we have we multiply that with $\emptyset_{ui}$ which can be seen here this is the first differential equation. Now, and in the differential equation in the terms for u_0 we substitute j is equal to 1 to M $\emptyset_{uj}q_{uj}$ and for w we substitute j is equal to 1 to N $\emptyset_{wj}q_{wj}$ and in this substituted form we multiply $\emptyset_{ui}$. So, this is $u_i$this is $\emptyset_{ui}$ok.

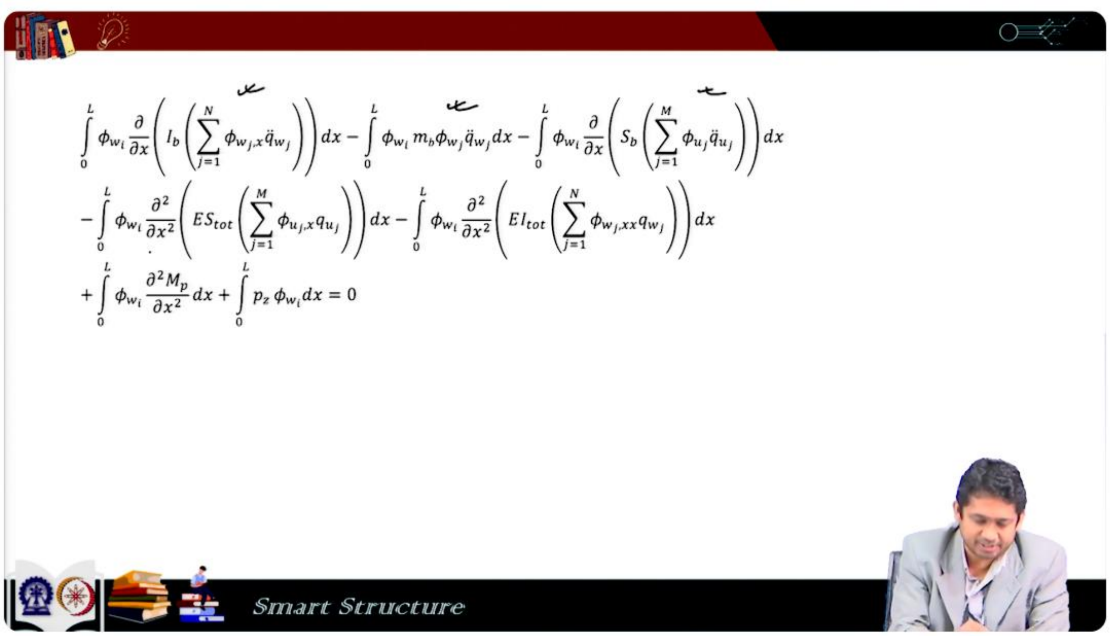
Now, we have this term here we do not have any spaced derivative here we have a spatial derivative, but we do not do anything with that here we have a spatial derivative which we shift which we did before also here we have a spatial derivative which we shift and we also shift this derivative. The first term comes as it is the second term comes as it is in the third term which we already did before for the static case again we are doing integration by parts. So, this is considered as the first function. So, this multiplied by integral of this which is this at the limits 0 and L and then with opposite sign the derivative of this and integral of this which is this integrated from 0 to L dx. same thing is done from this term here this comes as it is.

(Refer Slide Time: 010:29)



Now, comes our second differential equation. In the second differential equation again, we substitute the same things and we multiply the equation with $\emptyset_{wj}$. And here we can see that this term has a spatial derivative which we will be shifting this does not have any spatial derivative and this has a spatial derivative which will also be shifting and here we have spatial derivative to shift here also we have spatial derivative to shift by integration by parts same thing here. So, that is what we do.

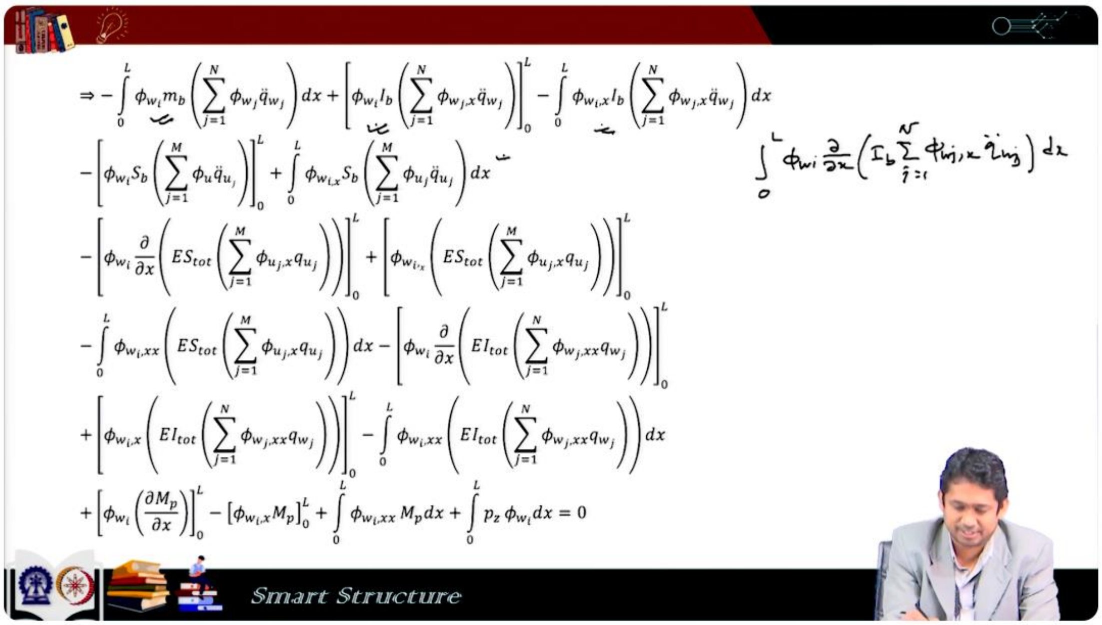
(Refer Slide Time: 011:25)



This term comes as it is and then we shift the derivatives and we get this term.

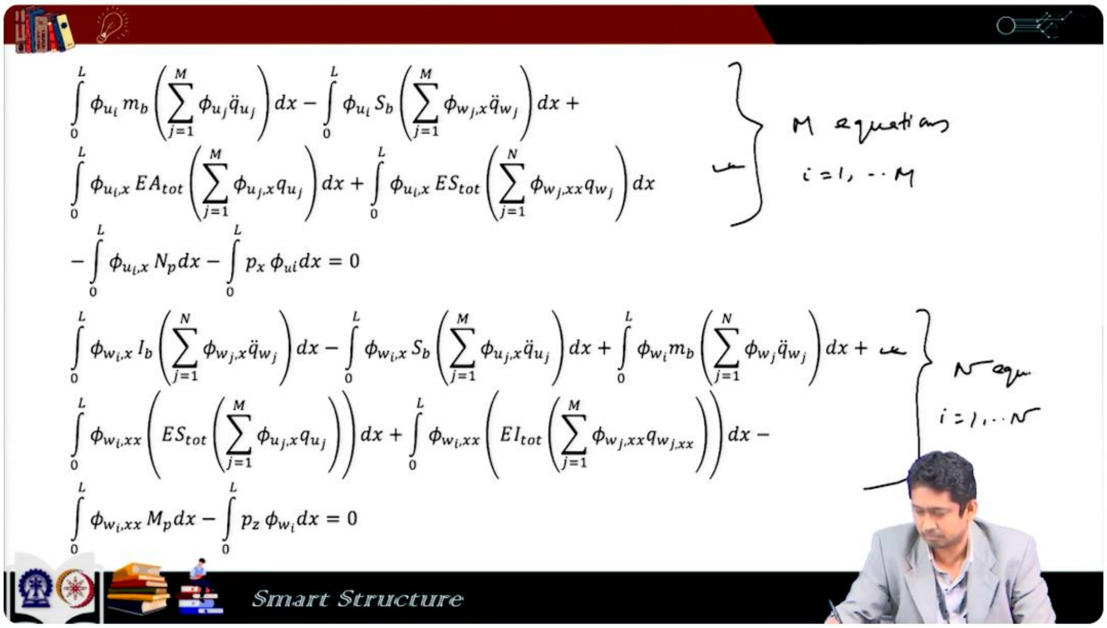
So, we shift the derivatives here. So, this is the first term, this is the second term and we can see that the derivatives have been equally distributed between $\emptyset_{wi}$ and $\emptyset_{wj}$ and then we do the same thing for the other term in the among the inertia terms and with us do the same things in the other terms as well and finally, after get after finally, we get this expression.

(Refer Slide Time: 012:19)



So, we here we have a set of boundary terms and the integral terms and then after dropping out the boundary terms we get these two equations. And then finally, this gives me a set of m plus n equations. So, here this is M equations because $\emptyset_i$, i goes from 1 to M and here we have n equations because i goes from 1 to N and as we have done before this is this can be written as a set of ordinary differential equations as this multiplied by q dot plus this stiffness matrix multiplied by q and that is equal to the force vector F.

(Refer Slide Time: 013:01)



This we already know while solving the static problem we get we got this system a set of linear simultaneous algebraic equations and again because it is a dynamic problem we get an inertia term. So, it becomes a set of coupled ordinary differential equations.

$$\begin{bmatrix} [M_{uw}]_{M\times M} & [M_{uw}]_{M\times N} \\ [M_{wu}]_{N\times M} & [M_{ww}]_{N\times N} \end{bmatrix} \{\ddot{q}\}_{(M+N)\times 1} + \begin{bmatrix} [K_{uw}]_{M\times M} & [K_{uw}]_{M\times N} \\ [K_{wu}]_{N\times M} & [K_{ww}]_{N\times N} \end{bmatrix} \{q\}_{(M+N)\times 1} = \{F\}_{(M+N)\times 1}$$

Here $M_{uu_{ij}}$So, this is again M by M, this is M by N, this is N by M, this is N by N. So, these also M by M, this is M by N, this is N by M and this is N by N and this is m plus N by 1, this is also N plus N by 1, this is also M plus N by 1. So, $M_{uu_{ij}}$becomes $m_b\phi_{ui}\phi_{uj}dx$ M_{uwij}becomes minus of $s_b\phi_{ui}\phi_{w_j,x}dx$ and M_{wuij} becomes minus of $-s_b\phi_{wi,x}\phi_{uj}dx$ and M_{wwij} is 0 to l $(m_b\phi_{wi}\phi_{wi} + I_b\phi_{wi,x}\phi_{wj},x)dx$ integrated from 0 to l. This matrices k and f we already know. So, if you have to if we just write it once more it is $K_{unij}$is equal to 0 to l $EA_{\text{tor}}\,\phi_{ui,x}\phi_{u_j,x}dz$, $K_{uvij}$is equal to $\int_0^L ES_{tor}\phi_{u_i,x}\phi_{wi},xddx$ and K_{wuij} is equal to $ES_{\text{tot}}\,\phi_{wi,xx}\phi_{ui,x}dx$ and $K_{wwij}$is equal to $EI_{\text{tor}}\,\phi_{wi,xx}\phi_{wj},xxdx$ and F_{ui} we wrote as F_{ui}. So, this has m component this has n component. So, $F_{wi}$is $(M_p\phi_{wi},xx + p_2\phi_{wi})dx$ So, this completes this equation we can solve this equation as an initial value problem and we will get q as a function of time.

$$M_{uu_{ij}} = \int_0^L m_b \phi_{ui} \phi_{uj} dx \; M_{uwij} = -\int_0^L s_b \phi_{ui} \phi_{w_j,x} dx$$

$$M_{wuij} = \int_0^L - s_b \phi_{wi,x} \phi_{uj} dx \; M_{wwij} = \int_0^L \left(m_b \phi_{wi} \phi_{wi} + I_b \phi_{wi,x} \phi_{wj}, x \right) dx$$

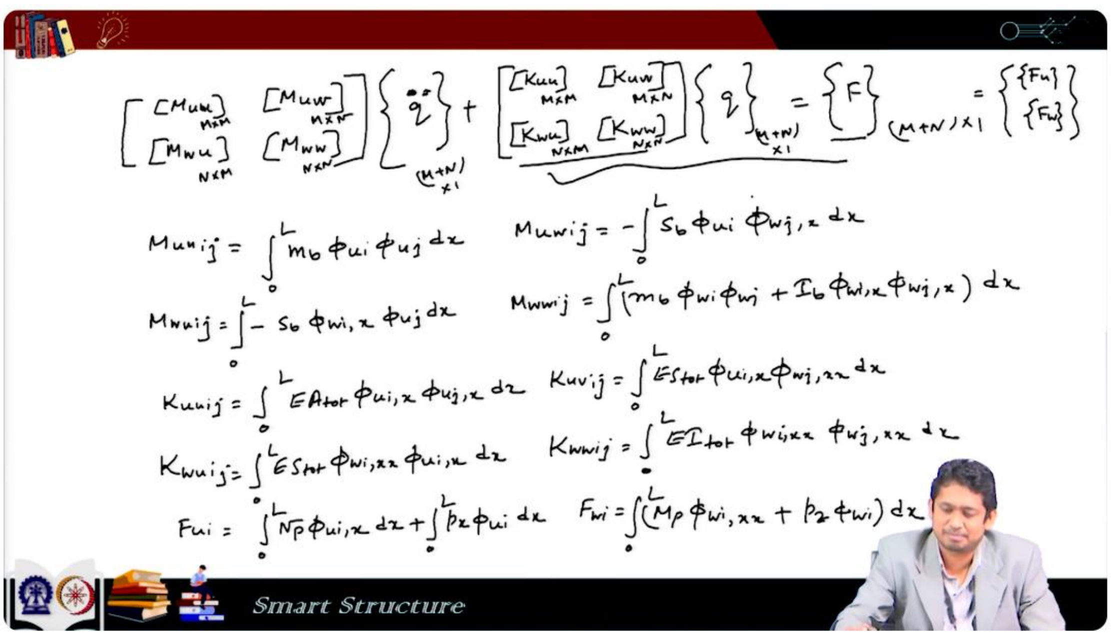
$$K_{unij} = \int_0^L EA_{\text{tor}} \, \phi_{ui,x} \phi_{u_j,x} dz \; K_{uvij} = \int_0^L ES_{tor} \phi_{u_i,x} \phi_{wi}, xddx$$

$$K_{wuij} = \int_0^L ES_{\text{tot}} \, \phi_{wi,xx} \phi_{ui,x} dx \; K_{wwij} = \int_0^L EI_{\text{tor}} \, \phi_{wi,xx} \phi_{wj}, xxdx$$

$$F_{ui} = \int_0^L N_p \phi_{ui} x dx + \int_0^L p_x \phi_{ui} dx \; F_{wi} = \int_0^L \left(M_p \phi_{wi}, xx + p_2 \phi_{wi} \right) dx$$

So, that will tell us at each time step what is my q and if we get the value of q at each time step then we can back substitute to the approximation and that will give us what is the value of u_0 and w at each time step.

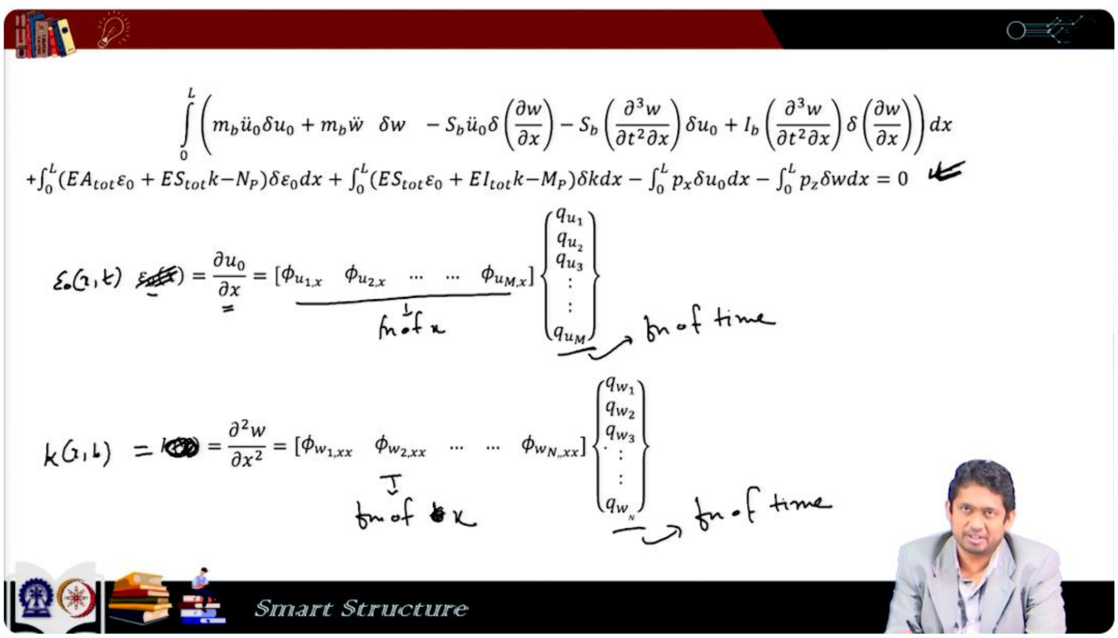
(Refer Slide Time: 019:41)



Now, we will look into the other approach where we directly put the approximations in the variational indicator and solve it. So, this is what this is our variational indicators. So, we started from the Hamilton's principle and then we got rid of the time integrals and we came to this expression to we came to this intermediate expression which is kind of analogous to our virtual work expression. So, in this expression now we will direct if we put the put the approximations directly then can we get the solutions.

Again, we have the same approximation for $u_0$and w and we know that $\varepsilon_0$is δ (u_0)/ δ x. Now here it has to be modified now it is no more a function of only x. So, ε_0 (x) is modified to be ε_0 x and time. So, this takes care of the time part a space part special part this this takes care of the time parts we can write function of x and this is function of time. So, ε_0 (x) can be written as δ (u_0)/ δ x and which can be written as this and then K kappa, kappa is also a function of x and time. So, it is this is not this and then again this is a function of function of x and this is function of time. So, we are familiar with this expression we did it before also the only difference is that now this q's are function of time.

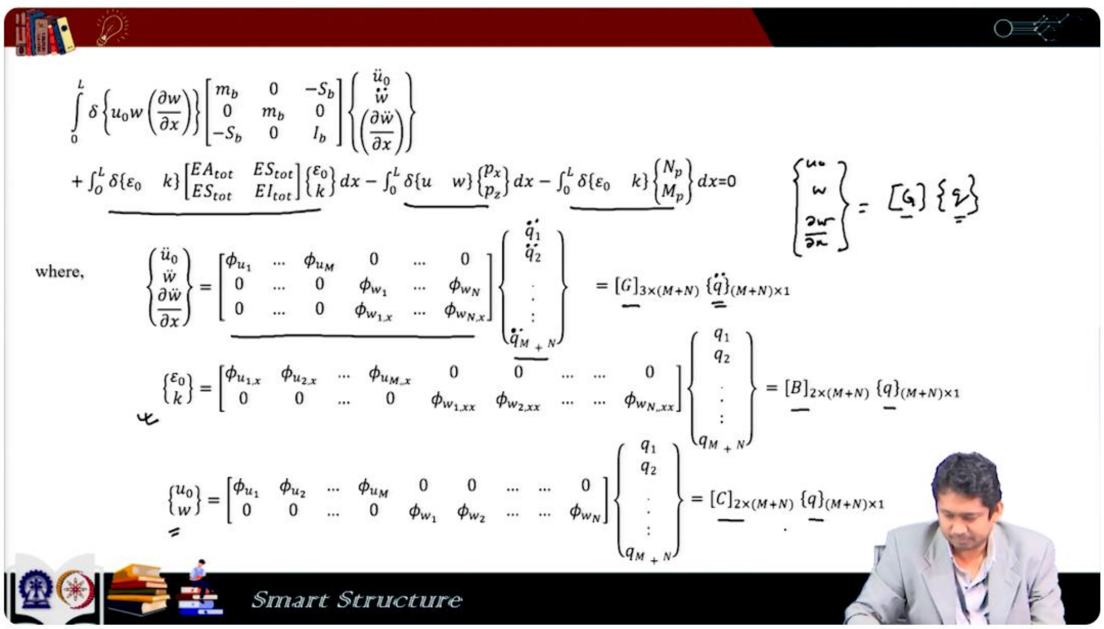
(Refer Slide Time: 022:00)



Now if we look at the previous expression that we wrote here this expression it can be written in a compact form here. So, we can see that if we multiply this vector of this double dots with this matrix and then we multiply here we get all the terms related to the inertia here. Similarly, from this term all these multiplications and from all these multiplications we get whatever we had here and from this multiplication we get all these terms.

Now, if we look at this u_0 double dot, w double dot and δ w double dot by δ x this can be written as this this matrix of Ø's multiplied by this q's. So, here q's also have to be dotted because we have put double dot here and this also have to be dotted. Similarly, if you want to write u_0 w and δ w/ δ x this will become that same G matrix multiplied by q. So, when we have u_0 double dot, w double dot, del w double dot by δ x it is G matrix multiplied by q dot when we do not have the dots here it is G multiplied by q. This we have already done now this we got from the from the static cases we already did this and here we have B multiplied by q u_0 w we have already done it and we got this. Now, what we do is we put these expressions in the in this expression we put all these approximations in this expression and from there we get the solutions.

(Refer Slide Time: 024:11)



Now, to do that we can write the first term which has all the inertias as 0 to L delta of {q} transpose multiplied by [G] transpose multiplied by [H] multiplied by [G] multiplied by {q double dot} *dx. Because we have seen that u_0 w δ (w)/ δ x is equal to G multiplied by {q}. So, u_0 w δ (w)/ δ x transpose which is a row vector that becomes {q} transpose [G] transpose and then if we take a variation here the variation comes here. And then plus as we wrote for the static problems it is Δ {q} transpose multiplied by [B]transpose multiplied by [D] multiplied by [B] *d x and then we have 0 to L Δ{q} transpose [C] transpose $p_x p_z * dx$ and then we have 0 to L Δ{q} transpose [B] transpose multiplied by $N_p M_p dx$ and that is equal to 0.

$$\int_0^L \delta\{\varepsilon\}^\top [G]^\top [H][G]\{\ddot{q}\}dx + \int_0^L \delta\{q\}^\top [B]^\top (D)[B]\{q\}dx - \int_0^L \delta\{q\}^\top [c]^\top \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} dx - \int_0^L \{q\}^\top [B]^\top \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} dx = 0$$

So, again we have delta q transpose everywhere. So, after getting rid of this we get an equation of this form. We can directly write the equation in the matrix form and the equation looks like this. [M] {q} double dot plus [K]*{q} is equal to F where [M] is integral of 0 to L integral [G] transpose [H] [G] *dx and then we have the K matrix that is B transpose D B d x and then we have the force vector as 0 0 to L [C] transpose $\{p_x \; p_z\}$* dx plus 0 to L [B] transpose $\begin{Bmatrix} N_p \\ M_p \end{Bmatrix}$ * dx. Now, if you go back, the [G] matrix is 3x(M+N) /3.

[H] matrix is 3 x 3 matrix this we term as [H] matrix this matrix let us term that as [H] matrix. So, it is a 3 x 3 matrix. So, it is a 3 x 3 matrix and then [G] matrix is 3x(M+N). So, if we multiply this it becomes a (M+N) x(M+N) and this is also (M+N) x(M+N) and this is (M+N)x1. So, again we have a system of coupled or ordinary differential equations solving which we can get our solution.

$$[M] = \int_0^L [G]^\top [H][G] dx$$

$$[K] = \int_0^L [B]^\top (D)[B] dx$$

Now, in all the problems we have got the equation in this form where we have inertia term, we have a stiffness term and the force vector. But in what happens is in real life there is always a damping associated with this which we cannot get directly from where we started. So, what is generally done is artificially damping is added. One common very common way of adding the damping is writing the damping matrix as a combination of the mass matrix and the stiffness matrix. Now, this alpha and beta these factors can be found out when the damping ratios are known and this is called Rayleigh damping.

$$[M)\{\ddot{q}\} + [C]\{\dot{q}\} + (K]\{q\} \quad = \{F\}$$

So, it is artificially added damping and then this entire combination of the equations are solved and the solution is obtained. So, with this I would like to conclude this lecture here. Later on we will see how to how these problems are solved, how these ordinary differential equations are solved and we will see some more cases. Thank you!

(Refer Slide Time: 030:40)

$$\int_0^L \delta\{q\}^T [G]^T [H] [G] \{\ddot{q}\} \, dx + \int_0^L \delta\{q\}^T [B]^T [D] [B] \{q\} \, dx - \int_0^L \delta\{q\}^T [C]^T \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} dx - \int_0^L \delta\{q\}^T [B]^T \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} dx = 0$$

$$\begin{Bmatrix} u_0 \\ w \\ \frac{\partial w}{\partial x} \end{Bmatrix} = [G]\{q\}$$

$$\delta \begin{Bmatrix} u_0 \\ w \\ \frac{\partial w}{\partial x} \end{Bmatrix}^T = \delta\{q\}^T [G]^T$$

$$\boxed{[M]\{\ddot{q}\} + [K]\{q\} = \{F\}}$$

$$\underset{(M+N)\times(M+N)}{[M]} = \int_0^L \underset{(M+N)\times 3}{[G]^T} \, \underset{3\times 3}{[H]} \, \underset{3\times(M+N)}{[G]} \, dx$$

$$\underset{(M+N)\times(M+N)}{[K]} = \int_0^L [B]^T [D] [B] \, dx$$

$$\underset{(M+N)\times 1}{\{F\}} = \int_0^L [C]^T \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} dx + \int_0^L [B]^T \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} dx$$

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F\}$$