Smart Structures

Professor Mohammed Rabius Sunny
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Week - 05

Lecture No - 25

Dynamic Analysis of Beam for Induced Strain Actuation Using Energy Principle

Welcome to the fifth week.

In the last week we looked into the energy method based analysis of beams with piezoelectric material, but for the static cases. Now, we will start looking into energy based analysis of beams with piezoelectric materials for dynamic cases. Now, when we were in the static cases we saw that we had a principle where it says that the variation of the total potential energy is 0 for equilibrium and then we also saw the virtual work internal is equal to external virtual work virtual work. Now if you want to look into dynamic problems here we have time dependence and here we have kinetic energy also. So, here we have something called Hamilton's principle which says that the variational integrator delta of integral of T minus U plus V between any two arbitrary time step t 1 and t 2 is 0.

So, T is kinetic energy, U is strain energy and V is the potential of the applied load. Now, so we have to so U plus V is the total potential energy it has two components strain energy and the potential of the applied load. So, T minus U minus V this quantity if it is integrated between any two arbitrary time steps t 1 and t 2 that variation of that integral is 0 that is what this principle says for this to be in equilibrium. Now here please understand that t 1 and t 2 are arbitrary time steps this is also written as t 1 integral between t 1 and t 2 of delta t minus delta U plus delta W where if our force is non conservative.

$$\delta \int_{t_1}^{t_2} [T-(U+V)]\, dt = 0, \int_{t_1}^{t_2} [\delta T - \delta U + \delta W]\, dt = 0$$

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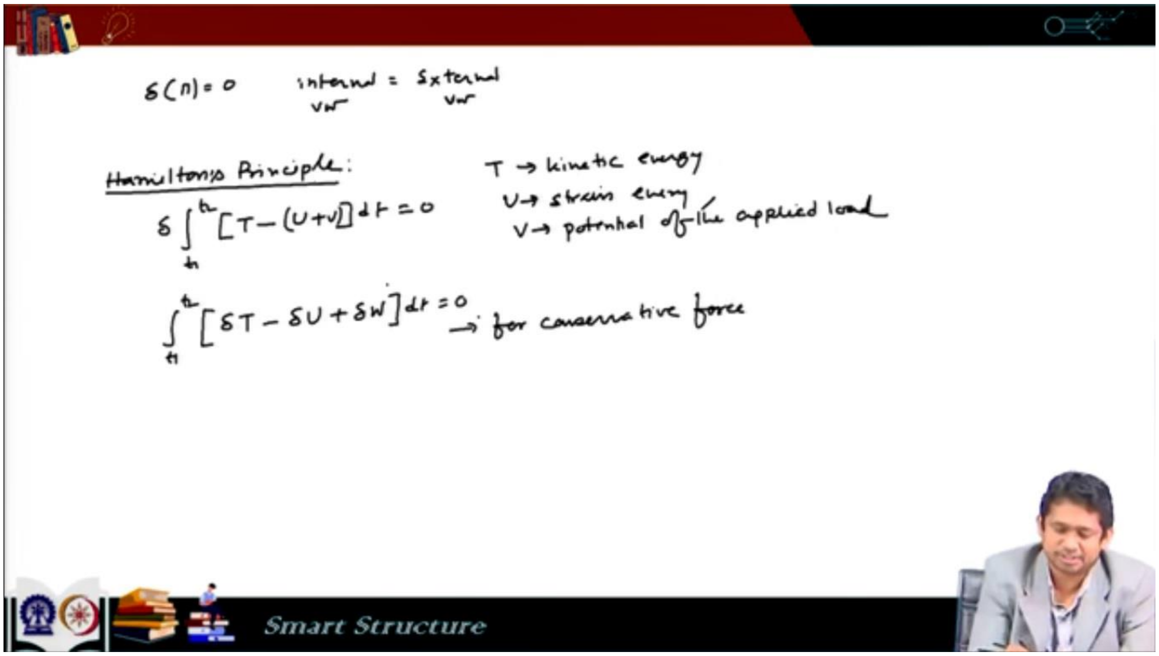

So, if the if the if the force can be derived from a potential V we can write this, but if the force is non conservative then we cannot define a potential in that case we have to write it in this form. Now, we will apply this principle for a very simple case first and then we will look into the beams with piezoelectric materials. So, as we did before let us again look into a bar. So, for the static case the bar was under a static distributed force P. Now this bar can be under a distributed force P which can be a function of time and this results in displacements and that is also a function of time.

Now we have to write T minus U minus V. So, t if we write delta of the integral of T minus U minus V dt is equal to 0 then we have delta of t 1 to t 2 or we can take the delta first inside and then we can proceed. So, delta of T is equal to 0 and then our kinetic energy T can be written as integral over the volume of half rho u dot square dv because u is a displacement along x direction. So, if the velocity is u dot. So, if I take half rho u dot square that is kinetic energy per unit volume and then if we integrate it over the volume that becomes our total kinetic energy.

$$\delta\left[\int_{t_1}^{t_2}(T-U-V)dt\right]=0 \Rightarrow \int_{t_1}^{t_2}(\delta T-\delta U-\delta V)\,dt=0$$

$$\int_{t_1}^{t_2}\left(\delta\left(\int_V \frac{1}{2}\rho\dot{u}^2 dV\right)-\delta\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)+\delta(pu)\right)dt=0$$

$$\int_{t_1}^{t_2}\left[\delta\int_0^L m\dot{u}^2dx-\delta\int_0^L\frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2dx-\int_0^L p\delta u dx\right]dt=0$$

$$\Rightarrow \int_{t_1}^{t_2}\left[\int_0^L m\dot{u}\delta\dot{u}dx - \int_0^L EA\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)dx - \int_0^L p\delta u\right]dt = 0$$

$$\left[\int_0^L m\dot{u}\delta udx\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\int_0^L m\ddot{u}\delta udxdt + \int_0^L EA\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right) - \int_0^L pudx\right)dt = 0$$

$$\int_V \frac{1}{2}\rho\dot{u}^2 dV = \int_0^L \int_A \frac{1}{2}\rho\dot{u}^2\, dAdx = \int_0^L \frac{m}{2}\dot{u}^2 dx$$

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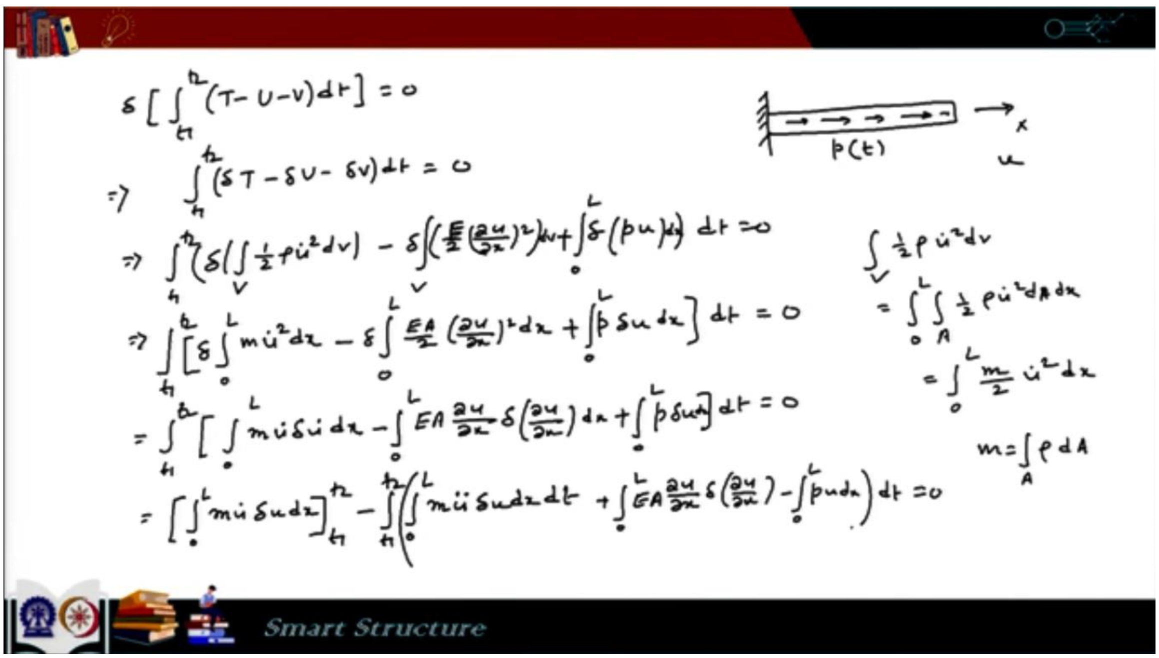

And then here we have half into del u by del x square and then here we have delta of p into u theta plus. dt equal to 0. Now, we have to put the volume integral here. So, it is dv. Now, this expression again can be broken down into a volume integral and an area and an integral over the length.

So, we can write this as half rho u dot square dA dx and then it becomes 0 to L it is our rho integrated over the cross section. So, if the beam has a cross section of area A and if we integrate rho over that cross section we can call it m, m is mass per unit length and then it becomes m by 2 u dot square dx where m is equal to rho integrated over the cross section. So, this can be written as dx and then we have this we already know that this can be written as and we have E here also the elastic modulus. So, this can be written as E A by 2 del u by del x square dx and the integral ranges from 0 to L. So, we have we already did it before while doing the static analysis we integrated it over the cross sectional area and that gives us A and then we integrate it over the domain x equal to 0 to L.

And then here we have p can be taken out of the variation and we have p into delta u and this is also an integral it goes from dx. So, here it goes from dx. So, this entire quantity is 0. Now, this can be rewritten as t1 to t2 we have del delta can be taken inside u delta del u dot dx and this can be written as E A del u by del x into delta of del u by del x dx and this remains same p into delta u and this is integrated from T 1 to T 2. Now what we do is here we have derivatives, but these derivatives are with respect to time.

So, it integrated parts with respect to time. If we do that then we get m u dot delta u dx. So, we are taking this as a first function and then we are integrating it with respect to time. So, it is delta u and this entire thing evaluated between t 1 and t 2 and then we have integral from t 1 to t 2 derivative of this and this. So, 0 to L if I take derivative of u dot with respect to time it becomes u double dot and we have del u del x del t and then we have rest of the terms.

So, the term here the symbol here the sign should be plus and rest of it we can put inside it. So, inside the time integral. So, it becomes 0 to L E A del u by del x del u by del x and we have 0 to L p u this also dx and dx dt that is equal to 0. Now if I compare these two terms we can see that this term is defined only at t1 and t2. So, the value of value of this term depends only at time t1 and t2 the value at the time t1 and t2 whereas, this term is an integral.

So, along the time axis if I have t1 and t2 and the and the quantity is very like this. So, this term depends on only the value here and the value here and these terms depend on the values at the entire domain which means if they are some has to be 0 that is only possible when this term is individually 0 and this term is individually 0 because it has to hold for any arbitrary t1 and t 2. If t1 and t2 are some fixed time interval then we could not say this, but because it is an arbitrary time interval then this can only hold true when this term is 0 and the other term which is defined as an integral is individually 0. So, that gives me that this term this integral is 0 again this integral is between I mean in this integral is defined in between T 1 and T 2 which is arbitrary. So, again this integral can be 0 only when the integrand itself is 0.

So, we can say that the term inside the bracket is 0. So, now we can get rid of the time and we can simply write this equal to 0 and again excuse me. So, we have delta here. So, we will rewrite this. So, now we have got rid of got rid of the time and we have established that 0 to L mu double dot del u plus 0 to L E A del u by del x multiplied by the variation of del u by del x minus 0 to L P del u d x equal to 0.

$$-\int_0^L (-m\ddot{u})\,\delta u dx + \int_0^L EA\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)dx - \int_0^L p\delta u = 0$$

$$\int_0^L m\ddot{u}\delta u dx - EA\frac{\partial u}{\partial x}\delta u|_0{}^L - \int_0^L \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)\delta u dx - \int_0^L p\delta u = 0$$

$$\Rightarrow \int_0^L \left(m\ddot{u} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) - p\right)\delta u dx = 0$$

$$m\ddot{u} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) - p = 0$$

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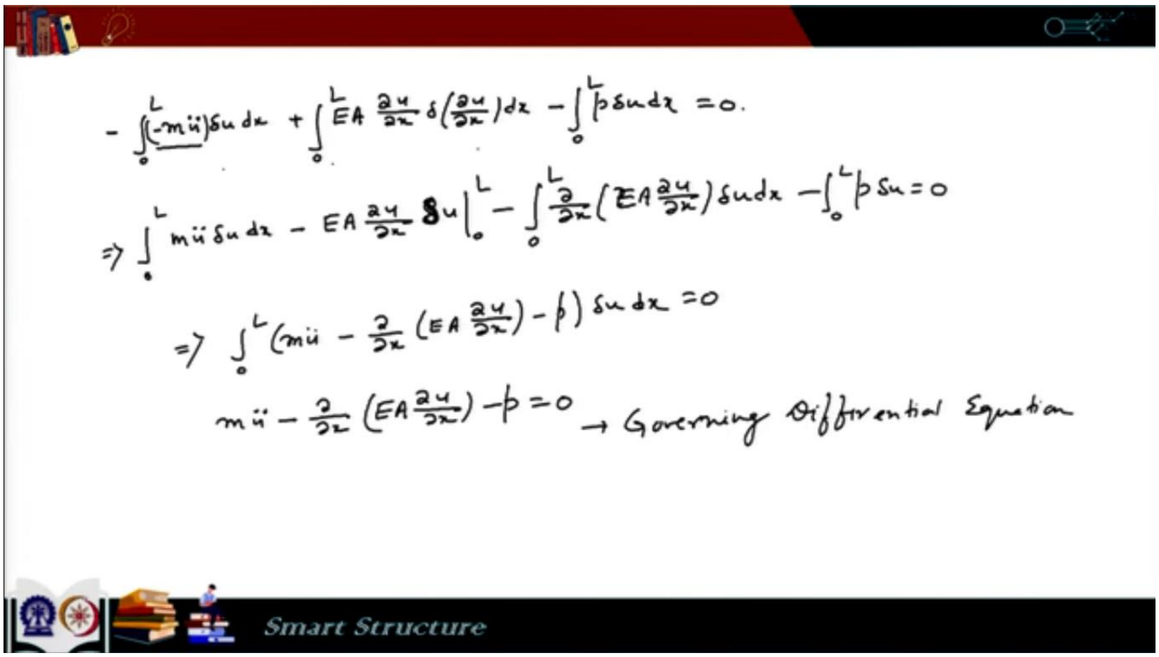

Now, this term if I neglect this then rest of it this minus this equal to 0 this is nothing, but our virtual work principle and if we look at the first term there also we see that delta u is multiplied. So, which means we can also treat this term as something analogous to this term and we can do this if I put a negative sign here and a negative sign here that gives the entire thing positive, but this then this we can take as a pseudo force minus mu double dot which we know that when something is in a is under acceleration we can we can consider that there is a pseudo force acting in the negative direction of it and the magnitude of that force is mass multiplied by that acceleration. So, this is also we can write this is also we can consider as a virtual work expression given the fact that we have a pseudo force minus mu double dot. So, while solving the problems we can also directly write this and start our solution from here. So, we have now mu double dot del u d x and then this we have to just integrate by part once we have done it before while solving the static problem and this is del u evaluated between 0 and L evaluated at 0 and L and then we have d x minus and given the boundary condition this term is 0.

So, we are left with this minus this minus this and that gives us a that gives us again delta is arbitrary. So, we can write mu double dot minus delta del by del x of E A del u by del x minus p is equal to 0. So, that is our governing differential equation. So, this is our governing differential equation. Now, we can develop solutions from here we can again we can follow the same thing we can take this equation we can multiply it with test function and get the solution or we can directly write the energy expression the Hamilton's principle or we can write this intermediate expression and from there we can develop the solutions.

Now, we will derive the governing differential equation for a beam which has piezoelectric patches and there is dynamic actuation which means the input voltage can be function of time and there can be distributed load also which can also be function of time. So, there can be distributed load p x and there can be distributed load p z and there is input voltage V 1 it can be same it can be different V 1 V 2 and these can all be function of time and naturally the response also becomes function of time and this is a induced strain actuation problem. So, we know that we are giving some voltage v and because of that we want to find out the response of the beam. So, we will write the expressions here. So, we started with Hamilton's principle and from there we got an expression like this write the similar expression here.

So, we will write the mass into acceleration and the integral of that and we considering all the displacement components. So, we know so, this is rho v is the density. So, density multiplied by acceleration along the z direction and its integral which is this and here we have the acceleration along the x direction because we know that the displacement along x is u 0 minus z into del w by del x and that gives me that u double dot is equal to u 0 double dot minus z into delta of w double dot by del x which can also be written as del 3 w by del t 2 del x. So, we have two displacements in two directions we found out the corresponding inertia forces and we wrote this term and then plus we have the internal virtual work which we already know we wrote it while solving the static problem and then we have the external virtual work. Now, we integrate this expression within the cross sectional area if we do that then rho v u 0 double dot delta of u 0 on being integrated over the cross section gives us m b.

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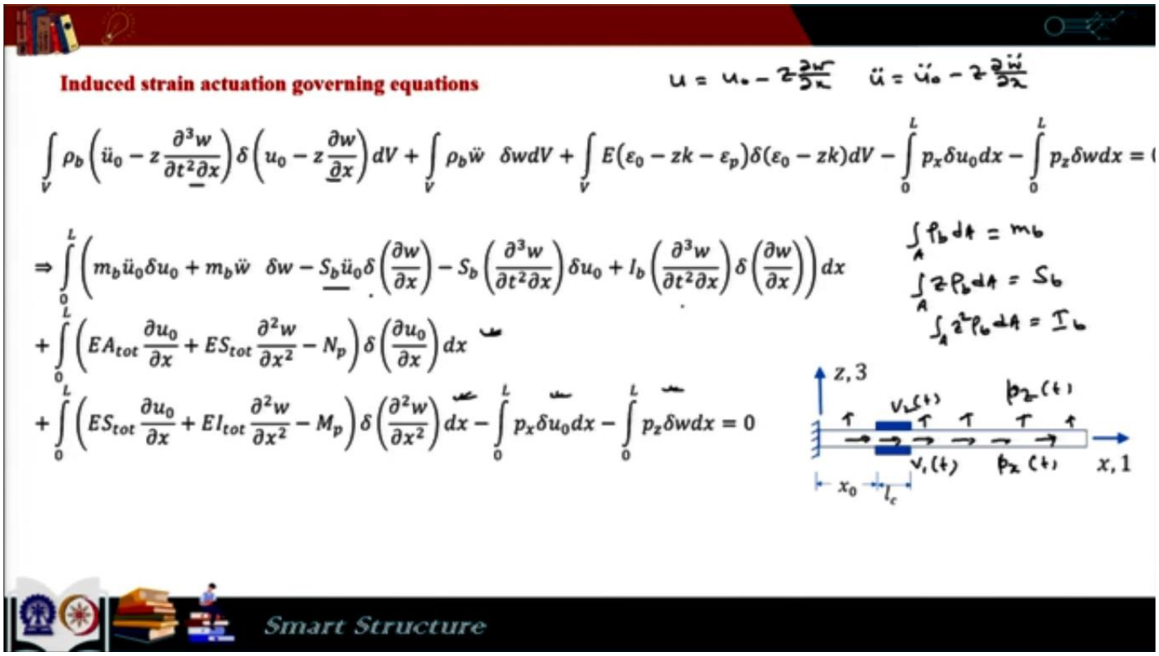

$$u = u_0 - z\frac{\partial w}{\partial x}, \ddot{u} = \ddot{u}_0 - z\frac{\partial \ddot{w}}{\partial x}$$

$$\int_A \rho_b \, dA = m_b, \int_A z\rho_b \, dA = S_b, \int_A z^2 \rho_b dA = I_b$$

So, if I integrate rho b within the cross section it gives me m b similarly from here also we get an m b m b w double dot del w and here we have a term rho v u 0 double dot multiplied by minus of z into delta w by delta x minus of and its variation and if we integrate that term we get S b u 0 double dot then variation of del w by del x. So, if we integrate z rho b over the cross sectional area that gives me S b and then we have another term rho b multiplied by minus z then this term del 3 del 3 w by del 2 del t 2 del x multiplied by variation of u 0. So, this gives me this and again we have S b is equal to z rho b integrate over the area and then finally, we have this rho v multiplied by this term multiplied by this term. So, here we have z and z z square. So, z square on being integrated after being multiplied with rho v and on being integrated over the area gives me I b.

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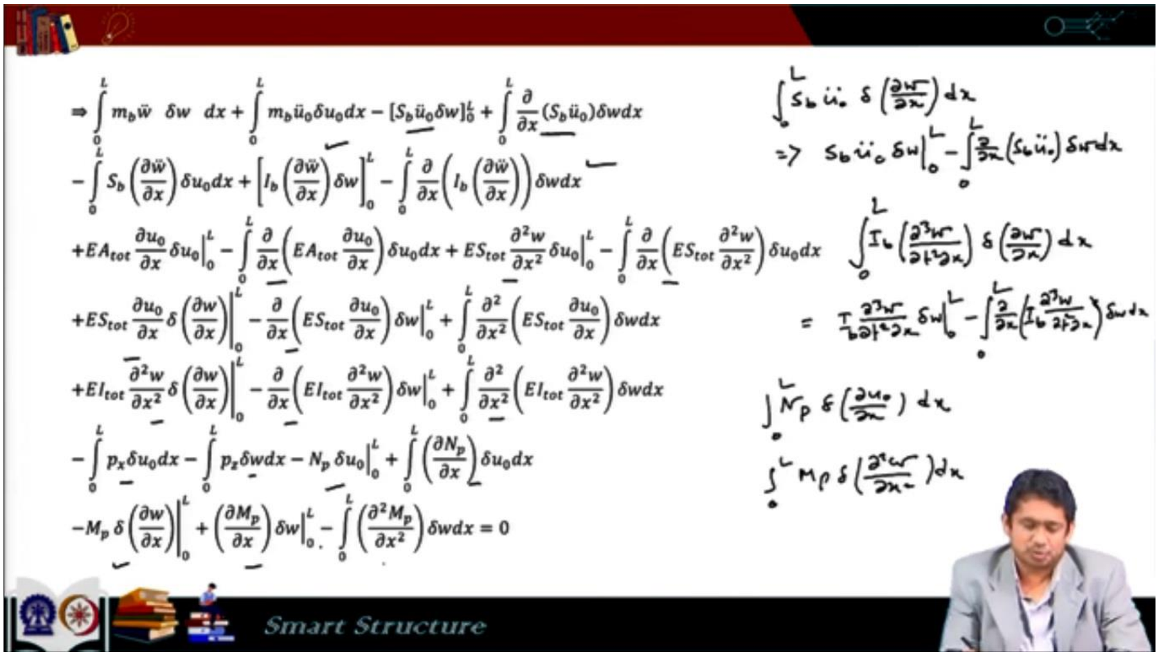

$$\int_0^L S_b\, \ddot{u}_0 \delta\left(\frac{\partial w}{\partial x}\right) dx \Rightarrow S_b \ddot{u}_0 \delta w|^L{}_0 - \int_0^L \frac{\partial}{\partial x}(S_b \ddot{u}_0)\delta w dx$$

$$\int_0^L I_b(\frac{\partial^3 w}{\partial t^2\, \partial x})\delta\left(\frac{\partial w}{\partial x}\right) dx = I_b \frac{\partial^3 w}{\partial t^2\, \partial x}\bigg|_0^L - \int_0^L \frac{\partial}{\partial x}\left(I_b \frac{\partial^3 w}{\partial t^2\, \partial x}\right)\delta w dx$$

$$\int_0^L N_p\, \delta\left(\frac{\partial u_0}{\partial x}\right) dx, \int_0^L M_p\, \delta\left(\frac{\partial^2 w}{\partial x^2}\right) dx$$

So, we have this rest it is then rest of the term comes from the internal virtual work expression and we already know that. So, this we have done while solving the static problem we defined E a total E s total N p. So, this is just same thing applied here and this remains same. After that our job is to do integration by parts now here in this term we do not have any space derivative here we do not have any special derivative here we have a special derivative here we have a special derivative and here we have a special derivative. So, we will shift some of the derivatives and similarly here we have special derivatives.

So, we do we will do some integration by parts. So, m b w dot w double dot del w came as it is because there is no special derivative and similarly this term also came as it is and then we had a term S b u double dot multiplied by variation of del w by del x and this was integrated from 0 to L. So, if we do integration by parts that gives me S b u 0 double dot delta w evaluated at x equal to 0 and L minus of del by del x of S b u 0 double dot multiplied by variation of w. So, this is what we can see here here here and here and then we had this term this came as it is the derivative was in del w by del x we did not shift it and then we had one more term which was I b del 3 w del t 2 del x multiplied by variation of del w by del x integrated within 0 to L and this gives me I b del 3 w del T 2 del x del w evaluated at L and 0 and subtracted and then we have del del x of I b del 3 w del 2 T del x multiplied by del w then integrated. So, this I b so, this term is written here here then rest of it is we have already done we integrated by parts and got this terms this terms this we have already done for the static cases this is already done and then we have this and then again like we did before we wrote this also 0 to L this we also integrated by parts like before and then we wrote this and similarly we integrated by part this term also like we did before and wrote here.

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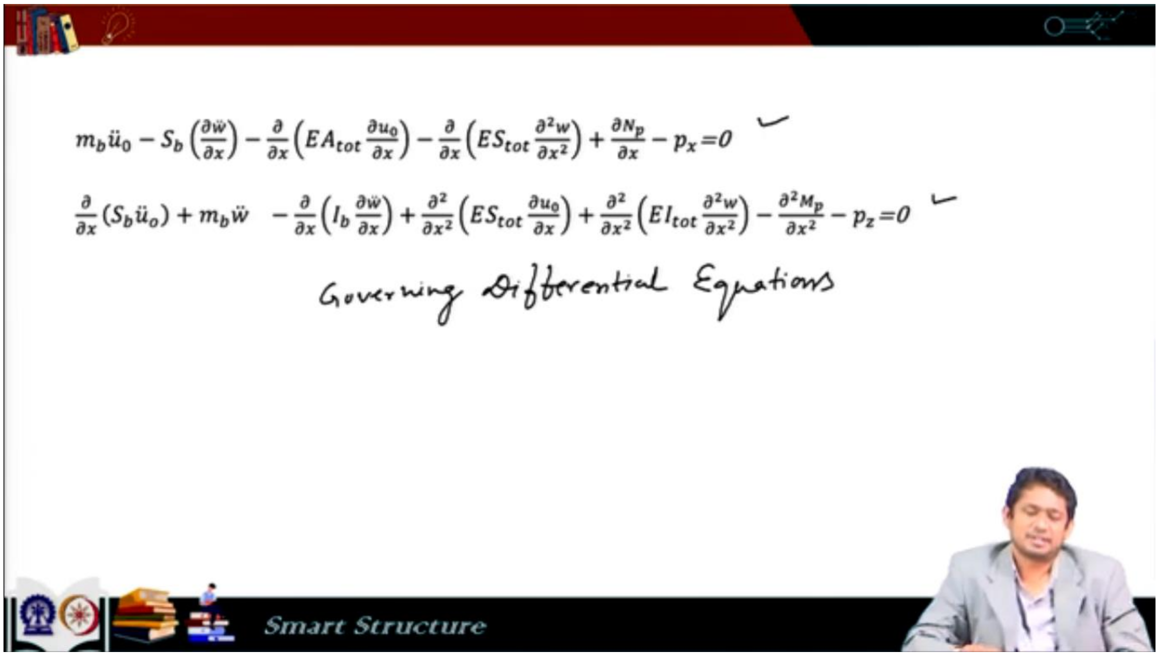

So, we had to do it twice. So, after doing all these things here we have separated the boundary terms and here they are the integral terms and again these variations are arbitrary all this del u 0 and del w variations and del variation of del del w by del x they are all arbitrary and that tells me that the integral has to be 0 and the integrand has to be 0. So, after making the integrand 0 we get these two equations. So, these two are now my governing differential equations. So, these are the governing differential equations for an induced beam induced strain actuation problem for a beam for a dynamic case.

So, we will end the lecture here in the next lecture we will see how to solve these problems.

Thank you.