Smart Structures
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Week - 05
Lecture No - 24

Static Analysis of beam for Induced Strain Actuation using Energy Principles Numerical

So, far we have seen how to perform static analysis for induced strain actuation cases using energy-based methods. Now, today we will see some of the numerical problems related to this. This is the first problem. So, we have this beam and it has two identical piezoelectric patches at the two sides and these two patches are actuated with same same electric field. So, it is a symmetric actuation. So, we have to find out the axial response due to this.

Because the actuation is symmetric it is going to cause only axial response there will be no will be no bending response. Now, these are the parameters that is given the entire length of the beam is L, where L is 0.6 meter and width of the beam as well as the piezoelectric patch is 0.05.

So, it is x axis this is z and this we can call y axis the axis 2. So, width is a dimension along this axis and that is 0.05 for both piezo and the beam and t b the thickness of the beam is 2 millimeters. So, this is my t b and t c is 1 millimeter this is t c again this is t c. And as far as the material properties are concerned both the beam and the piezoelectric patch has same elastic modulus which is 70 GPa and d 3 1 for the piezoelectric material is for both the piezoelectric materials minus 171 multiplied by 10 to the power minus 12 m by V.

Now, to solve it we will use the Galerkin technique where we do not shift the derivatives. So, for that we know the differential equation of the system is del x 2 minus del N p by del x plus p x equal to 0 and here we have del 2 W by del x 2 E s total del U 0 by del x plus del U 0 by del 2 by del x 2 multiplied by E I total del 2 by del x 2 minus we have del 2 m p 2 by del x 2 minus p z. Now, here this is a symmetric beam symmetric with respect to both material and geometric properties. So, E s total is 0 and also, we have M p is equal to 0 because it is a symmetric actuation and p z and p x are 0 anyway. So, effectively it leads to the fact that this equation is uncoupled.

So, if I just solve for this minus this is equal to 0 our solution is done. Now, here to solve it we will need this quantities E A total now E A for E A total we need E b and E c. So, we will if we calculate them separately we know how to calculate E a E b and E a b and E a c and if we if we calculate them separately we know how to calculate E a b and E A c. E A b comes to be 7 multiplied by 10 to the power 6 Newton and E A c considering the 2 piezo's also comes to be 7 into 10 to the power 6 Newton and we have N p N p comes as minus 59 comma 85 a point 85 Newton. Now while calculating N p we have taken voltage to be 50 volts and we will take the same voltage everywhere throughout all the problems.

Now to solve this let us make some approximation we approximate U 0 as one term approximation. So, phi U 1 multiplied by q U 1 and for phi U 1 we write x by L minus half into x by L whole square and multiplied by q U 1. So, this is our phi U 1. So, if we take that differential equation and then what we do is we replace U 0 by this approximation phi U 1 into q U 1. So, it is phi U 1 comma x multiplied by q U 1 and we multiply it by phi U 1 integrate from 0 to L dx and then we have here phi U 1 multiplied by del N p by del x dx equal to 0.

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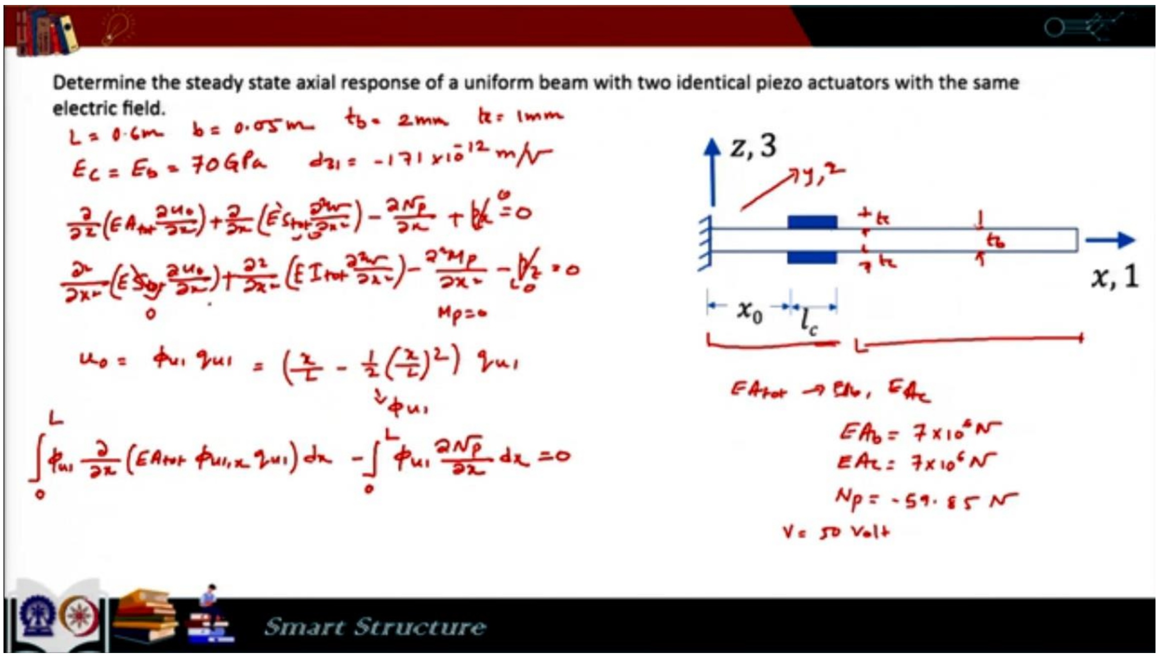

$$\frac{\partial}{\partial x}\left(EA_{tot}\frac{\partial u_0}{\partial x}\right)+\frac{\partial}{\partial x}\left(ES_{tot}\frac{\partial^2 w}{\partial x^2}\right)-\frac{\partial N_p}{\partial x}+p_x=0$$

$$\frac{\partial^2}{\partial x^2}\left(ES_{tot}\frac{\partial u_0}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(EI_{tot}\frac{\partial^2 w}{\partial x^2}\right)-\frac{\partial^2 M_p}{\partial x^2}-p_z=0$$

$$EA_b=7\times10^6N, EA_c=7\times10^6N, N_p=-59.85N, V=50Volt$$

$$u_0=\phi_{u1}q_{u1}=\left(\frac{x}{L}-\frac{1}{2}\left(\frac{x}{L}\right)^2\right)q_{u1}$$

$$\int_0^L \phi_{u1}\frac{\partial}{\partial x}\left(EA_{tot}\phi_{u1,x}q_{u1}\right)dx-\int_0^L \phi_{u1}\frac{\partial N_p}{\partial x}dx=0$$

Now we generally shift the derivatives here. So, we put so, we take derivatives in phi and I take N p out of the derivative by doing integration by parts and after doing all these things we get our expression to be this. So, q U 1 is equal to F sorry k multiplied by q U 1 is equal to F where k is equal to this and F is equal to this multiplied by phi U 1 comma x dx. Now our job is to evaluate these quantities and put it here and get the solution. So, if we find out k.

$$Kq_{u1} = F$$

$$K = \int_0^L EA_{tot}\,\phi_{u1,xx}dx$$

$$F = \int_0^L N_P\,\phi_{u1,x}dx$$

$$K = \int_0^{x_0} EA_b\,\phi_{u1,xx}dx + \int_{x_0}^{x_0+l_c}(EA_b + EA_c)\,\phi_{u1}\phi_{u1,xx}dx + \int_{x_0+l_c}^{L} EA_b\phi_{u1}\phi_{u1,xx}dx$$

$$= \frac{EA_c l_c\left({l_c}^2 + 3l_c x_0 + 3{x_0}^2\right)}{6L^4} - \frac{EA_c l_c(3l_c + 6x_0)}{3L^3} - \frac{EA_b}{3L} = 4.0025 \times 10^6$$

$$F = 4.3641, q_{u1} = \frac{4.3641}{4.0025 \times 10^6}, u|_{x=L} = q_{u1}\phi_{u1|_{x=L}} = -5.452 \times 10^{-7}m$$

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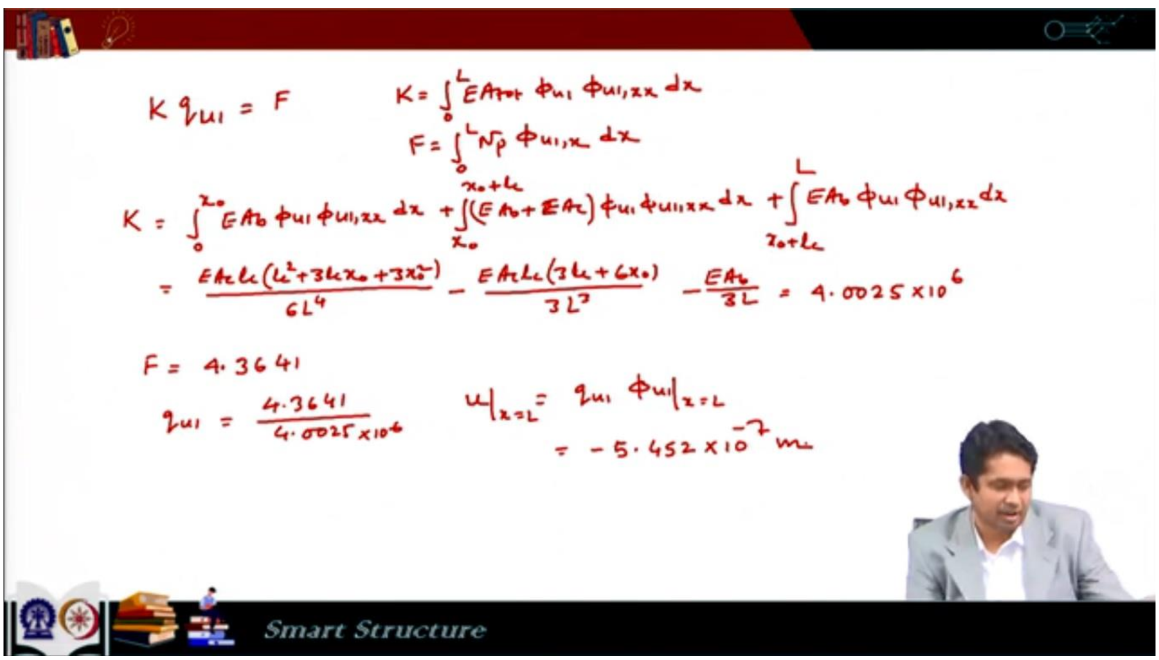

$$u(x) = \left(\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right) q_{u1} + \left(\frac{x}{L}\right)^3 q_{u2}$$

$$[K]_{2\times 2} = \begin{bmatrix} 0.4634 & 0.2958 \\ 0.2958 & 2.1003 \end{bmatrix} \{q_u\} = \begin{Bmatrix} 4.364 \\ 0.242 \end{Bmatrix}$$

$$\begin{Bmatrix} q_{u1} \\ q_{u2} \end{Bmatrix} = \begin{Bmatrix} 0.1027 \\ -0.0133 \end{Bmatrix} \times 10^{-5}$$

$$u|_{x=L} = -3.8 \times 10^{-7} m$$

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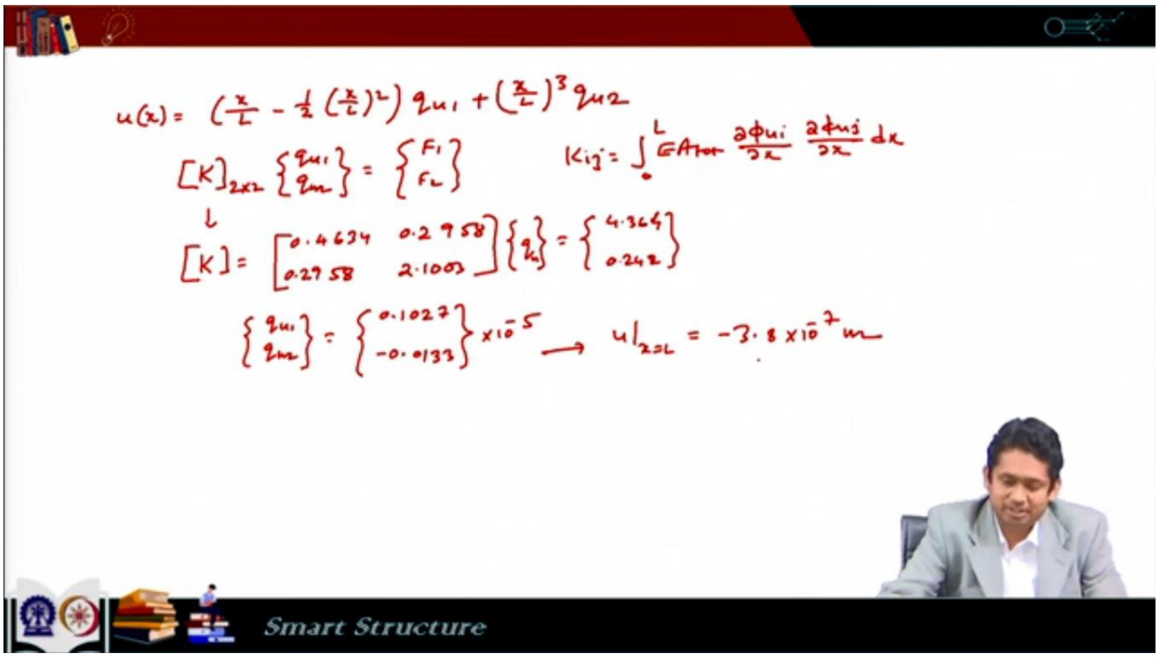

So, k we can find out like this within 0 to 0 to x 0 we have E A total as E A b and then we multiplied by phi 1 and then we multiplied phi U 1 comma x comma x dx plus we have from x equal to 0 from x 0 to x 0 plus L c we have E A b plus E A c multiplied by phi U 1 phi U 1 comma x x dx and then from x x 0 plus L c to L we have again E A b multiplied by phi U 1 multiplied by phi U 1 comma x x dx. And after doing all this we get our expression as E A c L c L c square plus 3 L c x 0 plus 3 x 0 square divided by 6 L to the power 4 and then minus E A c L c multiplied by 3 L c plus 6 x 0 divided by 3 L cube minus E A b divided by 3 L. And the value that comes is 4.0025 into 10 to the power 6 and F comes as 4.3641. So, our q U 1 that we get from these two are from here we find out q U 1 and that comes as 4.3641 divided by 4.0025 into 10 to the power 6. Now, from here if I want to find out U at x equal to L that is our q U 1 multiplied by phi U 1 evaluated at x equal to L, we know the expression for phi U 1. So, we just put x equal to L there and after doing that the x the result comes to be minus 5.452 multiplied by 10 to the power minus 7 meters. So, this is the axial displacement at the tip. Now, we will solve it by multiplying phi to the equation by, but by shifting the derivatives. And we have seen that if we shift the derivatives and we wrote the final expression for all the terms. So, it looks like this q U 1 to q U m q W 1 to q W n is equal to we have a force matrix here and then here we have k U u k U w k W u k W w.

$$\begin{bmatrix} [K]_{uu} & [K]_{uw} \\ [K]_{wu} & [K]_{ww} \end{bmatrix} \begin{Bmatrix} q_{u1} \\ q_{uM} \\ q_{w1} \\ q_{wN} \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

$$Kq_{u1} = F, K = -\int_0^L EA_{tot}\left(\frac{\partial \phi_{u1}}{\partial x}\right)^2 dx$$

$$= \int_0^{x_0} EA_b\left(\frac{\partial \phi_{u1}}{\partial x}\right)^2 dx + \int_{x_0}^{x_0+l_c} (EA_b + EA_c)\left(\frac{\partial \phi_{u1}}{\partial x}\right)^2 dx + \int_{x_0+l_c}^{L} EA_b\left(\frac{\partial \phi_{u1}}{\partial x}\right)^2 dx$$

$$=-\left[\frac{EA_b}{3L} + \frac{EA_c l_c\left(3L^2 - 3Ll_c - 6Lx_0 + l_c{}^2 + 3l_c x_0 + 3x_0{}^2\right)}{3L^4}\right] = -4.634 \times 10^6$$

$$u_0|_{x=L} = -4.709 \times 10^{-7} m$$

$$EA_{tot}\varepsilon_0 = N_P \Rightarrow \varepsilon_0 = -4.275 \times 10^{-6}, u|_L = -2.138 \times 10^{-7}$$

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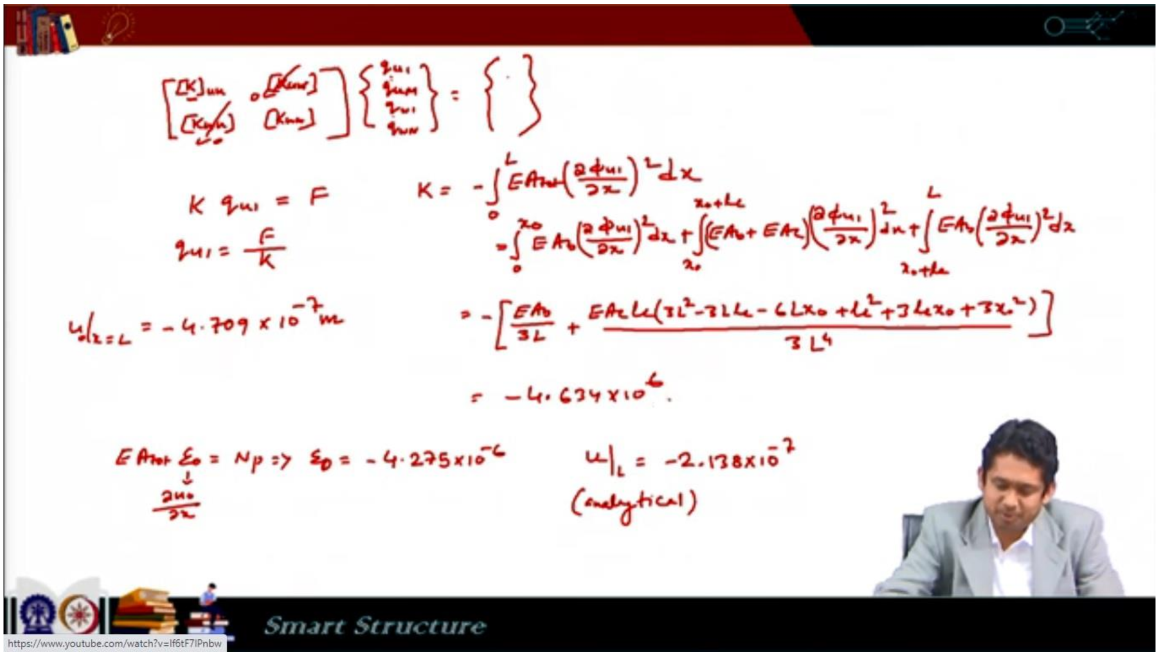

Now, this matrix involves E S total and this also involves E S total, but again E S total is 0. So, finally, we have we have just this x this multiplied by the by the q the q's q of U's is equal to the force corresponding to the U's and if we solve it, we get our solution. So, this

0 this 0 that uncouples the equation. So, we just get the the q U separately. Now, since it is an uncoupled problem the result the response is purely axial ok.

So, again let us have the same approximation using the same approximation our finally, the equation looks like this. We just have one equation k multiplied by q U 1 is equal to F. So, k is just this matrix. So, here also again we have only one approximation. So, it just one term here it is not a matrix and q U 1 is just this F is the first component of it corresponding to just the one approximation.

Now, for this the k is minus of 0 to L E A total del phi 1 by del x or we can write phi 1 comma x also square dx. You already derived this while discussing this theory. So, from there we are directly taking it and for F we have the same expression f does not change. Now, again if we evaluate it separately. So, it becomes 0 to x 0 E A b multiplied by phi q derivative of phi U 1 with respect to x square of that and then we have x 0 from x 0 to x 0 plus L C and we have E A b plus E A C multiplied by the same thing and then we have from x 0 plus L C to L E A b multiplied by same thing.

Now, after simplification the expression becomes minus 3 minus E A B by L minus E A B by 3 L plus E A C L C 3 L square minus 3 L L C minus 6 L x 0 plus L C square plus 3 L C x 0 plus 3 x square and that divided by 3 L to the power 4 and the value that we get for this is this minus 4.634 into 10 to the power 6. And again so, we get q U 1 as F by k and from here if we find out U at U 0 at x equal to L that comes to be little less than what we got previously minus 4.709 multiplied by 10 to the power minus 7 meters. Now, so, this is what we get by solving it approximately.

Now, say suppose if you want to solve it analytically like we did in the second week in that case we just solve this E A total epsilon 0 which is del U 0 by del x and is equal to N p and from here epsilon 0 comes to be minus 4.275 multiplied by 10 to the power minus 6 and with that the final result comes to be minus 2.138 multiplied by 10 to the power minus 7. So, this is analytically obtained result. So, we can see that there is substantial amount of difference between what we get from the approximate analysis and what we get from the analytical solution.

There is a reason for that we have just we just have one term approximation for this. So, we need to improve our approximation and by improving we can be more closer to the actual solution. Now, to do that let us add one more term to it. So, let us make an approximation like this. So, we have just added this term here.

Now need if we follow the same procedure then our K becomes a 2 by 2 matrix. So, previously it was 1 by 1 now it becomes 2 by 2 and we have two terms here F 1 F 2. Now the K matrix in this case becomes 0.4634 0.2958 0.2958 please note that the matrix is symmetric we are using the formulation where we shifted the derivatives and Q u and F term becomes 4.364 and 0.242. So, our K ij is E a total del phi u i by del x del phi u j by del x dx. Now if we solve it our q u 1 and q u 2 comes to be 0.1027 minus 0.0133 multiplied by 10 to the power minus 5. And then from here if we find out the value of u at x equal to l by at x equal to l u comes to be minus 3.8 into 10 to the power minus 7 meters. So, we can see that by adding one more term we have gone little closer to the analytical solution.

In that way we can keep improving our approximation and we will go closer to the analytical solution.

Now we will solve a similar problem, but here the actuation is an anti-symmetric actuation. So, all the parameters remain same the width, length, the material parameters everything is same voltage is same, but here the actuation is opposite. So, if I have a positive voltage here the voltage is negative here, if I have positive voltage at the bottom piezo, I have a negative voltage at the top piezo. Now to do this problem again it is an anti-symmetric actuation. So, it will not induce any axial response it will induce only it will induce only a bending response.

So, there will be no u 0 because again here also E s E s total is 0. Now so, E a total is also not needed for us. So, what we need is E Ib and E Ib is for this case 2.33 and E Ic is we are not writing the dimensions now we will write them finally, in the actual solution I mean the units now we will write them directly in the solution. Again, the solution that we get is in the form of this here we are using the solution technique where we multiply phi, but shift the derivatives.

$$K = \int_0^L EI_{tot}\left(\frac{\partial^2 \phi_{u1}}{\partial x^2}\right) dx = 68.4156$$

$$F = \int_0^L M_P \frac{\partial^2 \phi_{w1}}{\partial x^2} dx = -0.0249$$

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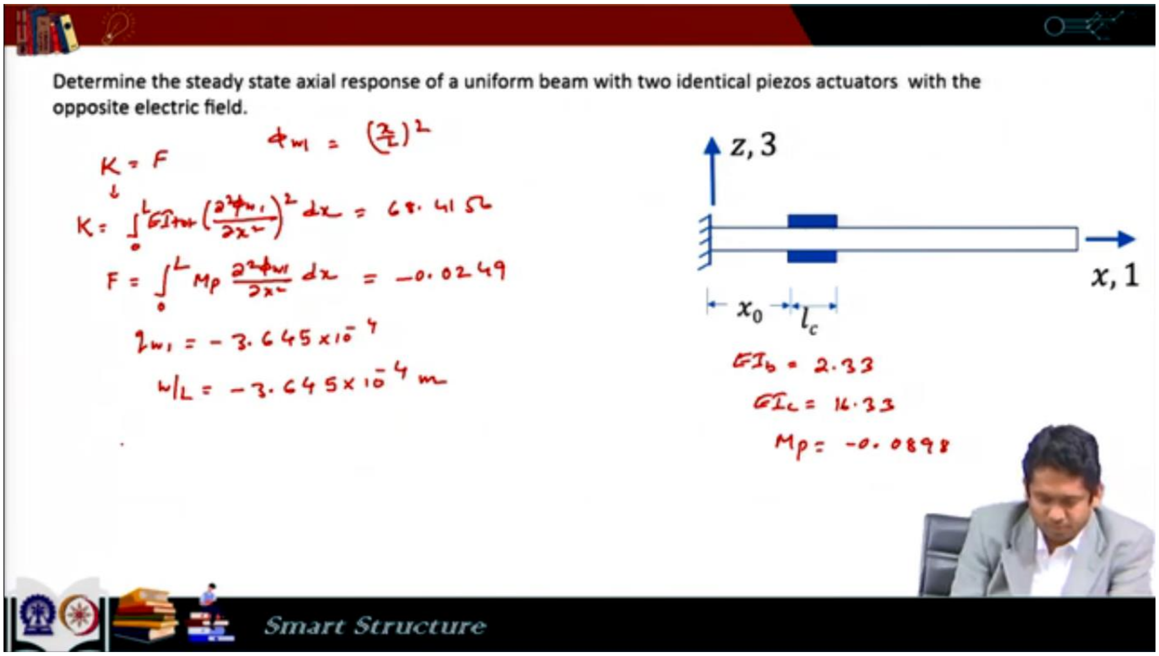

So, in our case k is E I total into del 2 phi 1 w del w 1 del 2 phi w 1 del x 2 square dx and F is dx. Now for this we have to make the assumption the assumption that we make here is Q w 1 is equal to x by L square. So, with that assumption our k matrix k comes to be it is not a matrix now because we have only one approximation. So, it is 68.4156 and F comes to be minus 0.0249. Then we can solve for our q w and by solving we get minus 3.645 into 10 to the power minus 4 and then if we put the approximation back to the I mean if we put q w 1 back to the approximation and then if we put x equal to L we get our was minus 3.645 multiplied by 10 to the power minus 4 meter. So, this is what we get with one term approximation. Now let us enrich the approximation by adding more terms.

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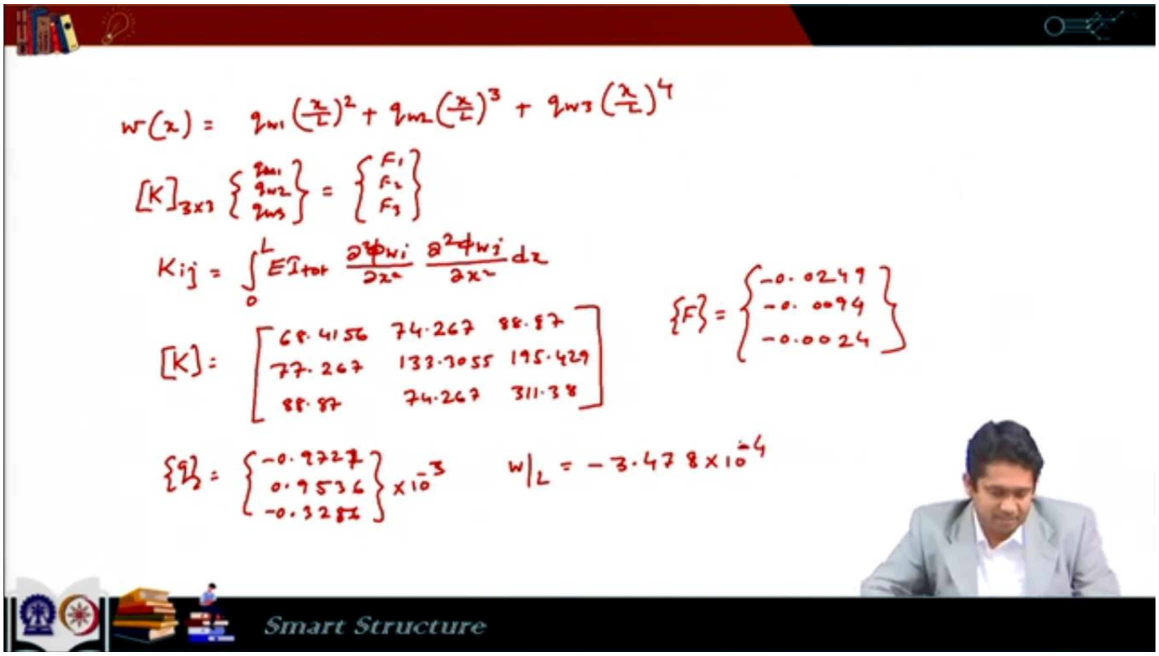

$$w(x) = q_{w1}\left(\frac{x}{L}\right)^2 + q_{w2}\left(\frac{x}{L}\right)^3 + q_{w3}\left(\frac{x}{L}\right)^4$$

$$[K]_{3\times3}\begin{Bmatrix} q_{w1} \\ q_{w2} \\ q_{w3} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$K_{ij} = \int_0^L EI_{tot}\frac{\partial^2\phi_{wi}}{\partial x^2}\frac{\partial^2\phi_{wj}}{\partial x^2}dx$$

$$[K] = \begin{bmatrix} 68.4156 & 74.267 & 88.87 \\ 77.267 & 133.3055 & 195.429 \\ 88.87 & 74.267 & 311.38 \end{bmatrix}$$

$$\{q\} = \begin{Bmatrix} -0.9727 \\ 0.9536 \\ -0.3286 \end{Bmatrix} \times 10^{-3}$$

$$\{F\} = \begin{Bmatrix} -0.0249 \\ -0.0094 \\ -0.0024 \end{Bmatrix}$$

$$w|_L = -3.478 \times 10^{-4}$$

So, let us approximate w as so, we are adding two more terms x by L cube x by L whole cube and x by L whole to the power 4. And then our k becomes a 3 by 3 matrix and we have q u 1 q w 1 q w 2 and q w 3 and in the right-hand side we have F 1 F 2 F 3 we may call this as F w 1 F w 2 F w 3 also here we have only w. And K ij here is 0 to L E I total del 2 phi i by del x 2 del 2 del 2 phi w j by del x 2 dx and with this we get our K matrix to be 64.4156 74.267 88.87 77.267 133.3055 195.40 88.87 it is symmetric 7 and finally, we have 311.38. And the force vector becomes minus 0.0249 minus 0.0094 minus 0.0024 and finally, after solving this we get our q as minus 9727 0.9536 and minus 0.3286 multiplied by 10 to the power minus 3. And then if we put it back to the expression and evaluate our w at x equal to L the value that we get is minus 3.478 multiplied by 10 to the power minus 4. And so, you can keep improving the term by adding I mean improve the improving the solution by adding more terms. Now, finally, we will look into one more case where we have a where we have an asymmetric problem. So, we have only one piezo and that is excited. Now, this structure is asymmetric with respect to the mid portion of the beam. So, if we consider with respect to that we have both EA total and EA total.

So, here we need EAb EAb remains same, but we need to calculate EAc EAc just takes contribution from one of the piezos. So, it becomes half we need ESc which we did not calculate before. So, we need ESb is 0 because beam is symmetric with respect to its own center and Es EAc ESc is 5250 Eib again remains same that we got before and EIc has contribution from one piezo.

So, it is 16.0417. Now, with this we have to calculate I mean we have to. now, it will induce both bending and axial response. So, the equation that we solve is this K multiplied by qu 1 qw 1 is equal to let us call it fu 1 fw 1 provided we take one term solution for both. So, let us assume ux is equal to x by L minus half into x by L whole square multiplied by qu 1 and wx is equal to x by L whole square multiplied by qw 1. So, this is our phi u 1, this is our phi w 1. Now, here in this expression for K, our K 1 1 is 0 to L Ea total phi u del phi u 1 by del x whole square and K 2 1 is equal to K 1 2 that is Es total del phi u 1 by del x del phi w 1 by del x dx and we have K w K 2 2 is equal to Ei total.

$$[K]\begin{Bmatrix} q_{u1} \\ q_{w1} \end{Bmatrix} = \begin{Bmatrix} F_{u1} \\ F_{w1} \end{Bmatrix}$$

$$u(x) = \left(\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right) q_{u1}$$

$$w(x) = \left(\frac{x}{L}\right)^2 q_{w1}$$

$$K_{11} = \int_0^L EA_{tot}\left(\frac{\partial \phi_{u1}}{\partial x}\right)^2 dx$$

$$K_{21} = K_{12} = \int_0^L ES_{tot}\frac{\partial \phi_{u1}}{\partial x}\frac{\partial^2 \phi_{w1}}{\partial x^2} dx$$

$$K_{22} = \int_0^L EI_{tot}\left(\frac{\partial^2 \phi_{w1}}{\partial x^2}\right)^2 dx$$

$$EA_c = 3.5 \times 10^{-6}, ES_b = 0, ES_c = \Omega 50, EIc = 16.0417$$

$$F_{u1} = \int_0^L N_P \frac{\partial \phi_{u1}}{\partial x} dx, F_{w1} = \int_0^L M_P \frac{\partial^2 \phi_{w1}}{\partial x^2} dx$$

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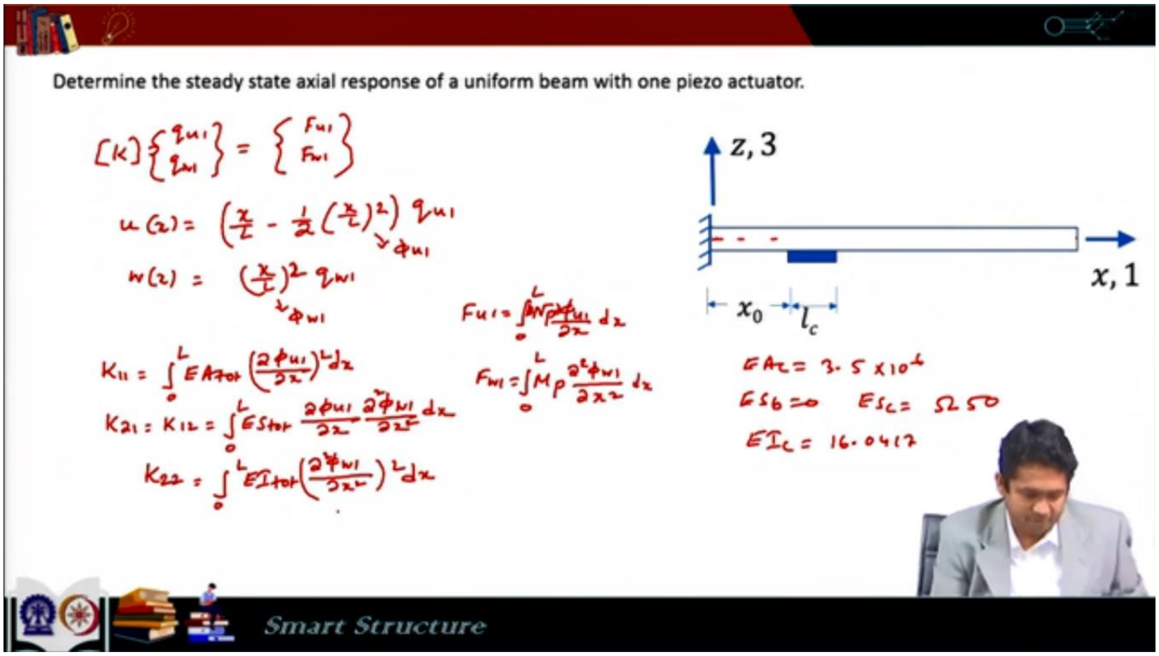

We have a second order derivative here and we know our fu 1 is 0 to L Np phi Np multiplied by del phi 1 by del x and fw 1 is 0 to L Mp x. So, after evaluating everything we get now it is to be understood that Np and Mp are also going to be reduced here because we have only one piezo.

$$\begin{bmatrix} 0.426 \times 10^7 & -1.043 \times 10^7 \\ -1.043 \times 10^7 & 16.9914 \end{bmatrix} \begin{Bmatrix} q_{u1} \\ q_{u2} \end{Bmatrix} = \begin{Bmatrix} -2.182 \\ 0.0062 \end{Bmatrix}$$

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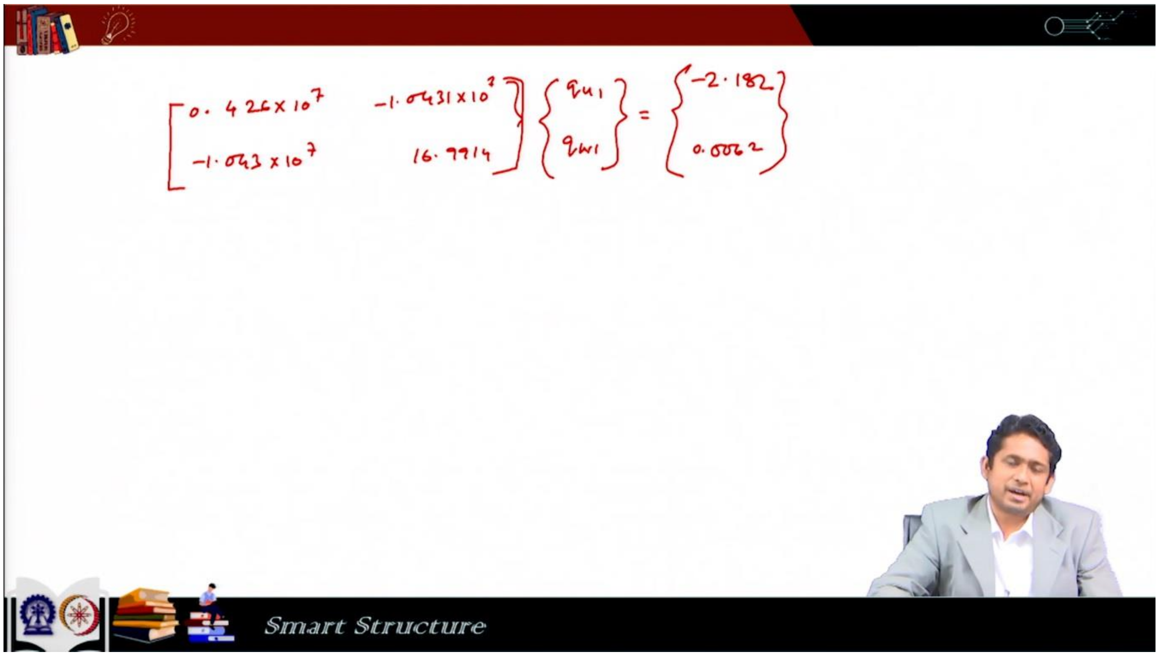

So, Np becomes minus 29.92 and Mp is minus 0.0499. So, finally, if we the expressions the matrices that we get is. So, if you solve this expression we get our q u 1 and q w 1 and then if you put it back to the approximation we get our axial displacement and the displacement along z direction. So, we have seen we have seen three different examples first two were uncoupled problems and third was third was a coupled problem. So, with that with this I would conclude this lecture.

Thank you.