Smart Structures
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Week 04
Lecture No: 23
Static Analysis of beam for Induced Strain Actuation using Energy Principles (continued)
Part 05

In the previous lecture, we derived the governing differential equation in an alternative form starting from the virtual work equation and then solve that equation using the Galerkin technique.

Today, we will also solve using the Galerkin technique, but in a slightly different way.

So, this is what we started previously. We had a governing differential equation in terms of u_0 w and we assumed our u_0 as, we assumed our u_0 as, summation of i equal to 1 to m and phi$_{ui}$ q$_{ui}$ and w is assumed as summation of j is equal to 1 to n phi w$_j$ q w$_j$.

$$u_0 = \sum_{i=1}^{M} \phi_{ui} q_{ui} \qquad\qquad w = \sum_{i=1}^{N} \phi_{wi} q_{wi}$$

And then, we put this approximation in the differential equation, in both the differential equations and then we multiplied each of the differential equation, the first by the phi's. So, here what we will do is we will – So, there are total M plus N number of equations, sorry there are total two equations, and the first equation if I write it. The first equation is – there are two equations, but M plus N number of variables. So, the first equation is E A$_{total}$, del u_0 by del x and it differentiated, plus del w by del x E S$_{total}$ differentiated twice and then, we have minus of del N$_p$ by del x plus p$_x$ equal to 0.

$$\frac{\partial}{\partial x}\left(EA_{tot}\frac{\partial u_0}{\partial x}\right) + \frac{\partial}{\partial x}\left(ES_{tot}\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial N_p}{\partial x} + p_x = 0$$

So, what we do is we multiply and so, if I put this u_0 here and this approximation for w here, I get an equation with M plus N number of unknowns. So, I multiply this equation with this phi's. So, there are total phi$_{ui}$'s. So, there are total M number of phi$_{ui}$'s. So, I multiplied with phi$_{ui}$ times. So, first we multiply with phi$_{u1}$ and integrate from 0 to L and equate that to 0, that gives me one equation. And then, I multiplied with phi$_{u2}$ and integrate from 0 to L that gives me two equations. So, I like i is multiply with phi$_{ui}$, i going from 1 to M. So, that gives me total M number of equations. Similarly, I have one more equation and the equation is del^{2}w by del x^2, E S$_{total}$ del u_0 by del x, plus del^{2}w by del x^2, E I$_{total}$ del^{2}w by del x^2, and then we have minus of M_p minus q_z. So, here I multiply this with phi w_j. So, there are total n number of j's and likewise integrate from 0 to L.

$$\int_0^L \left(\frac{\partial^2}{\partial x^2}\left(ES_{tot}\frac{\partial u_0}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(EI_{tot}\frac{\partial^2 w}{\partial x^2}\right)-M_p+q_z\right)\phi_{w_i}dx$$

So, that will give me total N equations. So, I get M equations from here and I get N equations from here.

Now, in each of these equations now we shift the derivatives. So, this is written here. So, it is multiplied with phi_{ui} and all these u_0 's and w's are replaced with their approximations. In the previous case, we did not shift the derivatives. So, here we can see that phi_{uj} is differentiated twice. I have a differentiation here; I have a differentiation here. So, comma x means differentiation with respect to x. Similarly, here comma double x means differentiation with respect to x twice. So, here the phi w_j has been differentiated three times whereas, this phi_{ui} which is multiplied with it has not been differentiated. So, instead of keeping it like that we shift the derivative and we try to distribute the derivative between this phi and this phi. So, to do this, we do the integration by parts. We have this function phi_{ui} multiplied with this. So, phi_{ui} into integral of this term that is here, this is the integral of this term inside this del by del x and evaluated at 0 and L. And then minus the differentiation of this multiplied by integral of this. Here, we do the same thing. We take this and take integral of this minus differentiation of this which is here into integral of this. Here, also we shift the derivatives. So, phi_{ui} is here, integrated and we get N_p and we differentiate this and then, get integral of this N_p. And p_x phi_{ui} remains as it is. So, it is u. So, p_x phi_{ui} remains as it is.

Now, if we again look at the boundary terms, we can see the boundary term vanishes because at the boundary we have phi_{ui} multiplied by – So, at the boundary we have, I mean if we sum up these two terms this and this, we have phi_{ui} multiplied by E A_{total} into summation, I am not giving the limit in the summation now, q_{uj} plus E S_{total} into summation phi w_j comma xx, w_j and minus N_p.

$$\phi_{u_i}\left[EA_{tot}\left(\sum \phi_{u_{j,x}}\,q_{u_j}\right)+ES_{tot}\left(\sum \phi_{w_{j,xx}}\,q_{w_j}\right)-N_p\right]$$

Now, this is approximation for u_0, this is approximation for w. So, this term is very close to our N as we have seen before. And again, by the same logic that at x equal to 0 my displacement is 0 for the clamped end and accordingly phi is 0 because this is supposed to satisfy the essential boundary condition. And at x equal to L, there is no normal force. So, N is 0. So, that helps me get rid of these three terms. So, I am left with this.

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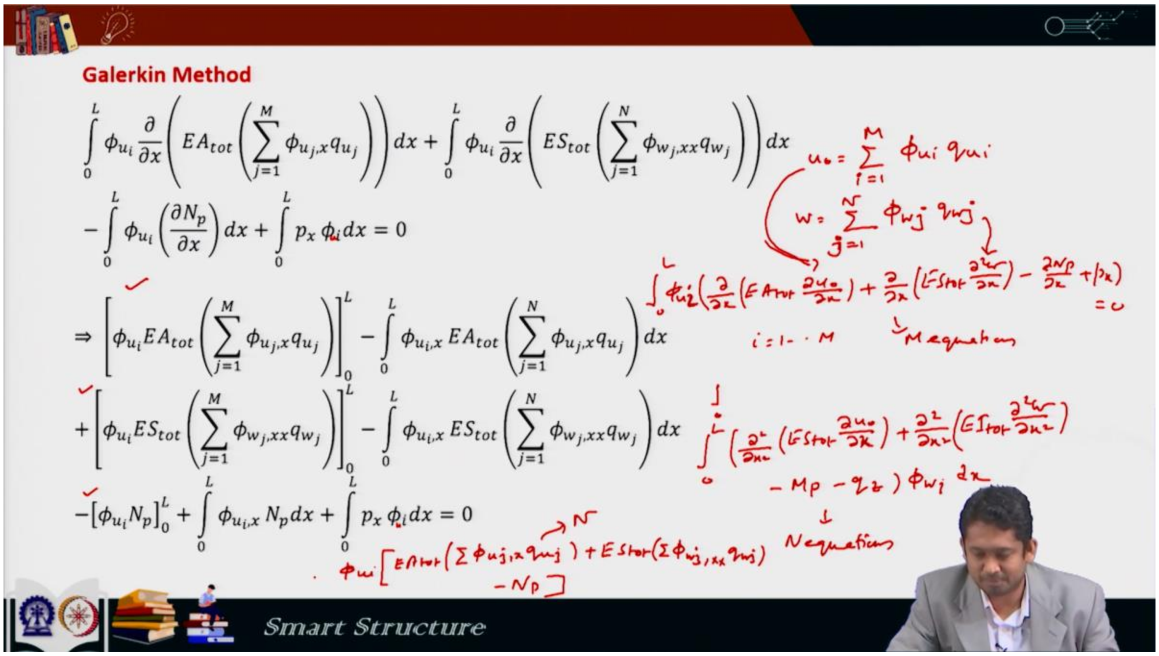

So, this gives me total, sorry I am left with rest of the terms and that gives me total M number of equations. Now, we will take the other equation and then multiply phi w_i and again we will do integration by parts and shift the derivatives.

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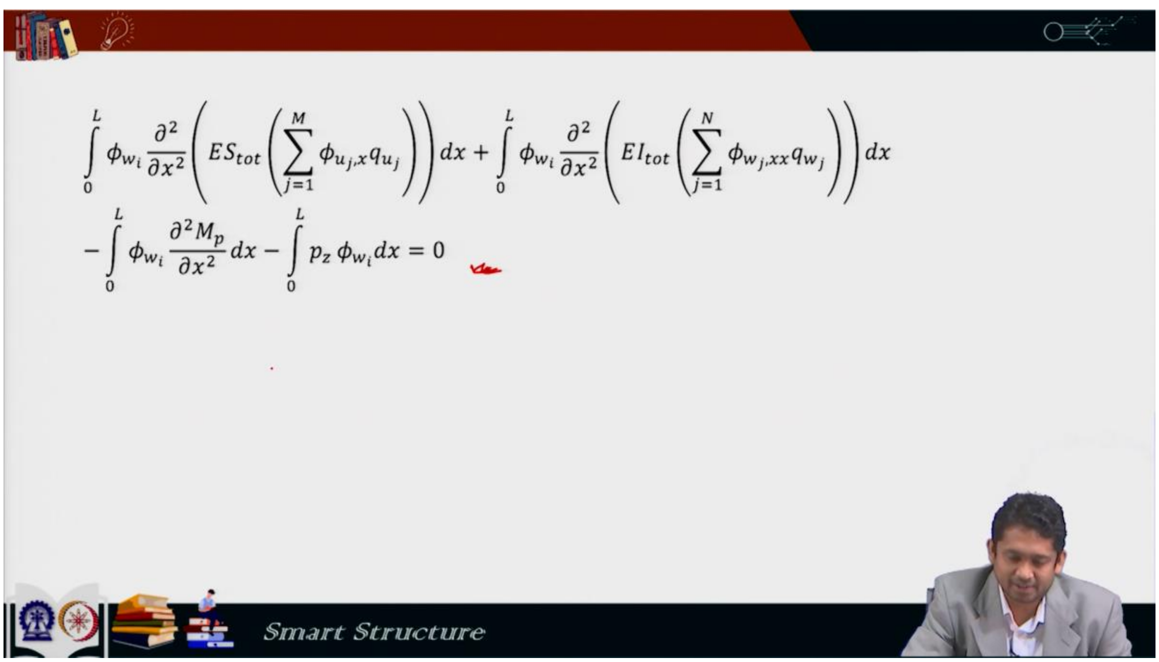

So, that is the first term, phi w_i, this term is integrated once and then we get this minus the derivative of phi w_i. And here, we have carried out the integration by parts twice and finally, phi w_i has got two derivatives and u has one derivative. In this equation we can see that here also we have second order derivative and here also we have second order derivative. So, the integration by parts has been done twice and that gives two derivatives to this w and this phi and two derivatives to this phi.

So, the integration by part is done twice here and here. And finally, we get in the integrals, we get phi w_i differentiated twice. Here also we have phi w_i differentiated twice, and these are the boundary terms. Similarly, in this expression which we have here also integration by parts has been done twice and these two derivatives have been given to phi w. And we can see it here and this p_z multiplied by phi w_i remains as it is. Again, if we look at the boundary terms, we see that, we have those terms becoming zero. So, the boundary terms are this, this which has phi w, this which has phi w, this which has phi w. And similarly, in the boundary term, we have this as phi w. So, if I combine these three terms that gives me, phi w_i multiplied by shear force and they are evaluated at x equal to 0 and L.

$$\left[\phi_{w_i} V\right]_0^L$$

And again, we have the same logic that this phi is supposed to satisfy the essential boundary condition. So, they are 0 at the x at x equal to 0 and at the other and there is no shear force. So, this is 0. So, this we can get rid of this term, this term and this term. Similarly, if I combine the other boundary terms which is this plus this, plus this and that would give me 0.

$$\left[\phi_{w_{i,x}} M\right]_0^L$$

So, again we apply the same logic at x equal to 0, slope is 0. So, the phi is supposed to satisfy this. And at x equal to L, bending moment is 0. So, M is 0.

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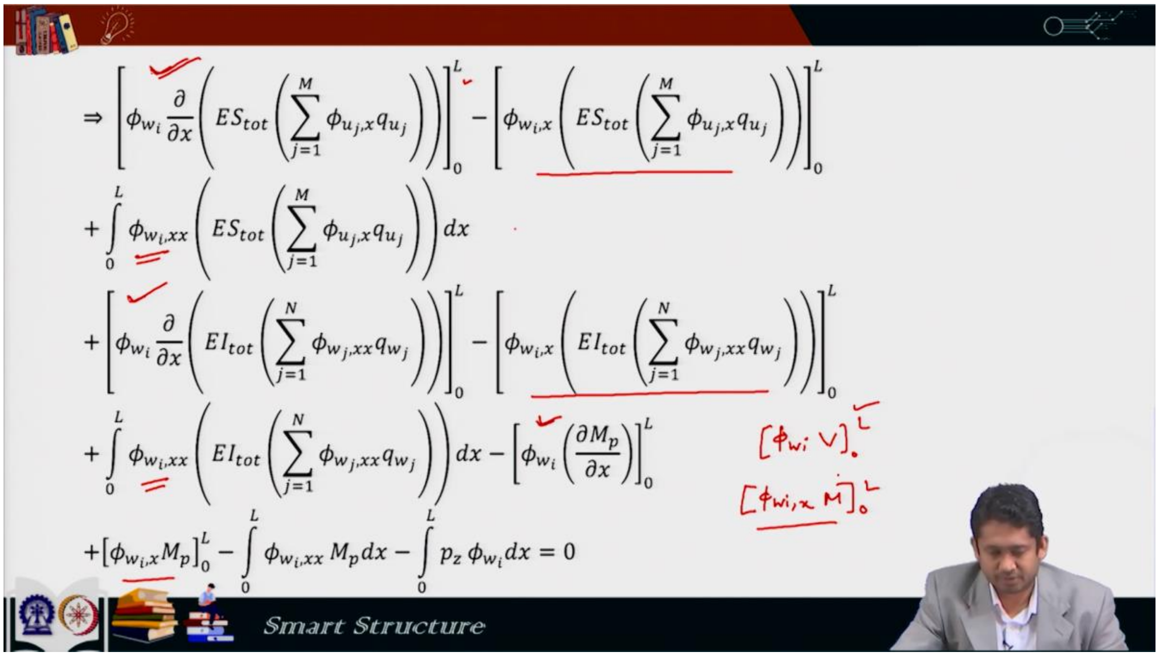

So, after getting rid of the boundary conditions, we get the final equations. So, here these boundary conditions are all summed up. So, like I said we have phi$_{ui}$ multiplied by this. So, this is nothing, but N. And here, we have phi w_i comma x, that is multiplied with this, that is M. And here, we have shear V and that is multiplied with phi w_i and that helps me get rid of all the boundary terms.

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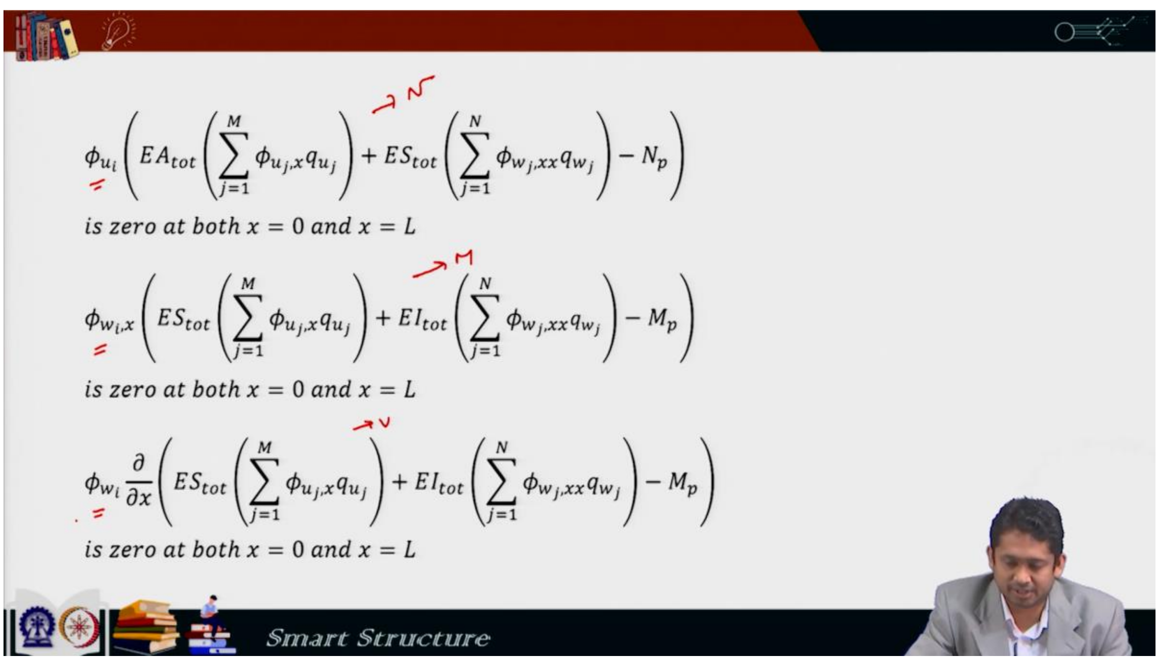

Finally, we have this equation and this equation. So, here I have M equations, and here I have N equations. These equations, M plus N equations can be written in a compact way as this. Now, please understand as we defined before this q vector has total M plus N components. Among them the first M's are – so, among them the first M components are q_u and then we have q_w. And then, that is equal to F, the force.

$$\begin{bmatrix} [K_{uu}]_{MxM} & [K_{uw}]_{MxN} \\ [K_{wu}]_{NxM} & [K_{ww}]_{NxN} \end{bmatrix} \{q\} = \{F\} \quad \text{where,} \{q\} = \begin{Bmatrix} \{q_u\}_{\text{Mx1}} \\ \{q_w\}_{\text{Nx1}} \end{Bmatrix}$$

K_{uu} is M by M. This is M by N. This is N by M. This is N by N. If we look at the i j th expression of K_{uu} this is 0 to L, E A_{total} multiplied by phi_{ui} comma x, phi_{uj} comma x, dx that we get from here.

$$K_{uu_{ij}} = \int_0^L EA_{tot} \phi_{u_{i,x}} \phi_{u_{j,x}} dx$$

Then K_{uw} which comes from this term, this gives us K_{uw} i j is equal to E S_{total}, then phi_{ui} comma x, multiplied by phi w_j comma xx, dx. So, this comes from this.

$$K_{uw_{ij}} = \int_0^L ES_{tot} \phi_{u_{i,x}} \phi_{w_{j,xx}} dx$$

Now, this gives us K_{wu}. So, we can write it here K_{wu} is 0 to L, E S_{total}, phi_{ui} comma xx, phi_{uj} comma x, dx.

$$K_{wu_{ij}} = \int_0^L ES_{tot} \phi_{w_{i,xx}} \phi_{u_{j,x}} dx$$

And this gives us K_{ww}, K_{ww} ij is E I_{total}, phi_{wi} comma xx, and phi_{wj} comma xx, dx.

$$K_{ww_{ij}} = \int_0^L EI_{tot} \phi_{w_{i,xx}} \phi_{w_{i,xx}} dx$$

And here, this is just phi_w, xx does not mean anything here. It does not differentiate. Likewise, we can write the forces. So, the force also has, we can see that, it has two major components, one is F_u which is M by 1 and F_w which is N by 1.

$$\{F\} = \begin{Bmatrix} \{F_u\}_{\text{Mx1}} \\ \{F_w\}_{\text{Nx1}} \end{Bmatrix}$$

So, if you want to write F_{ui} that is 0 to L, N_p phi_{ui} comma x, dx plus p_x, phi_{ui}, dx.

$$F_{u_i} = \int_0^L N_p \phi_{u_{i,x}} dx + \int_0^L p_x \phi_{u_i} dx$$

And similarly, F_{wi} may be written as M_p, dx plus 0 to L, p_z comma dx.

$$F_{w_i} = \int_0^L M_p \phi_{w_{i,xx}} dx + \int_0^L p_z \phi_{w_i} dx$$

So, this entire matrix and vectors are defined by solving this. We can find out q and then we can get our solution. So, here by shifting the derivative, the benefit that we get is the order of the differentiability requirement for phi reduces. In the previous case here phi had no derivative, but here phi had derivative of order 2. And similarly, here the phi had derivative of order of 4. So, accordingly we had to choose phi_u in such a way that it is differentiable at least two times. Here, we had to choose phi in such a way that it is differentiable at least 4 times. But here because we have distributed the derivatives, in this case – in the present case, the requirement of differentiability is just 2. So, now, phi w needs to be differentiable just at least two times. And similarly, phi_{ui} needs to be differentiable just at least one times. And also with this formulation, the matrix K becomes symmetric.

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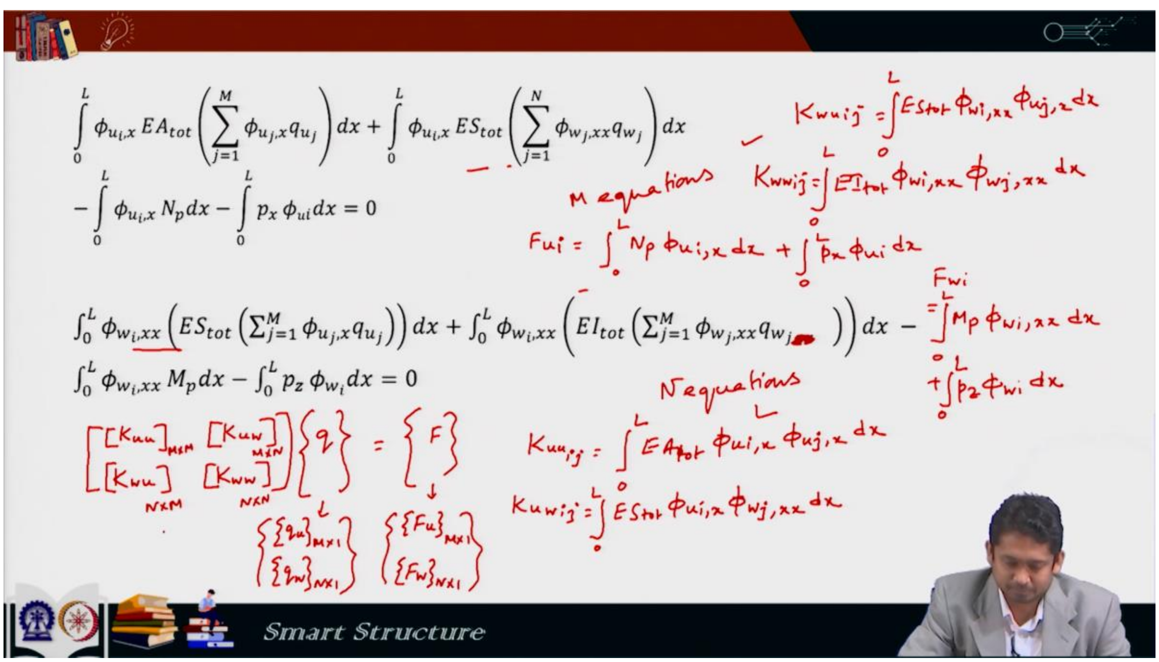

Now, we will look into another approach and we can call it is the Ritz method. It is known as Ritz method. Here, what we do is we look into the energy equation that we derived before or the virtual work equation and in that equation itself we put all the approximations and get the solution from there.

This is the virtual work equation. After doing the area integrals as we have seen, we get it in this form. This form is well known to us. Now in this form itself we will put our approximations. Again, we have the same approximation that we made before u_0 is equal to i and w is equal to this.

$$u_0 = \sum_{i=1}^{M} \phi_{u_i} q_{u_i}$$

$$w_0 = \sum_{i=1}^{N} \phi_{w_i} q_{w_i}$$

Now, if u_0 is this, we know that epsilon 0 is derivative of u_0 with respect to x. So, we can write this as this. So, epsilon 0 becomes our summation of i is equal to 1 to M, phi_{ui} comma x multiplied by q_{ui}.

$$\varepsilon_0 = \sum_{i=1}^{M} \phi_{u_{i,x}} q_{u_i}$$

Now, this can be written in the matrix and vector form as this. So, this row vector multiplied by this column vector gives me this. Similarly, kappa becomes j is equal to 1 to N, q w_i comma xx, multiplied by – we are using j here. So, let it be and j and again that can be written as this.

$$\kappa = \sum_{i=1}^{N} \phi_{w_{j,xx}} q_{w_j}$$

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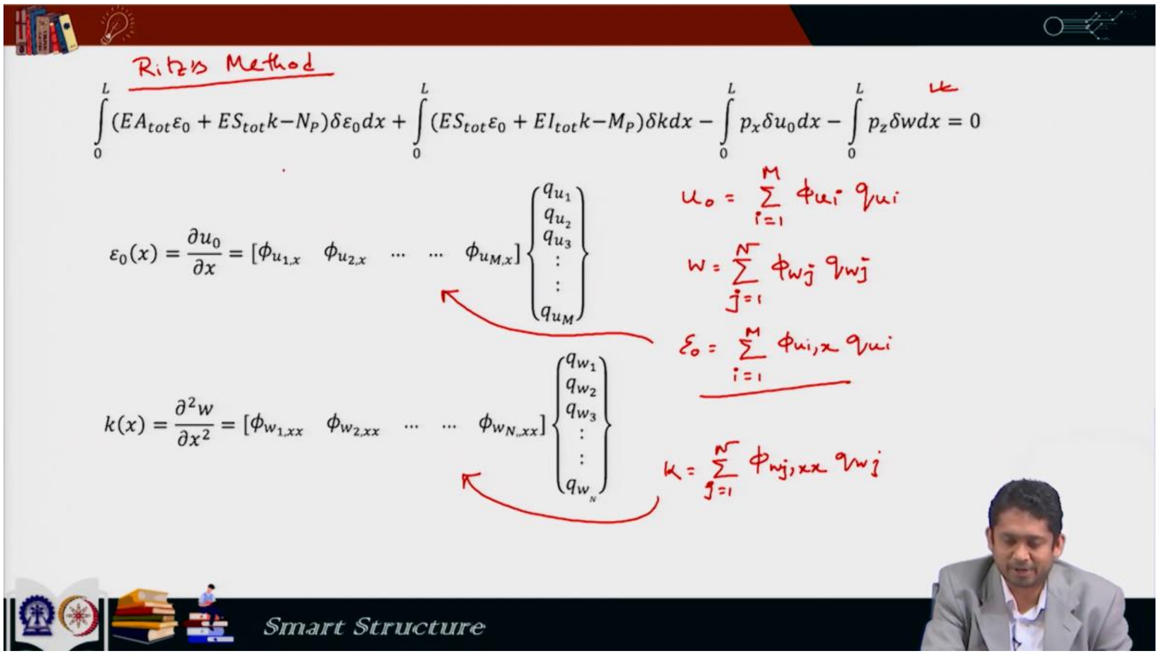

Then, this equation what we have, can be written in a more compact way using matrices as this. You can see it if we multiply this column with this row, if you multiply this column with this row and then finally, we multiply everything here, we get this and this and these are as it is. So, this entire expression is written in a more compact form as this expression. We have defined epsilon 0 and k. So, we can define a vector epsilon 0 and epsilon k and that can be written in this form.

So, if we take the approximations of epsilon 0 and k from just what we did here and we can combine them and write it in this form. And now let us give it a name. Let us call it a B matrix and which is of size 2 into M plus N. So, we have B matrix multiplied by this q vector. And this q matrix is the full q vector where the first M number of terms are q_u and rest of the N number of terms are q_w. We can see it here if I just multiply this column with this row, the first M number of terms which are q_u gets multiplied here and then rest of the terms which are q_w does not are multiplied with 0. So, on being summed up that gives me epsilon 0. And similarly, here also if I multiply this column with this row that gives me kappa.

So, epsilon 0 k vector is written as B matrix multiplied by the q vector. Similarly, let us define another column of u_0 w and then write it in this form. So, this 2 by M plus N matrix is called C matrix. So, we are using C multiplied by q and that is giving me epsilon u_0 w vector. So, what we have here is this which is written as this. And we have this vector has been written as this. To write this we have to do some transpose we will see that and we have u_0 v which is written as this. Let us give this matrix E A_{total}, E S_{total}, E S_{total}, E I_{total} a name maybe we can call it a D matrix, and D is 2 by 2.

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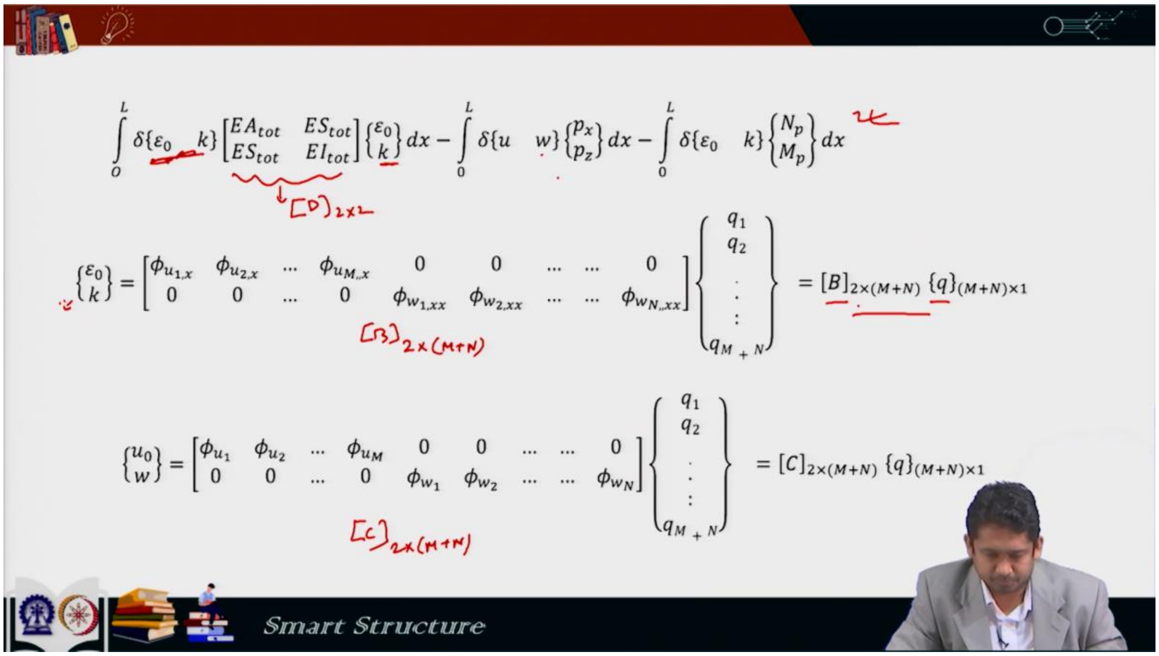

Now, we will rewrite this entire expression in a more compact form, not in a more compact form, but with the help of this B and C. We know that epsilon 0 and kappa when I write it as a row vector this will become q transpose multiplied by B transpose because epsilon 0 kappa when it is a column vector it is B into q. And if I take variation of this epsilon 0 and kappa, then it will become variation of q transpose multiplied by B transpose. Because B matrix consists of these derivatives of q's. So, it cannot be varied, whereas, this q's are unknown to me, it can be varied. So, variation of B is not possible, but variation of q is possible. So, the variation of epsilon 0 kappa, this row vector is just this.

$$\{\varepsilon_0 \quad \kappa\} = \{q\}^T [B]^T$$

$$\delta\{\varepsilon_0 \quad \kappa\} = \delta\{q\}^T [B]^T$$

Accordingly, we have u_0 and w and that is q transpose into C transpose. So, we have variation of u_0 w is variation of q transpose multiplied by C transpose.

$$\{u_0 \quad w\} = \{q\}^T [C]^T$$

$$\delta\{u_0 \quad w\} = \delta\{q\}^T [C]^T$$

Now with the help of this let me rewrite this expression. So, now we have 0 to L delta of q transpose multiplied by B transpose, and then we have the D matrix because if you go back it is followed by the D matrix and then we have epsilon 0 k as a column vector. So, D multiplied by epsilon 0 k which we know as B multiplied by q and dx it is followed by this term. So, we have minus 0 to L delta of q transpose multiplied by C transpose into P_x and

P_z, dx. And then we have minus 0 to L, the next term is this epsilon 0 kappa. So, that can be written as – and then we have the vector N_p and M_p, dx. And this is equal to 0.

$$\int_0^L \delta\{q\}^T[B]^T[D][B]\{q\}dx - \int_0^L \delta\{q\}^T[C]^T \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} dx - \int_0^L \delta\{q\}^T[B] \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} dx = 0$$

Here, we can see that everywhere we have delta q transpose sitting at the front and again delta q transpose means variation of q. So, the entire expression can be 0 only when we have rest of it equal to 0 because q is an arbitrary variation and they are variation of their components are independent.

So, if I get rid of this q, we have this equal to 0 to L multiplied by p_x p_z plus one transpose should be there. So, multiplied by B T, M.

$$\int_0^L [B]^T[D][B]\{q\}dx = \int_0^L [C]^T \begin{Bmatrix} p_x \\ p_z \end{Bmatrix} dx + \int_0^L [B] \begin{Bmatrix} N_p \\ M_p \end{Bmatrix} dx$$

And again, this comes to the same form that we derived before. So, this is M plus N multiplied by M plus N. This is M plus N multiplied by 1. This is M plus N multiplied by 1.

$$[K]_{(M+N)x(M+N)}\{q\}_{(M+N)x1} = \{F\}_{(M+N)x1}$$

And this K matrix is same as what we got in the previous approach where we use the Galerkin technique by shifted the derivatives. So, the matrix remains same here.

$$[K] = \begin{bmatrix} [K_{uu}] & [K_{uw}] \\ [K_{wu}] & [K_{ww}] \end{bmatrix}$$

And accordingly, we have F.

$$\{q\} = \begin{Bmatrix} \{F_u\} \\ \{F_w\} \end{Bmatrix}$$

So, K_{uu}, K_{uw}, K_{wu}, K_{ww}, F_u, F_w they are same as what we wrote for the previous solution approach where we had Galerkin technique where we shifted the derivatives. And again, we can solve it and get our q's and that completes the solution.

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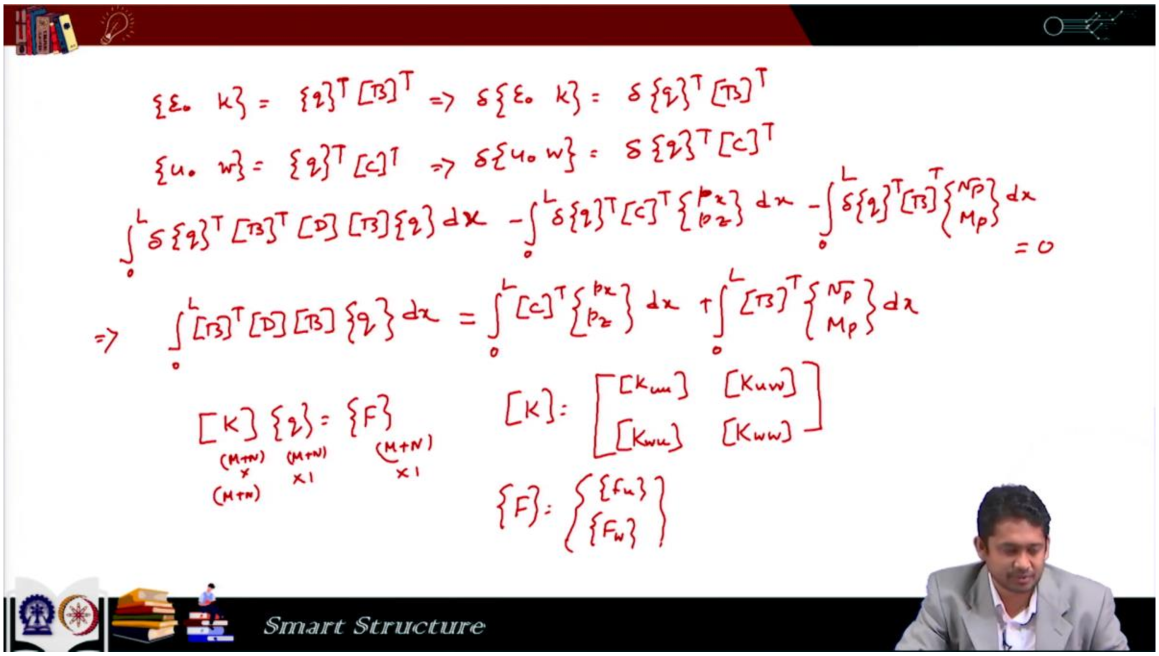

With this, I would end this lecture here.

In the next lecture, we will see some problems which we can solve using these approaches.

Thank you.