Smart Structures

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Week 04
Lecture No: 22
Static Analysis of beam for Induced Strain Actuation using Energy Principles (continued)
Part 06

In the last lecture, we started with the virtual work equation for this beam with piezo's and from those equations, we saw how we can get a governing equation that we are familiar with.

Now, we will see how by starting from the same virtual work equation, we can get a governing equation in a different form. So, again we look at equilibrium equation, but we will derive it in a different form. So, after we start with the virtual work equation, we obtained this form.

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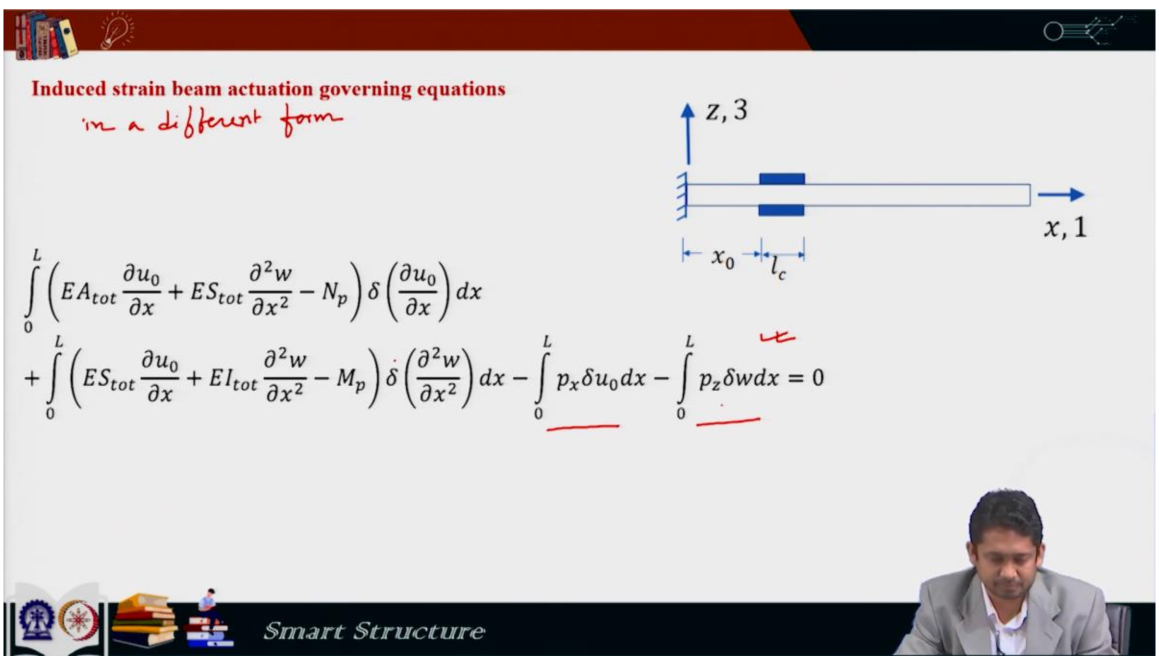

So, if I don't shift the derivatives here, if I don't do the integration by parts, if we written it as it was, we have it here and rest of it we know how to get. Now, what we will do is instead of integrating by parts here and here, we will do the integration by parts here and here, and that will give us a governing differential equation in a different form. So, if we look at the first term which was our integral E A total, if I integrate by parts, it gives me: E A total, del u_0 by del x multiplied by integral of this and integral of this, is nothing but delta of u_0 and it is evaluated at x equal to 0 and L and then we have minus of differentiation of this. So, del del x of E A total, del u_0 by del x, multiplied by integral of that. So, delta of u_0 0 to L dx. So, we can see it here, these terms. So, these terms come from this.

$$\int_0^L EA_{tot}\frac{\partial u_0}{\partial x}\delta\left(\frac{\partial u_0}{\partial x}\right)dx \;\Longrightarrow\; EA_{tot}\frac{\partial u_0}{\partial x}\delta u_o\Big|_0^L - \int_0^L \frac{\partial}{\partial x}\left(EA_{tot}\frac{\partial u_0}{\partial x}\right)\delta u_0 dx$$

Similarly, we can do the same thing for the next term where we have delta of del u_0 by del x multiplied by E S total, into del 2 w by del x 2. After we do the integration by parts, we get this term. And then, we had Np multiplied by delta of del u_0 by del x and that were integrated from 0 to L. So, if I do integration by parts here, it gives me Np multiplied by the integral of this, which is delta of u_0 0 to L minus del Np by del x into variation of u_0, and which we can see here. So, this had a negative sign before. So, that is why it was negative, it is positive, what we see here.

$$\int_0^L N_p\delta\left(\frac{\partial u_0}{\partial x}\right)dx \Longrightarrow N_p\delta u_o\Big|_0^L - \int_0^L \frac{\partial N_p}{\partial x}\delta u_0 dx$$

Then we had a term like E S total, multiplied by del u_0 by del x, multiplied by variation of the second order derivative of w and this was integrated from 0 to L. If I integrate this by part, that gives me E S total, del u_0 by del x variation of delta w by del x, which is evaluated at the limit 0 and L. And then, minus I would have derivative of this and integral of this. So, now, the other term that I get as minus, I am writing here, multiplied by variation of del w by del x. But again, if I integrate this by parts, this would give me minus del by del x, E S total, multiplied by variation of w and that gets evaluated at 0 to L. And then, plus this differentiated twice, E S total. So, here this term is missing, it is E S total, del u_0 by del x multiplied by delta w. And here, we have the same thing, but differentiated once more multiplied by delta w dx. So, that is how we get this term, this term and this term. Accordingly, we can find out rest of the terms by doing the integration by parts and we get this entire expression.

$$\int_0^L ES_{tot}\frac{\partial u_0}{\partial x}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)dx$$
$$\Longrightarrow\; ES_{tot}\frac{\partial u_0}{\partial x}\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L - \left[\frac{\partial}{\partial x}\left(ES_{tot}\frac{\partial u_0}{\partial x}\right)\delta w\right]\Big|_0^L$$
$$+\int_0^L \frac{\partial^2}{\partial x^2}\left(ES_{tot}\frac{\partial u_0}{\partial x}\right)\delta w dx$$

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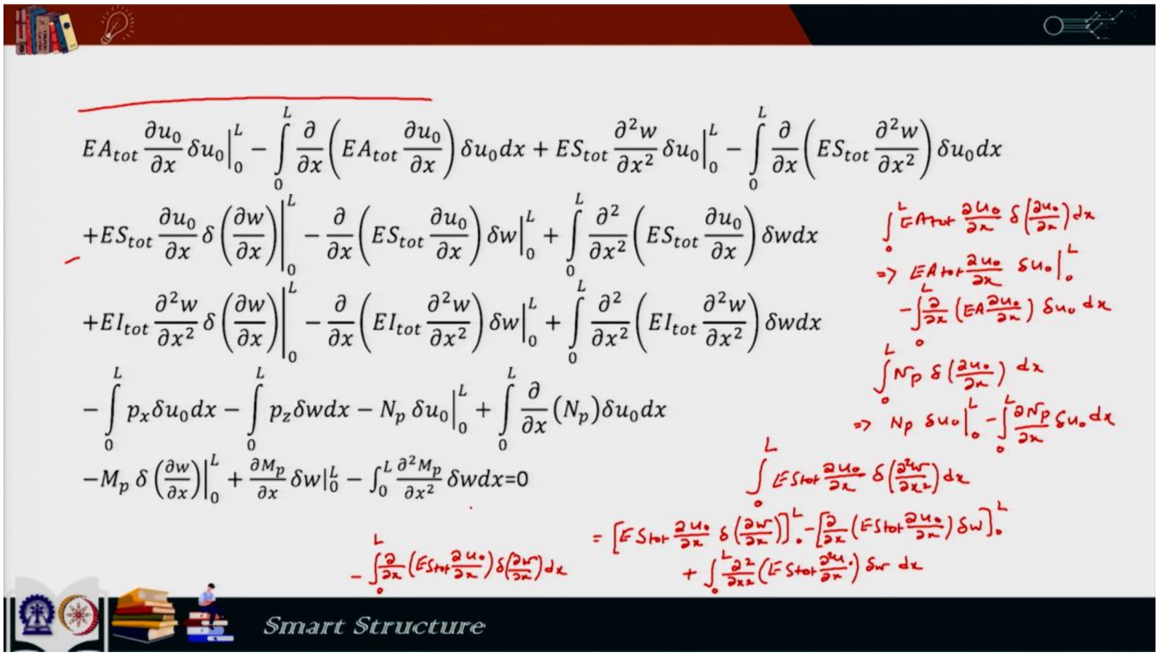

Now, if we look at this entire expression, we can see that the variations are in del u_0 w, del w by del x, and that is all. So, if we separate out the components, that are multiplied with delta of u_0, delta of w, and delta of del w by del x, we get this. In this expression, we have these two terms that are integrated, whereas, these expressions are: these three expressions are all defined at the boundaries. Now, if we look at these boundary terms, we can see that, this is nothing but our N, normal force N.

So, this thing inside the bracket is normal force N. Why? Because, we have seen so far, that E A total, plus E S total, multiplied E A total, del u_0 by del x, plus E S total, del 2 w by del x 2, minus Np, minus N, equal to 0, which means that this equal to N. So, this is equal to N here.

$$EA_{tot}\frac{\partial u_0}{\partial x} + ES_{tot}\frac{\partial^2 w}{\partial x^2} - N_p - N = 0$$

Similarly, using the same reason, this is my M. And this is nothing but del M by del x, which gives me shear force. Now again, we can make similar argument and we can say that these terms are 0 for our case because at x equal to 0, we have the axial displacement specified which is u_0, which means that delta of u_0 is 0. At the other end the normal force is 0.

Similarly, here we have the slope del w by del x specified, which means the variation of the slope is 0 at this end. And at x equal to L, we do not have the bending moment. So, the bending moment is 0 at x equal to L. And at x equal to 0, we have the displacement specified. So, del w is 0 here and at x equal to L, we do not have any shear force. So, that helps me get rid of all these terms and we are left with only this and this. And again, we know that delta u_0 and del w are independent and arbitrary variation, which tells me that these integrands must be individually 0.

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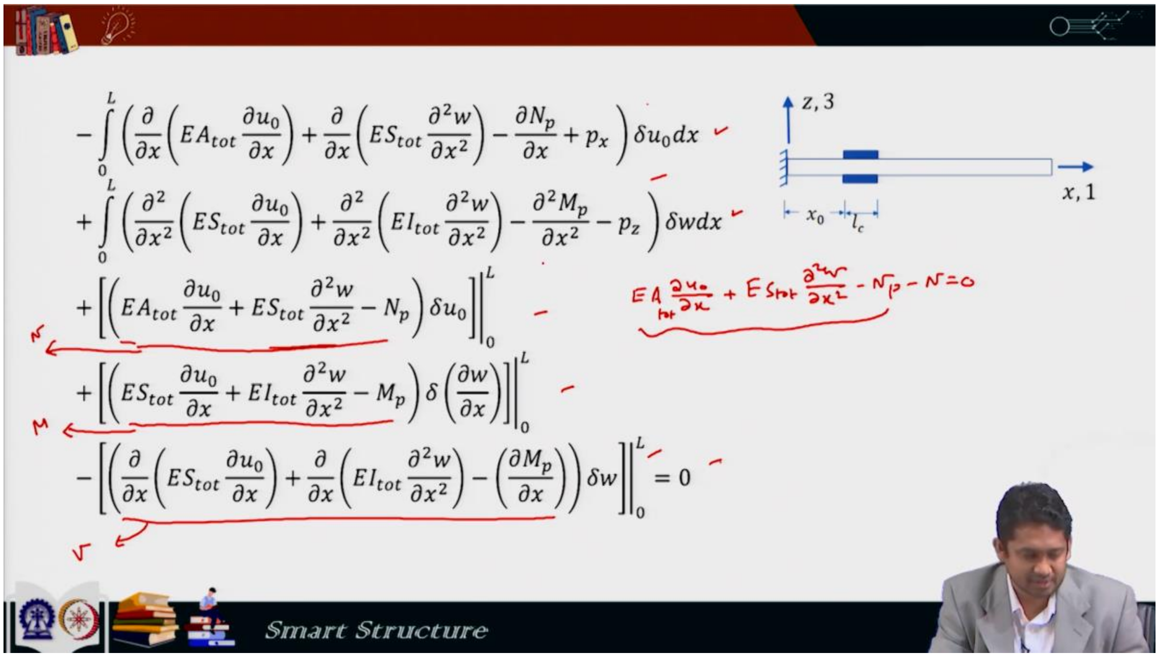

And from that same argument we can say that this is our one governing differential equation, and this is our other governing differential equation. So, we get the governing differential equations in a different form. Now, if I compare this form with our previous form, the difference is that while getting the previous form, we did integration by parts in these terms where Np and Mp were present, sorry, where Px and Pz were present whereas, here we did integration by parts in the other terms. And there, it was written in terms of variation of epsilon 0 and kappa. Here it is written in terms of u_0 and w. And from there, we get these two equations. Now, we will see using these two equations, how we can get the solution to the problem.

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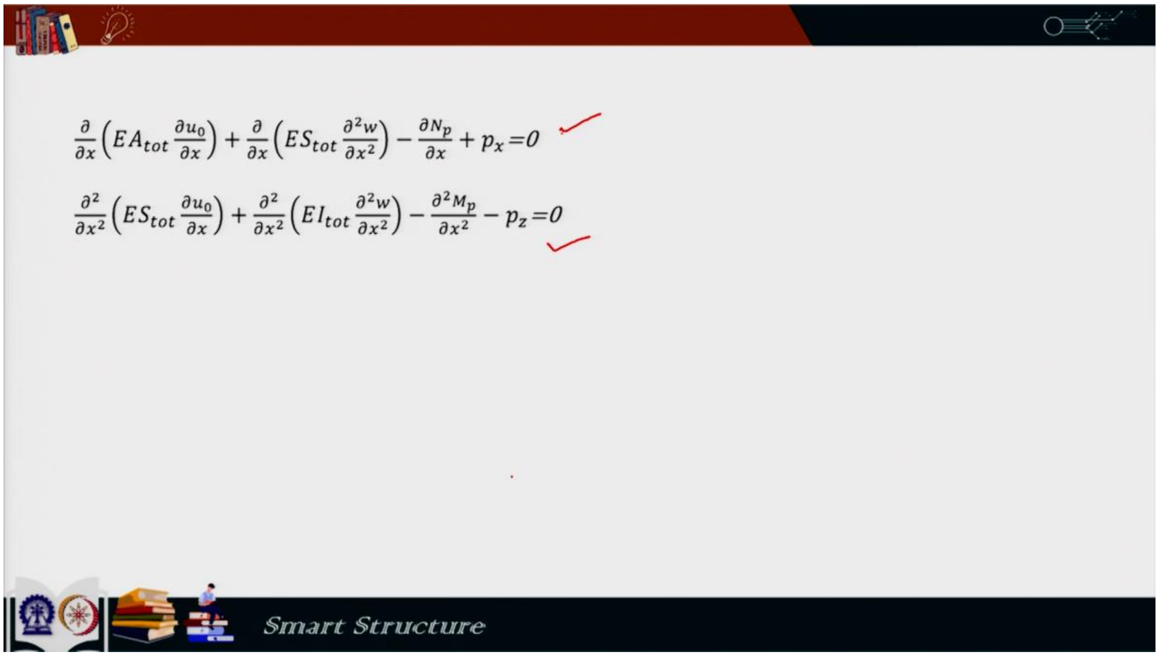

Now, these two equations are now written in a compact form as this. So, where we have this operator. So, u_0 comes here, and it gets operated here, w comes here and it gets operated here and we get the first equation. Similarly, u_0 comes here and w comes here and we get the second equation. We are going to solve it by using a technique named the Galerkin technique.

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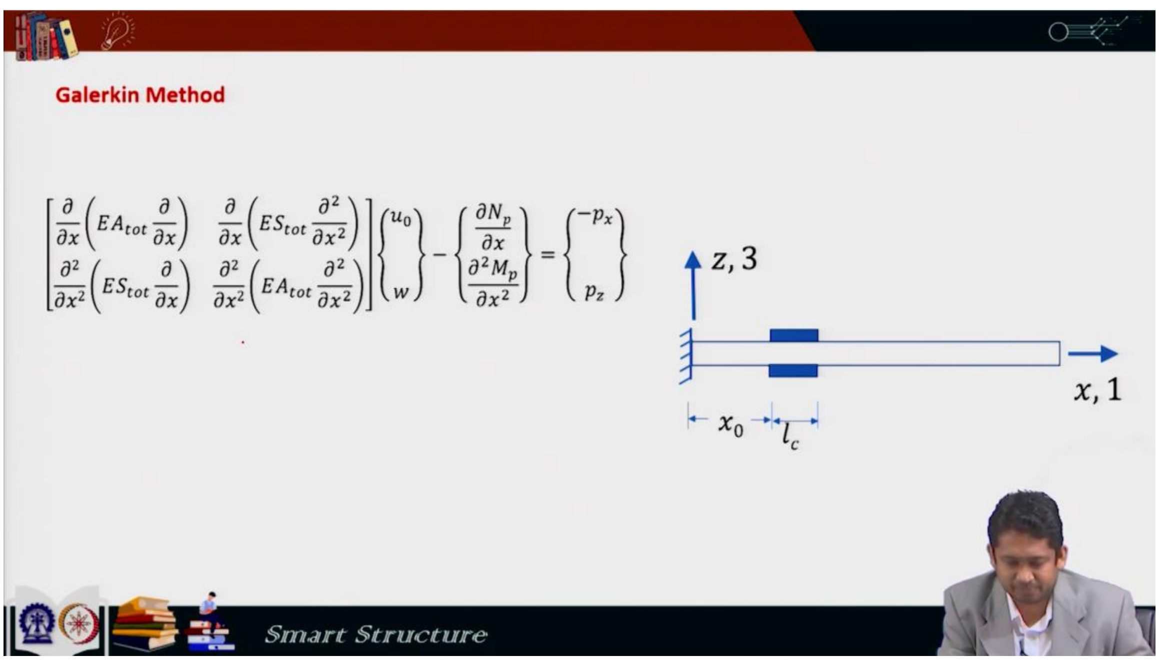

So, let us apply the technique. Again, let us assume that we have u_0 as this. So, u_0 is written as a combination of this product and within this product, we have phi and q. So, phi ui is chosen function, chosen function to approximate u_0 and similarly phi wi is chosen to approximate w. If we can choose in such a way that they satisfy both essential and natural boundary conditions, I mean the geometric and force boundary conditions, that is good, but at the bare minimum they should satisfy the essential boundary conditions. So, these phis are all known to us, because we are choosing them and they are being multiplied with unknown constants q ui and this q wj. Now, q ui, q wj are unknowns to be found out. Now, we have assumed that our u_0 has this m number of terms and w has this n number of terms.

So, the vector q u ranges from 1 to m and the size of the vector q w is n. Now, we can write a vector u which has u_0 and w. And then, we can write it in terms of this phi's and this q's as this. Here we have first m terms, which is populated with this q u's and then rest of the n terms are 0. And here, we have first m term 0 and then rest of the n terms populated with this q w's. So, this is of size 2 multiplied by m plus n. And this is of size m plus n multiplied by 1. Now, we can call this entire vector as one q vector with size m plus n. So, within this q vector the first m number of q's are q u's and rest of the n number of q's are q w's and it is multiplied with the same thing. So, in a compact way we can call this as phi matrix multiplied by q matrix. So, this is our phi matrix and this is our q vector.

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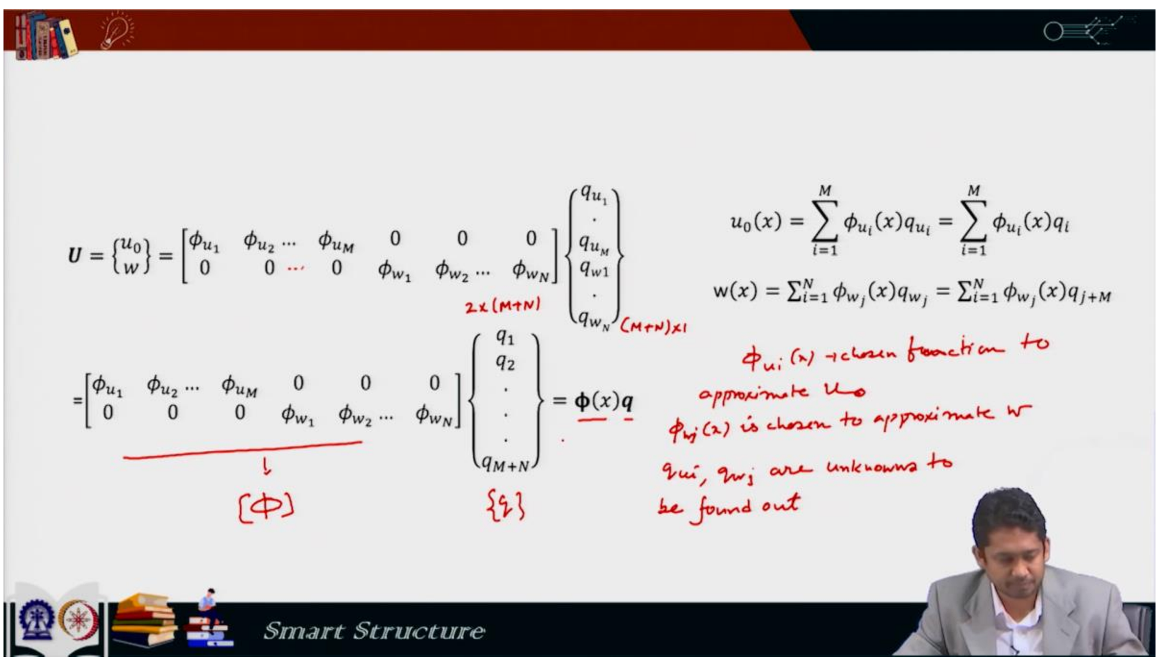

So, again the governing equation rewritten here. So, if we put the approximations of u_0 and w so, that becomes our phi multiplied by q. Now, because they are approximations, they may not be exact with u_0 and w. So, if I take the right-hand side to the left-hand side, and if we substitute u_0 w vector as q as phi q, then we can say that, this entire thing may not be exactly 0 because phi q may not be exactly equal to u_0 w. So, there is can be some error which is remaining and that error is this. So, the close we are to the real value of u_0 and w this is the error.

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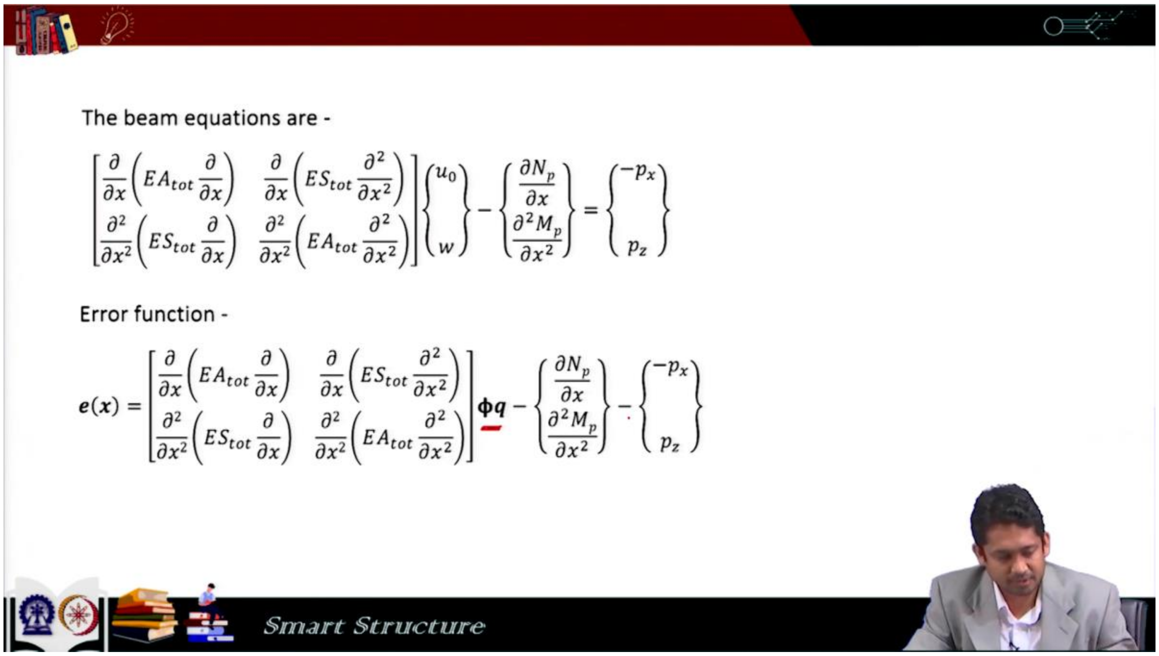

Then what we do is: we multiply this error function by this approximation phi's and then, so, if we go back to our phi vector. So, this is our phi 1, this is our phi 2, this is our phi 3 phi m and so on and we can go up to phi n. So, we may say that, we may like to call this as our phi 1 vector and we can go on. So, this is our phi m vector and we may call each of them as a vector and we can say that, this phi matrix consists of, this capital phi matrix consists of all this small phi vectors.

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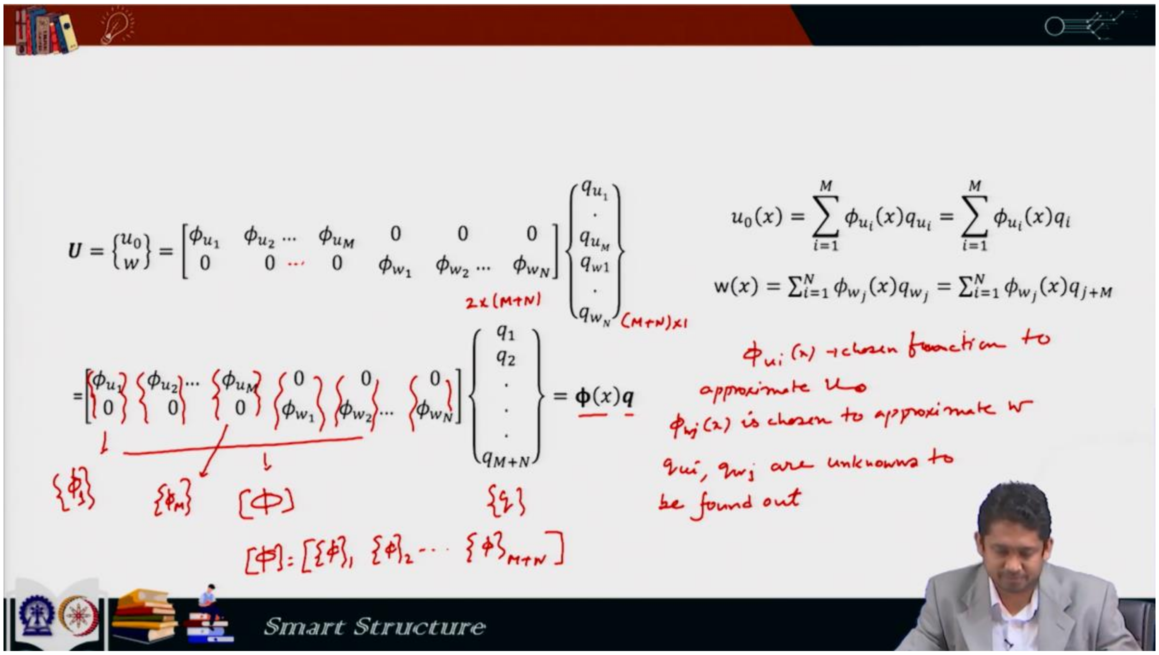

Now, in this error function, if I multiply the - what we are doing here, if I multiply the i th phi, the transpose of that with the error function and then integrate from 0 to L and equate that to 0 and we can do it for all these phi's. So, that gives me total m plus n number of equations. And those m plus n number of equations would have these q's as the unknowns and that will give me. So, it will give me m plus n number of equations with m plus n number of unknowns and that can be written in a matrix form as this. Now, this we call stiffness matrix and this we call force vector. In the stiffness matrix, the generalized i th component can be written as this. So, the so k ij is equal to phi i multiplied by this matrix multiplied by phi j. So, we know that it is 1 by 2 because the i th column of phi is 2 by 1, and similarly this is 2 by 1 and this is 2 by 2.

So, after this multiplication this gives me one value the ij th value of the Kth matrix. Let us assume that our phi j, let us assume that, we are taking from our phi matrix, we are taking this as i and this as j. So, let us assume that i is equal to 1and j is equal to 2. So, in this case, it would look like this: phi u1 0 multiplied by this vector del by del x, E A total del by del x, and then, we have this phi u1 and 0. So, when I multiply this column with this rho, I have del del x of E A total, del phi 1 del x, plus 0. And here, I have del 2 del x 2 of E S total, del phi 1 del x as 0. So, as we said, let us assume i is equal to 1 j is equal to 2. So, it is 2. So that gives me phi u1 0, and here, we have del del x of E A total, del phi ut by del x. And here, we have del 2 by del x 2 E A total, del phi u2 by del x. And of course, this has to be integrated from 0 to L. This also has to be integrated from 0 to L. Again, we multiply this rho with this vector and after we multiply, we are remaining with only one term. So, this term multiplied with by this term, that will give me just one value. This multiplied by 0 will give me 0. So, finally, we will get 1 by 1 term which is our k ij. So, in this particular case, which is k 12. So, accordingly we can just put any i here and any j here, and accordingly, we will get the k ij and this entire matrix will be populated.

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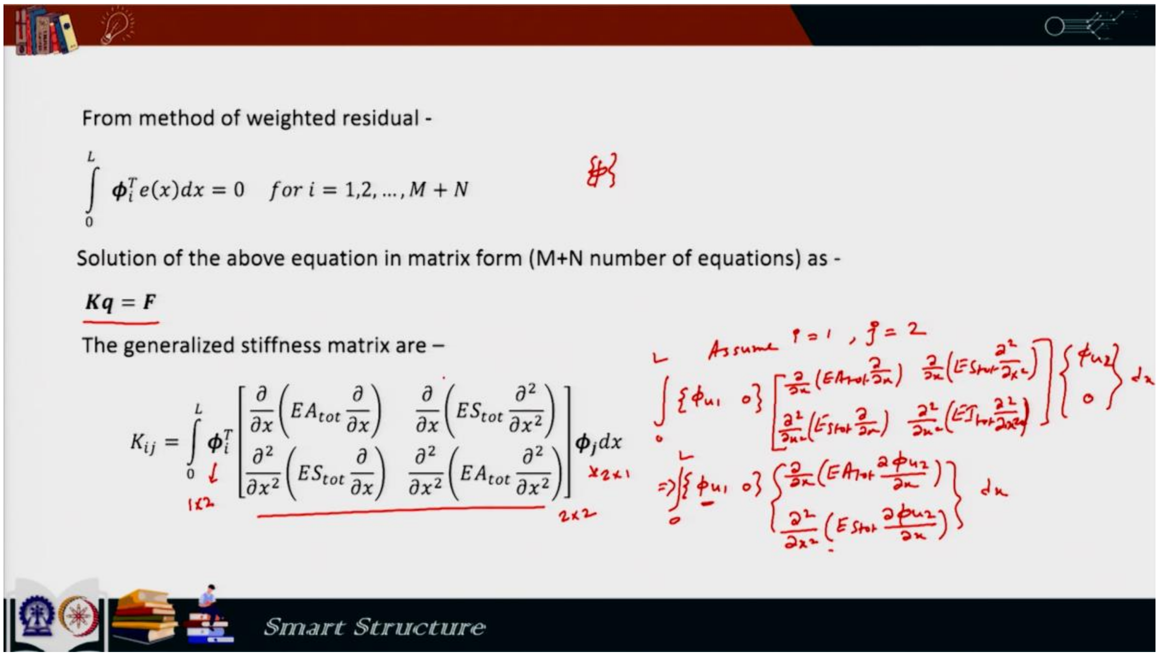

Similarly, when I find out phi I, this becomes the expression. So, phi i is multiplied with del Np by del x, and del 2 Mp by del x 2, and then we have phi i transpose. So, here also we have phi i transpose and here also we have phi i transpose multiplied with minus Px, Pz vector. Now, generally while evaluating this, we don't directly evaluate this, we shift the derivatives here also by integrating by parts and then we see that the boundary terms go 0 and the rest of the integral evaluated. So, instead of the differentiation being here, we get the differentiation here and we evaluate. So, later on when we do an example, we will see it in more details, how the derivative is shifted and how this term is evaluated in a better way.

So, with this we get the k matrix and the force vector and by solving we get our q. And once we, and once we get our q. So, we have q u, we have q w's. So, we get q ui, q wj from here, and then we put it back in our approximation that u_0, as a function of u is summation of phi ui into q ui, and w as a function of x is equal to phi wj into q wj. So, we just put it here, and we put it here, and that gives us the desired solution. So, we get our solution. So, that is one way to solve these differential equations.

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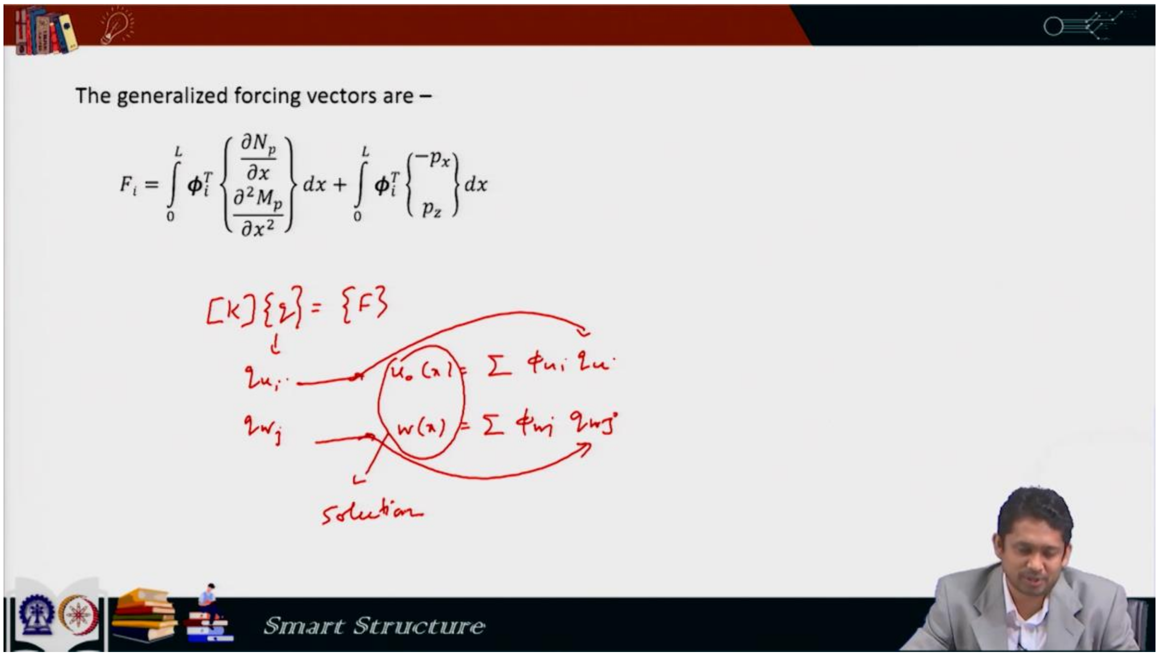

So, here we started with the differential equation and then, we multiplied them by each of these approximation functions and finally, this differential equation was converted to a set of algebraic equations. And then by solving the algebraic equations, we get our solution.

So, with this I would conclude it here. We will look into some other similar techniques in the next lecture.

Thank you.