Smart Structures
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Week - 04
Lecture No - 21

Static Analysis of beam for Induced Strain Actuation using Energy Principles

Welcome to the third lecture on Energy Based Techniques.

So, so far we just looked into the basic of the energy principles and we applied it to one simple structural problem. Now, we will see how we can apply this technique to problems where piezoelectric patches are fitted in a structure. So, again we have come to this known structure where I have a beam with piezoelectric patches. So, for this beam if I want to write down the virtual work form it looks like this sigma multiplied by delta epsilon minus dz multiplied by variation of the electric field minus. So, let us assume that it has forces distributed along x some external forces external load and that we call P x and then there are distributed forces along z direction also.

The distribution can be anything they need not be uniform and that let us call that p z. So, we have here p x multiplied by variation of u 0 d x and that goes from 0 to L 0 to L P z variation of w d x minus surface integral of charge multiplied by variation of the potential delta v and that is over the surface. So, to understand this in this problem we have the piezoelectric patches and they can be actuated with voltage and there can be external loads which are shown as p x and p z. And as we have seen that depending on the forces or and the depending on the type of actuation there can be extension as well as vertical displacement.

So, u 0 is pure axial displacement. So, axial displacement at the mid of the beam and w is vertical displacement. And we know that u at any x and z is u 0 minus z into del w by del x and here u is 0 is function of only x and w is a function of only x. So, here we are using again the Euler Bernoulli based formulation we will do Euler Bernoulli assumption we will take and we will formulate it accordingly. So, in this case the unknown stresses and strains are only s sigma x x and sigma and epsilon x x.

So, because that is the only unknown I mean the only nonzero component unknown as well as nonzero we do not consider other stress or strain components those are the only existing components. So, I did not put any subscript. So, just sigma and epsilon. So, this is sigma x x and this is epsilon x x and in the piezos there is electric field is in the third direction the z direction. So, E 3 that is E z because z is three directions and similarly d z is equal to d 3.

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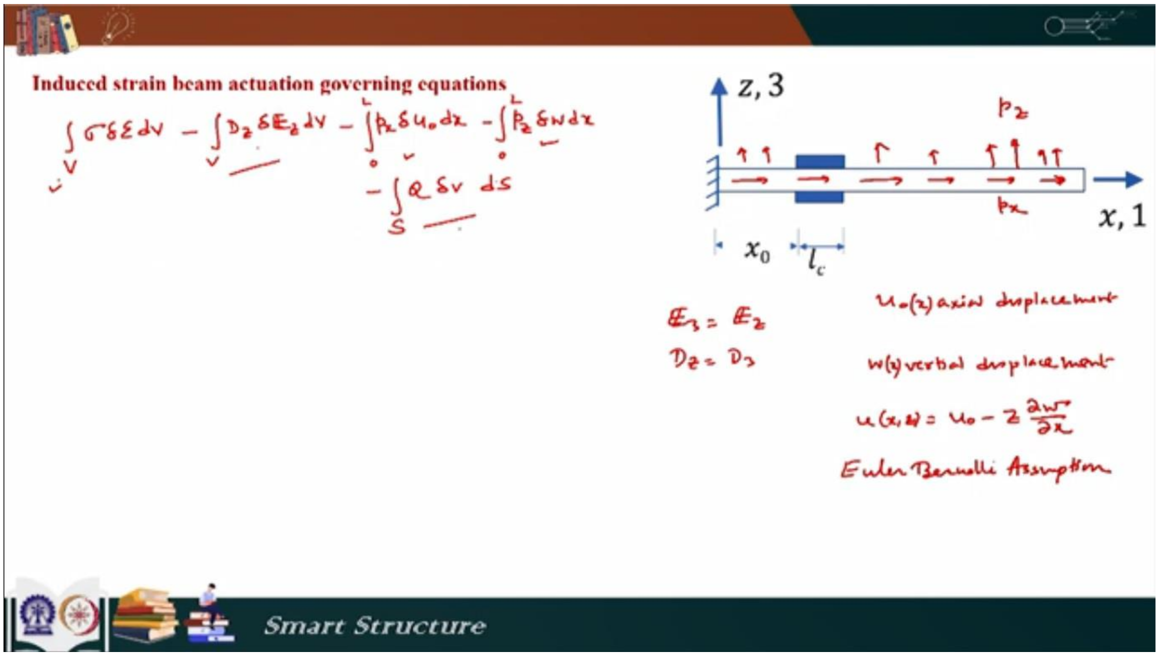

$$\int_V \sigma\delta\varepsilon dV - \int_V D_z\,\delta E_z dV - \int_0^L p_x\,\delta u_0 dx - \int_0^L p_z \delta w dx - \int_S Q\delta V dS$$

$$u(x,z) = u_0 - z\frac{\partial w}{\partial x}$$

$$E_3 = E_Z$$

$$D_z = D_3$$

So, we can see that apart from these components which come from the mechanical quantities we have components which come from the electrical quantities as well. So, if I just only consider this this and this that is that is quite familiar to us we have the internal virtual work and these two gives us the external virtual work, but we have introduced now these two quantities that come from the mechanical electrical part of it and we will see how we how it came to it why this is negative and so on. So, let us spend some time on deriving this. So, this is our virtual work equation. Now, before directly using it for our other formulations. I mean let us spend some time on deriving this.

So, we will derive it for a generic 3D case and then whatever we did can be a specific case of it. So, we can write sigma i j comma i multiplied by variation of u j dv integral over the volume plus D i i multiplied by the variation of the potential v equal to 0 as our equation to start with. Now, how from where did this equation come? Sigma i j comma i is just the conservation of linear momentum. So, sigma i j comma i equal to 0 is just the conservation of linear momentum. So, from the conservation of linear momentum that we saw before we have sigma i j comma i plus B j anybody force and that is equal to rho u j double dot.

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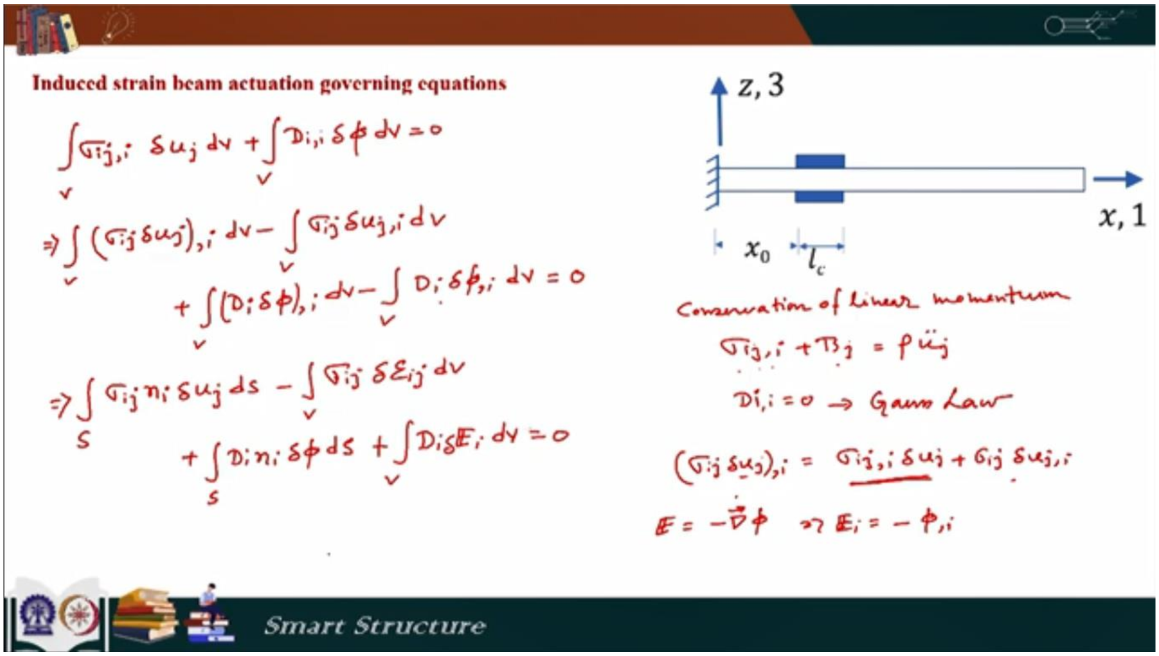

$$\int_V \sigma_{ij,i}\, \delta u_j dV + \int_V D_{i,i}\, \delta\phi dV = 0$$

$$\Rightarrow \int_V \left(\sigma_{ij}\delta u_j\right)_{,i} dV - \int_V \sigma_{ij}\, \delta u_{j,i} dV + \int_V (D_i\, \delta\phi)_{,i} - \int_V D_i \delta\phi_{,i} dV = 0$$

$$\Rightarrow \int_S \sigma_{ij}\, n_i \delta u_j dS - \int_V \sigma_{ij}\, \delta\varepsilon_{ij} dV + \int_S D_i n_i\, \delta\phi dS - \int_V D_i \delta E_i dV = 0$$

$$\sigma_{ij,i} + B_j = \rho \ddot{u}_j$$

$$D_{i,i} = 0$$

$$(\sigma_{ij}\delta u_j)_{,i} = \sigma_{ij,i}\delta u_j + \sigma_{ij}\delta u_{j,i}$$

$$E = -\vec{\nabla}\phi$$

$$\Rightarrow E_i = -\phi_{,i}$$

Now, here v j is the body force if we assume that there is no body force and we have anyway assume that there is no body force due to the interaction between the polarization and electric field and apart from that there is no other body force also. So, it is it is not there and it is a static problem. So, u j double dot is also 0. So, we are left with only sigma i j is equal to 0 after we set them equal to 0 and D i comma i equal to 0 comes from the Gauss law. So, the first equation which is the conservation of linear momentum sigma i j comma i equal to 0, if I multiply that with the variation of displacement and d i comma i if

I multiplied with the variation of the potential and then add this we get this and then we set it equal to 0.

Now, this entire this expression can be rewritten as sigma i j del variation of u j comma i d v minus sigma i j variation of u i j comma i d v. Similarly, this can also be rewritten as product of D i into variation of potential phi comma i minus D i into variation of potential comma i and then again d v equal to 0. Now, how could we write this? So, if I just look at this expression sigma i j del u j comma i. So, that is differentiation of this entire quantity inside the bracket with with respect to x i. So, it can be x y z depending on what is i i goes from x y x i goes along x y z.

So, if I differentiate it we first differentiate this keep this as same and then we keep this as same and differentiate the second quantity. So, it becomes del i j comma i sorry sigma i j comma i into variation of u j plus sigma i j multiplied by variation of u j comma i. So, this we had. So, this equal to this quantity minus this quantity that we have written here. In the similar way we can write this also as this.

Now, if you apply the Gauss divergence theory here it becomes a surface integral and we have sigma i j n i n i is the unit normal at the surface along direction i then d s and then we have sigma i j. So, u j comma i is epsilon j i and, but we know that the strain stress tensor is symmetric. So, that we can write as epsilon i j also. So, this becomes epsilon i j and then we have D i n i then variation of the potential phi. So, it is a surface integral and here we have D i.

Now, phi comma i is nothing, but negative of the electric field because we have seen that electric field as a vector is minus of the gradient of the potential phi. So, and which is equal to minus from here we can just write that E electric field along i direction is just minus of phi comma i. So, this quantity gives me D i multiplied by the electric field along i direction d v. So, this is how we derive it. Now, this is variation of the potential and we know that the voltage is just potential difference.

So, with all these this is the expression in the 3 dimensional system. So, for our system we can reduce this equation to this form and that explains why this quantity is negative where from these quantities came and so on. Now, just to explain one more thing D i n i is the charge. Now, in our problem when we have actuation we are actuating by these quantities this piezoelectric patches by voltage then the voltage here and the electric field here are known to us which means the variation of potential and the variation of the electric field are 0. So, we can discard this quantity and this quantity when we formulate it for the actuation problem.

So, we will have only this and this would suffice for us and also if sometimes the charge can be taken to be 0 in that case also this quantity is discarded. So, depending on the problem that we are solving we can keep some of the quantities we can discard some of the quantities and here this expression is written just in general if you want to write it for the 2 piezo separately then we have to write for this surface and this surface separately for this 2 piezo. So, as we said now we will discard these 2 quantities because we know the variation of potential and the electric field and then we will move on. So, now, let us write it once again. So, now, we will write the governing sorry the virtual work equation and then from there we will find out the from the virtual work equation we will find out the equilibrium equation that we are using last week.

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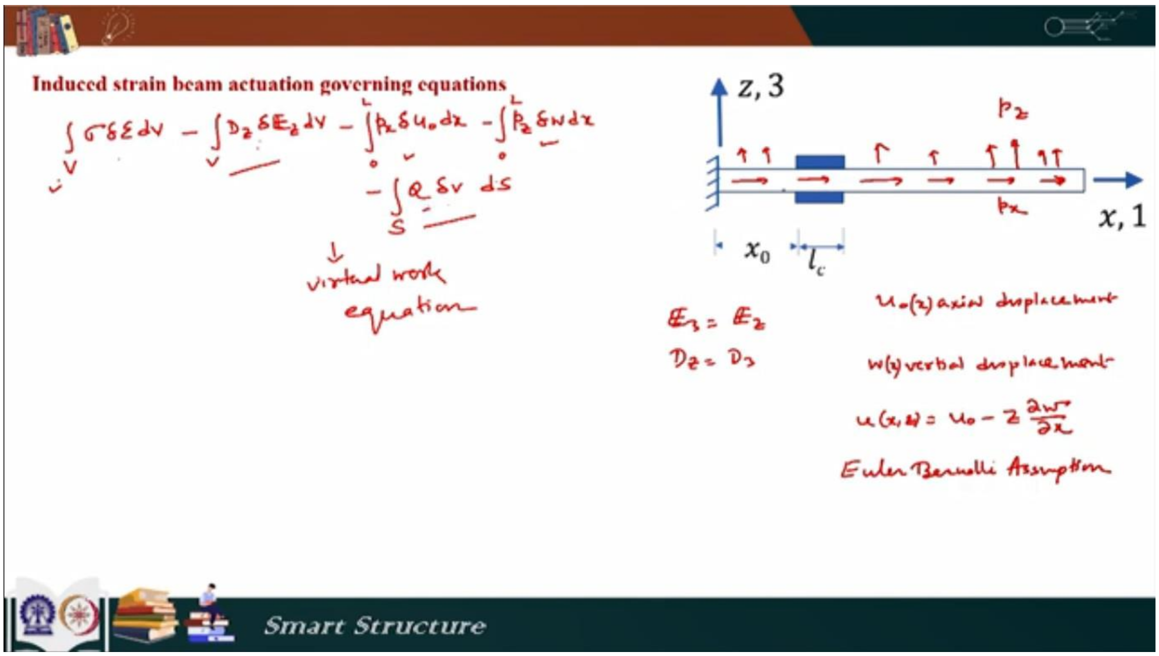

$$\int_V \sigma\delta\varepsilon dV - \int_V D_z\,\delta E_z dV - \int_o^L p_x\,\delta u_0 dx - \int_0^L p_z\,\delta w dx - \int_S Q\delta V dS$$

$$u(x,z) = u_0 - z\frac{\partial w}{\partial x}$$

So, the virtual work equation is sigma multiplied by variation of strain. So, here we have only one stress component that is sigma xx and one strain component that is epsilon xx. So, we are not using any subscript and then we have this as the external virtual work component and that is equal to 0. Now, we know that epsilon is epsilon 0 minus z kappa and sigma is E multiplied by epsilon 0 minus z kappa minus epsilon p. So, if we put it here we get E multiplied by epsilon 0 minus z kappa minus epsilon p minus epsilon 0 multiplied by z kappa and volume integral and then we have again px this remains as it is and that is equal to 0.

Now, what we do here is we separate out whatever is multiplied with delta epsilon 0 and whatever is multiplied with delta kappa. So, if we do that we have epsilon 0 minus z kappa minus epsilon p minus multiplied by delta epsilon 0 and then we have z into k we just take this z out and epsilon 0 multiplied by z kappa multiplied by epsilon p and that multiplied by delta kappa. And then we can break down this volume integral into a cross sectional integral and an integral over the line 0 to L. So, this is our area over the cross section A and this goes from 0 to L. Now, while treating this we will do integration by parts here.

So, let us take this first. So, we have delta of u 0 multiplied by integral of px dx and the entire thing evaluated at x equal to 0 and L plus we have differentiation of this. So, if I differentiate u 0 with respect to x that gives me epsilon 0. So, we have delta of epsilon 0 multiplied by px dx 0 to L and again we have we have to take care of this. So, we do the same thing delta w pz dx evaluated at 0 and L and then plus.

So, after this whatever the term is I am writing it here for a reason plus we get delta of del w by del x into pz dx evaluate integrated over 0 to L, but this can again be further separated. So, if we do that we get plus delta of del w by del x multiplied by pz dx dx and this entire thing evaluated at 0 to L and then minus integration 0 to L and then if I differentiate once more I get del 2 w by del x 2 which is nothing, but kappa. So, delta kappa multiplied by the pz integrated twice and then this entire thing integrated over 0 to L and that is equal to 0. Now, if we look here e epsilon 0 when integrated over A that gives me E A total. So, we have E A total multiplied by epsilon 0 minus z E on being integrated gives me E A total.

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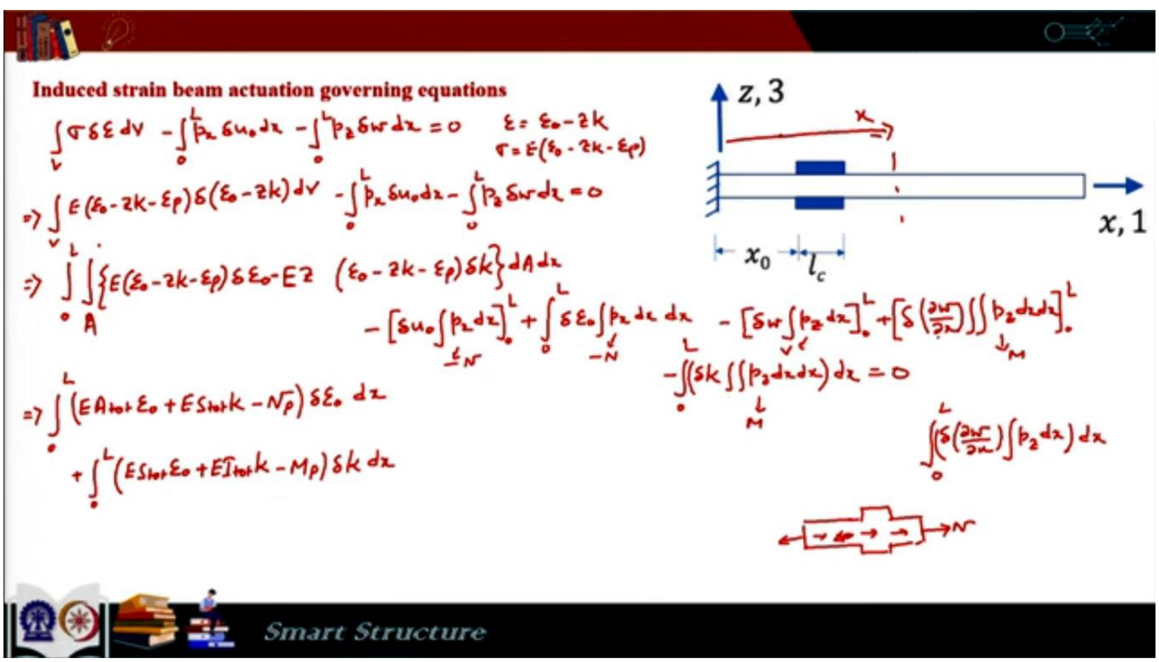

$$\int_V \sigma\delta\varepsilon dV - \int_0^L p_x\, \delta u_0 dx - \int_0^L p_z \delta w dx = 0$$

$$\varepsilon = \varepsilon_0 - z\kappa, \sigma = E(\varepsilon_0 - z\kappa - \varepsilon_p)$$

$$\int_V E(\varepsilon_0 - z\kappa - \varepsilon_p)\delta(\varepsilon_0 - z\kappa)dV - \int_0^L p_x\, \delta u_0 dz - \int_0^L p_z \delta w dx = 0$$

$$\int_0^L \int_A \{E(\varepsilon_0 - z\kappa - \varepsilon_p)\delta\varepsilon_0 - Ez(\varepsilon_0 - z\kappa - \varepsilon_p)\delta\kappa\} dAdx - \left[\delta u_0 \int p_x dx\right]_0^L + \int_0^L \delta\varepsilon_0 \int p_x dxdx - \left[\delta w \int_V p_z dx\right]_0^L + \left[\delta\left(\frac{\partial w}{\partial x}\right) \iint p_z dxdx\right]_0^L - \int_0^L \left(\delta\kappa \iint p_z dxdx\right) dx = 0$$

$$\int_0^L \left(EA_{tot}\varepsilon_0 + ES_{tot}\kappa - N_p\right) \delta\varepsilon_0 dx + \int_0^L \left(ES_{tot}\varepsilon_0 + EI_{tot}\kappa - M_p\right)\delta\kappa dx - \int_0^L N\delta\varepsilon_0 dx - \int_0^L M\delta\kappa dx = 0$$

So, we have E S total kappa and e epsilon p on being integrated gives me N p. So, this multiplied by delta epsilon 0 dx. Now when we treat this we have minus z into epsilon 0 integrate over dA. So, that will give me E S total into epsilon 0 minus z into minus z that is plus z and so, z square z square E integrated over integrated over A the area will give me the E I total. So, here I should have one more E also.

So, that is E the elastic modulus is missing here. So, it will give me E I total kappa and then and the kappa sign should not be there. Yeah instead of kappa you should see it over here. So, now, so that gives me E I total E z square kappa. So, E I total kappa and then I have minus M p that we get from these two terms and epsilon p and that multiplied by delta kappa dx.

Now if we look at these terms closely. So, px on being integrated it gives me minus of N. So, we can quickly check it here. So, suppose if I take a section somewhere somewhere here and then if we look at the structure from x equal to 0 to that. So, suppose it is some x and I take a section there and then if we look at the structure from x equal to 0 to that part and suppose there are reactions depending on the force there can be reactions some force I am just doing some showing some axial forces whatever it is.

So, please do not understand it is just to show any force these directions do not matter. So, if I sum up these forces and that force is equal to negative of this force N for equilibrium. So, integration of px gives me minus of N. So, this is minus of N and similarly integration of pz gives me the shear force V. So, this is V and again pz when it is integrated twice which means when shear force is integrated once that gives me bending moment M and this is bending moment M.

Now, if we look at this boundary term this term is 0 because at x equal to 0 I have u specified it is a clamped end. So, delta of u 0 is 0 at x equal to L I have no axial force at x equal to L I have no N here. So, it is 0. So, this term finishes. So, this goes to 0 this term again the boundary term at x equal to 0 W I know that it is specified which is 0 for our case.

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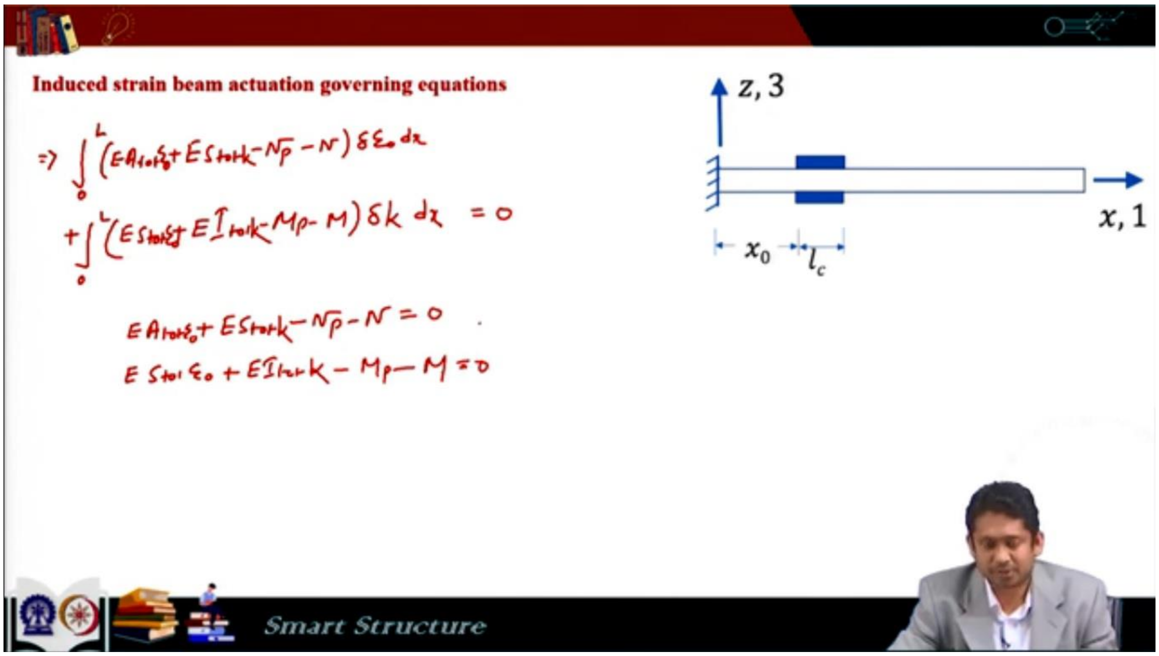

$$\Rightarrow \int_0^L \left(EA_{tot}\varepsilon_0 + ES_{tot}\kappa - N_p - N\right)\delta\varepsilon_0 dx + \int_0^L \left(ES_{tot}\varepsilon_0 + EI_{tot}\kappa - M_p - M\right)\delta\kappa dx = 0$$

$$EA_{tot}\varepsilon_0 + ES_{tot}\kappa - N_p - N = 0$$

$$ES_{tot}\varepsilon_0 + EI_{tot}\kappa - M_p - M = 0$$

So, delta W is 0 and at the tip there is no shear force. So, it is 0. Similarly again at the clamped end the slope is specified. So, delta of this delta W by del x is 0 and at x equal to L the no bending moment.

So, M is 0. So, this this terms are all going to 0. So, I have this and this remaining. So, that gives me minus of 0 to L N del epsilon 0 d dx minus 0 to L M del kappa dx equal to 0. So, plus Now, we know that this del epsilon 0 and delta kappa they are independent variations. So, they can only be 0 when this is equal to 0 and this is equal to 0.

So, that gives me again these two equations E A total plus E S total minus N P minus N and E S total epsilon 0 kappa. So, we have epsilon 0s here. So, E S total epsilon 0 plus E I total kappa minus M P minus M equal to 0. So, this is the equation that we dealt with last week. So, instead of balancing the forces we got it starting from the virtual work equation.

Now, with that I would like to conclude this video here.

Thank you.