Smart Structures
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Week - 04
Lecture No - 20

Introduction to Energy Principles for Structural Analysis (continued)

Welcome to the second lecture on Energy Based Principles.

Now, today we will look into a simple problem and we will try to understand the interrelation between the governing differential equation and the virtual work equation and the energy equation. And then from there we will gradually move on to the smart structure problems where we have structure with piezoelectric materials. So, before that we will solve a simple problem where there is no piezoelectric material. Now, let us consider a very simple structure like a bar and assume that the bar is under a distributed load P, distributed axial load P and this is our x axis and this we can call it z axis. And the cross-sectional properties of the bar the cross-sectional area is A, the material property Young's modulus is E.

$$\frac{\partial\left(EA\frac{\partial u}{\partial x}\right)}{\partial x}+p=0$$

$$\int_0^L\left(v\left(\frac{\partial\left(EA\frac{\partial u}{\partial x}\right)}{\partial x}\right)+pv\right)dx=0$$

$$vEA\frac{\partial u}{\partial x}\bigg|_0^L-\int_0^L\frac{\partial v}{\partial x}EA\frac{\partial u}{\partial x}dx+\int_0^L pvdx=0$$

$$\int_0^L EA\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx-\int_0^L pvdx=0$$

$$\int_0^L EA\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)dx-\int_0^L p\delta udx=0$$

$$\frac{\partial u}{\partial x}=\varepsilon_{xx}$$

$$EA\frac{\partial u}{\partial x}=\sigma_{xx}$$

Now, for this bar if I take a small portion of the bar and then if we look at the equilibrium of that small portion the governing differential equation comes to be this. So, where u is the displacement along x direction. So, it is a bar other effects all other effects are ignored. So, it is a one-dimensional problem and with just one unknown variable u.

Now, from here we will derive the virtual work equation and the energy equation. So, let us multiply both side of this equation by a variable v. So, let us consider another variable v and multiply both side of it and the new equation comes as and integrate it over the entire domain L. So, where L is the length of the bar. So, our domain is x equal to 0 to x equal to L within the domain we have integrated what we got after multiplying v with this equation.

Now, v is called a test function and it has to be consistent with the essential boundary conditions. So, here the essential boundary condition is at x equal to 0 my u is 0. So, v is also 0 at x equal to 0 and other than that it can be anything and it is also arbitrary it is arbitrary, but it satisfies it does not violate the essential boundary condition other than that it has been arbitrary. So, it has only one constraint here other than that v can be anything. So, now if I take this v multiplied by this and do integration by parts it gives me.

So, v multiplied by the integral of this integral of this quantity is nothing, but E A del u by del x evaluated at x equal to 0 and x equal to L minus the derivative of this. So, we have del v by del x E A multiplied by del u by del x and then we have p v. So, in p v none of them has any derivative. So, we do not do any integration by parts here. Now, if we look at this quantity for this problem, they are 0 because if I try to evaluate this at x equal to 0 at x equal to 0 we know that v is equal to 0 because that is where that is the point where we have the essential boundary condition.

So, where they are v is 0. So, v is 0 at x equal to 0 at x equal to L if I look at this product now this quantity E A del u by del x is my normal force N because del u by del x is my normal strain epsilon x x. So, E del u by del x is normal stress and when I multiply that normal stress with the cross-sectional area that gives me the normal force. So, if I look at this structure at x equal to L there is no force applied here. So, at x equal to L this quantity is 0 which means this entire thing is 0.

So, I am left with this. Now, we defined v as something which is arbitrary and it is consistent with the essential boundary conditions which means that delta of u which is variation of u can also be a v can also be a candidate for v. So, delta of u can also be taken as v because delta of u also satisfies the same condition it does not it remains consistent with the essential boundary condition and other than that it is arbitrary. So, if I replace v by delta of u then we get this. So, here we replacing v by delta u.

Now, if we closely look into this equation this is nothing, but the virtual work equation because this is our this, E del u by del x is our stress and this is strain and this is the external virtual work. So, this part is the internal virtual work this part is the external virtual work. So, this is internal virtual work and this is external virtual work. So, let us see this. So, here we have the forces distributed all over the domain I mean this force P is distributed all over the domain.

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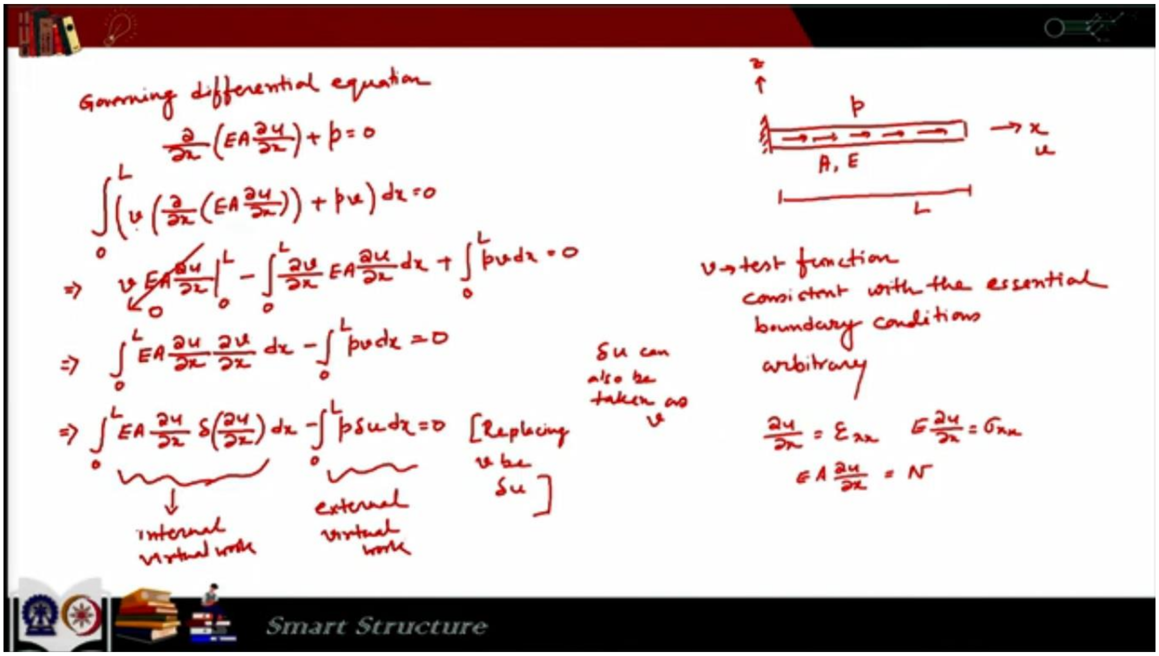

So, the external virtual work is our P multiplied by delta u and since it is distributed to sum it we have to integrate it over the domain and for the external internal sorry the internal virtual work we have sigma xx that is the only stress component that we have multiplied by delta of epsilon xx and if I integrate it over the volume we get this and the principle says that this equal to this which means if I subtract the external from the internal it is 0. Now, this volume integral can be broken into an integral over the domain x equal to 0 to L and the cross-sectional area A and then sigma xx we can write it as E into del u by del x and this as we know it is del u by del x and then it is an area integral and a integral over x equal to 0 to L. Now, within this integrand this del u by del x does not vary over the cross section E also does not vary over the cross section it I mean if you want we can vary it we can consider different values of E along the cross section that is also possible and then we have if we integrate it over the cross sectional area we get E A del u by del x multiplied by the variation of del u by del x dx minus P into del u dx. So, this is the equation that we got before. So, we can say that from the starting from the governing differential equation and then multiplying the variation of del w we get the virtual work equation.

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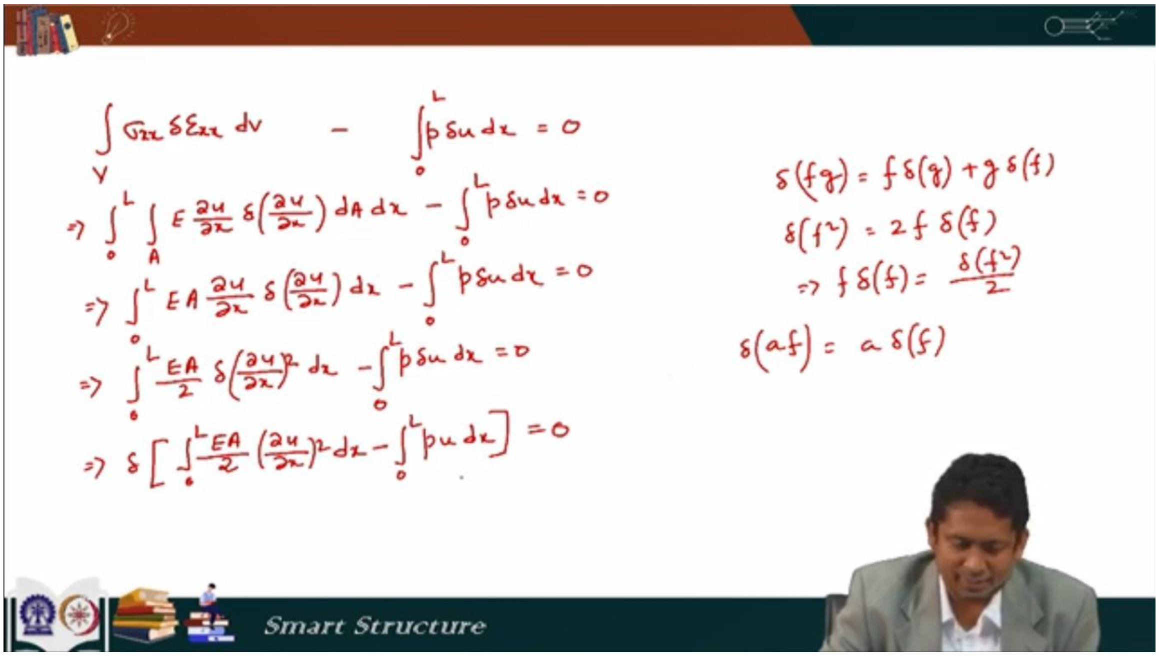

$$\int_Y \sigma_{xx}\, \delta\varepsilon_{xx} dv - \int_0^L p\delta u dx = 0$$

$$\int_0^L \int_A E \frac{\partial u}{\partial x} \delta\left(\frac{\partial u}{\partial x}\right) dA dx - \int_0^L p\delta u dx = 0$$

$$\int_0^L EA \frac{\partial u}{\partial x} \delta\left(\frac{\partial u}{\partial x}\right) dx - \int_0^L p\delta u dx = 0$$

$$\int_0^L \frac{EA}{2} \delta\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L p\delta u dx = 0$$

$$\delta\left[\int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L pudx\right] = 0$$

$$\delta(\pi) = 0$$

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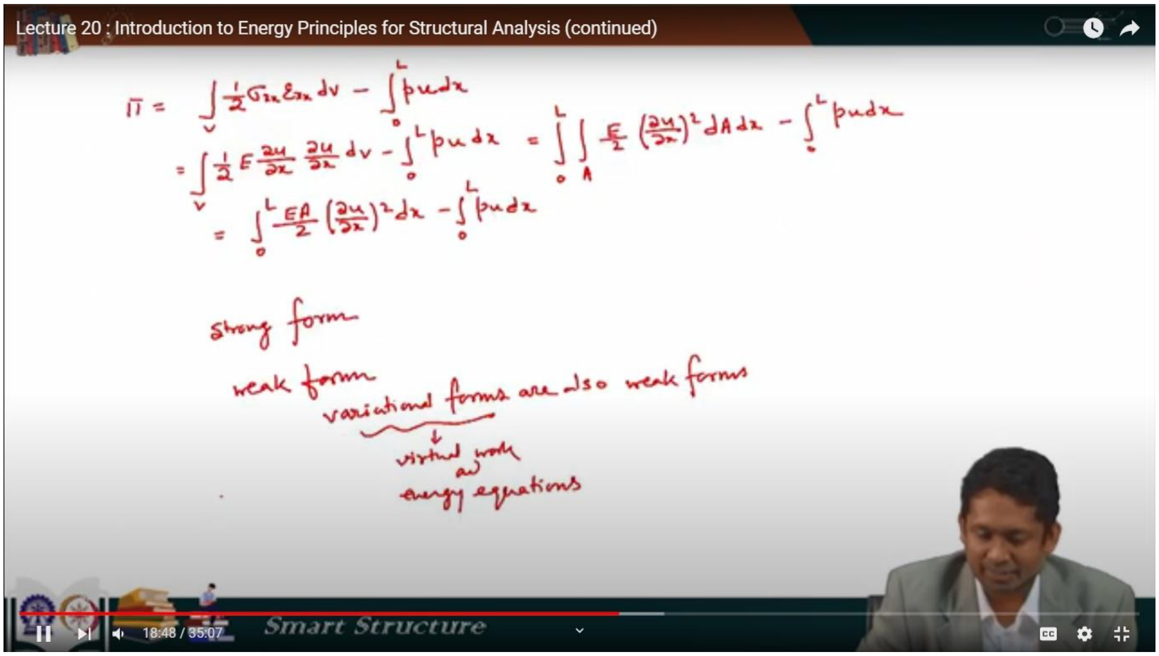

$$\pi = \int_v \frac{1}{2}\sigma_{xx}\varepsilon_{xx}dv - \int_0^L pudx$$

$$= \int_v \frac{1}{2}E\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}dv - \int_0^L pudx = \int_0^L \int_A \frac{E}{2}\left(\frac{\partial u}{\partial x}\right)^2 dAdx - \int_0^L pudx$$

$$\int_0^L \frac{E}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L pudx$$

Now, we can take it further and we have seen that if I have two functions f and g then the variation can be written as this. So, when we have these two f and g same we can write that as delta f or we can say that f delta f is equal to delta of square by 2. If we look here, then if we consider del u by del x as f here then we can apply the same formula and we can write this as E A by 2 multiplied by variation of del u by del x square dx and then we have P del u dx. Now, if I have f a variable and if I multiply with some constant an and then if I take its variation that is also equal to a multiplied by the variation of f. So, with that we can take this delta out of it we can take this delta out of it and finally, we can take delta out of the entire expression and then we can write this as equal to 0 and we will see that this is our total expression for the total potential energy.

This expression and it can be shown that this expression is this thing under the variational operator is our potential energy. So, if we if we want to write our expression for the potential energy it is pi as we saw integral over the volume it is half multiplied by sigma xx multiplied by epsilon xx dv and then we have the force P multiplied by u and integrated over 0 to L. Now, we know what is sigma? Sigma is E del u del x and epsilon is again del u by del x V minus 0 to L P u dx. Now, this integral can be broken down to area integral and volume integral. So, we have E by 2 del u by del x whole square and then this and the potential due to applied force.

Now again in this expression if I integrate this integrand over the area this gives me E a by 2 del u by del x square dx minus 0 to L P u dx. So, just to note this is x naught V. Now, if I just look back at the previous expression. So, this is what we have. So, this is nothing, but delta of pi equal to 0.

So, if we look back, we had the differential equation and from the differential with the differential equation we multiplied something called test function V and we got this and then we saw that delta u which is variation of u can also be a candidate for V and by doing that we got this equation which is our virtual work equation and then. So, this is our virtual work equation and then we came here and that is energy equation. Now, the differential equation it is said to be in weak form ah strong form this is called strong form of the problem and then what we get here is our weak form and this virtual work equation and the energy equations are also weak forms and because here we are using the variation of u as our test function, they are called variational forms also. So, we have strong form of the equation and then we have weak form and then we have variational forms are also weak forms and variational forms means they are virtual work and energy equations. Now, if we just compare these different forms, we can see here that the highest order derivative appearing here is 2 whereas, in the weak forms the highest order derivatives are always 1 because we have shifted the derivatives and this strong form satisfies at every point in the domain.

So, our domain ranges from x equal to 0 to L. So, at every point this strong form satisfies whereas, these weak forms satisfy in an integral sense. So, our integral from 0 to L that satisfies the equation here. So, whereas, this satisfies as each and every point and also this this material and geometric parameters E as they were inside a differentiation in the strong form in the weak form they come out of the derivative as we can see here, they are not differentiated anymore and these leads to various advantages we will see it as we go on. Now, we will solve a small problem using this principle.

So, let us again our domain is from x equal to 0 to x equal to L and within this domain we have to find out the variation of u that is the problem. So, we could have solved it by just solving the strong form, but we would not do it instead of that we will put we will solve it using one of these three different weak forms. So, let us assume to do this let us assume the solution u x is equal to something. So, let us assume it to be a polynomial maybe a 0 plus or let us use q as our that will be that way, we will be consistent with our later formulations. So, q 0 plus q 1 x plus q 2 x square let us assume that solution to be like this.

Now, our assumption has to be such that it satisfies the essential boundary condition. In our case the essential boundary condition. So, essential boundary condition says that u is equal to 0 at x equal to 0 because u is clamped fixed at x equal to 0. So, this ah this assumption should satisfy that. So, if we satisfy that satisfying the condition u 0 equal to 0.

So, if you put x equal to 0 and equate this expression to 0 we get q 0 equal to 0 and then we get our assumption as q 1 multiplied by x plus q 2 multiplied by x square. Now, in this entire expression first of all u is not known to us this is what we are trying to find out x is known to us x means just a straight line here x square is known to us this this is something that we are defining, but this quantities q 1 and q 2 which are being multiplied with this x s this x or x square the parts of the polynomial they are not known to us. So, if I can find out this q 1 and q 2 our solution is done. So, u is not known to be q 1 is not known to be q 2 q 2 is not known to me. So, if I take a variation of u.

So, because u is not known to me it is not something fixed, I can always take a variation of that if I take a variation of this then in this in between these two terms x is cannot be varied because x is already fixed q can be varied. So, we have x delta q 1 plus x square delta q 2. Now, we have u 1 and u 2 in terms of q 1 q 2 sorry we have u and delta u in terms of q 1 q 2 and delta q 1 and delta q 2. Now, let us put it in the virtual work equation. So, our so putting it in the virtual work equation putting the approximations of u x and delta u x in the virtual work equation we get.

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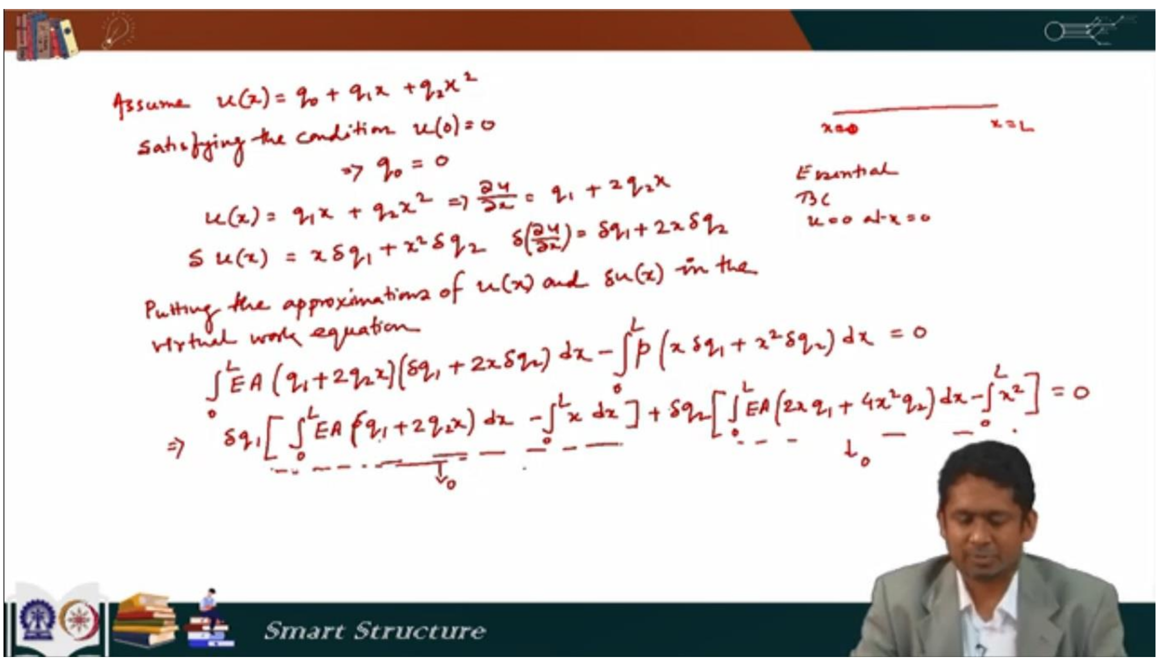

$$u(x) = q_0 + q_1 x + q_2 x^2$$

$$u(0) = 0$$

$$q_0 = 0$$

$$u(x) = q_1 x + q_2 x^2$$

$$\frac{\partial u}{\partial x} = q_1 + 2q_2 x$$

$$\delta u(x) = x\delta q_1 + x^2 \delta q_2$$

$$\delta\left(\frac{\partial u}{\partial x}\right) = \delta q_1 + 2x\delta q_2$$

$$\int_0^L EA(q_1 + 2q_2 x)(\delta q_1 + 2x\delta q_2)dx - \int_0^L p(x\delta q_1 + x^2\delta q_2)dx = 0$$

$$\delta q_1\left[\int_0^L EA(q_1 + 2q_2 x)dx - \int_0^L xdx\right] + \delta q_2\left[\int_0^L EA(2xq_1 + 4x^2 q_2 dx - \int_0^L x^2\right] = 0$$

So, we have del u by del x. So, if I just differentiate this, we will get this. So, here if I differentiate it del u by del x is my q 1 plus 2 q 2 x. Similarly here we get delta of del u by del x please remember that if I differentiate delta u with x that is same as taking variation of del u by del x we saw it before and that gives us del q 1 plus 2 x delta q 2 and then if you put it here we get q 1 plus 2 q 2 x multiplied by delta q 1 plus 2 x delta q 2 d x and then we have 0 to L p x delta q 1 plus x square delta q 2 d x and that is equal to 0. Now, what we will do is from these entire expressions we will separate out whatever is getting multiplied with delta q 1 and whatever is getting multiplied with delta q 2. Now, if I separate out delta q 1 I have this 0 to L E A multiplied by q 1 plus 2 q 2 x d x minus x d x and then here I have if I separate out delta q 2.

So, here we have E A then 2 x q 1 plus 4 x square q 2 d x minus 0 to L x square and that equal to 0. Now, in this expression we have delta q 1 here and delta q 2 here and they are arbitrary variations. So, they come from the variation of delta u we have a we have approximated u in terms of q 1 q 2. So, when you when we vary delta u we have this terms delta q 1 and delta q 2. Now, delta q delta u is an arbitrary variation.

So, delta q 1 and delta q 2 also can be arbitrary and they are independent. So, this entire term can be 0 only when the term inside the bracket here is 0 and inside the bracket here is 0. So, this equal to 0 that gives me one equation and this equal to 0 that gives me another equation. So, we have two equations and in these two terms there are two unknowns sitting over their q 1 and q 2. So, we have now two equations and two unknowns which we can solve and find out our solution.

So, now, if I write these two. So, this equal to 0 that gives me one equation. So, we call it equation 1 equation 1 maybe and that gives me another equation which we called equation 2. So, if I combine equation 1 and 2 and write in a matrix form, we can write this. So, equation 1 and equation 2 in matrix form may look like this.

And that is equal to this. There is we have to multiply this entire thing with the vector q 1 and q 2. And now we can evaluate the integral and this will come to be E AL L square L square 4 L cube by 3 q 1 q 2 is equal to P multiplied by L square by 2 here and L cube by 3 here. Now, we can solve this equation we can find out q 1 q 2. So, after finding out q 1 q 2 we can put it after finding out q 1 q 2 we can put it in the original approximation which is q 1 x plus q 2 x square and that will give me the solution. Now, here we can see there are few advantages first of all my this geometric and material properties they were outside the differentiation.

In the strong form if we go back, they were inside that they were they had to be differentiated, but in the weak form they do not need to be differentiated because we have shifted the derivatives. So, any variation in the properties if our beam or bar it if it looks like this if they have any variation in the properties even if there is any discontinuity that is easy to take care of, we can always do this we can always have the integral I have E A inside the integral and do the integration numerically. Similarly, if force has any abroad variation, we can take care of it easily and incorporation of static indeterminacy is not also a big problem if this beam was clamped here this problem would have been statically in determinate problem. In that case we would have to assume this polynomial in such a way that u is 0 here as well because in that case the boundary condition is u is 0 here and u is 0 here. In that case we had to choose this approximation in such a way that u is 0 at x equal to 0 and u is 0 at x equal to L and again they always need not be a polynomial we can have some other approximations other kind of approximation functions as well.

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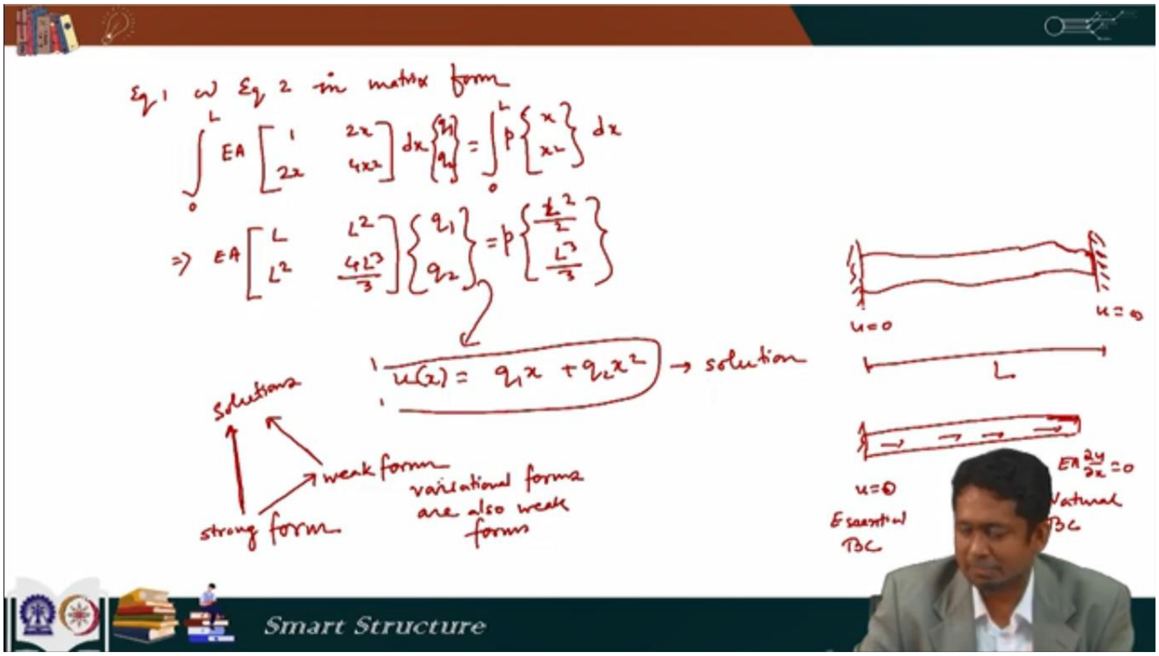

$$\int_0^L EA \begin{bmatrix} 1 & 2x \\ 2x & 4x^2 \end{bmatrix} dx \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \int_0^L p \begin{bmatrix} x \\ x^2 \end{bmatrix} dx$$

$$EA \begin{bmatrix} L & L^2 \\ L^2 & 4L^3/3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = p \begin{bmatrix} L^2/2 \\ L^3/3 \end{bmatrix}$$

$$u(x) = q_1 x + q_2 x^2$$

And apart from this one thing to notice that while doing these approximations we are satisfying only the essential boundary conditions we are not satisfying the natural boundary conditions here. The problem that we looked at the beginning had it look like this. So, we had a u equal to 0 here and at that tip there was no applied force. So, at the tip the normal force is 0.

So, that is a natural boundary condition. So, it is essential boundary condition. And this is natural boundary condition. So, this boundary condition we did not satisfy in our approximation we can set if we can approximate something which satisfy this also well and good otherwise it is not a mandatory requirement. If we have proper approximation with sufficient number of terms we can approximate the solution to a great extent. So, we approximated it in such a way that it satisfies the essential boundary condition just.

Now so, this was a simple demonstration of the energy based techniques. Again just to show the entire picture we have a strong form of the differential equation and from the strong form there is something called weak form. Now variational forms are also weak forms and from the weak form we make approximate solutions. Solution can be done from strong form also, but this going through this route is not always possible that is why we go via this route and in our problems most of the time the weak form is readily available. We saw that without even starting from the strong form we can directly write the Heuer-Schwabers equation or the energy equation.

So, we can simply just start with this and solve it without having to bother too much about the differential equation. So, that is another advantage we can just starting from this and we can get the solution and we can use the features of it that we can make geometric properties outside the differentiation even taking care of other boundary conditions are easier I mean any other essential boundary condition which makes the problem indeterminate that is easier we can make the I mean we can make those features we can take make use of this those features for our advantage and we can directly get the solution. So, this is the entire big picture. Now this was a simple demonstration for a structural problem. Now we will gradually move on to a problem where there are host structure which is inert with piezoelectric patches.

Now when we have piezoelectric patches we have apart from the elastic quantities we have electrical quantities also. So, naturally they take their place in the in these various forms and we will see that.

So, this brings us to the end of this lecture.

Thank you.