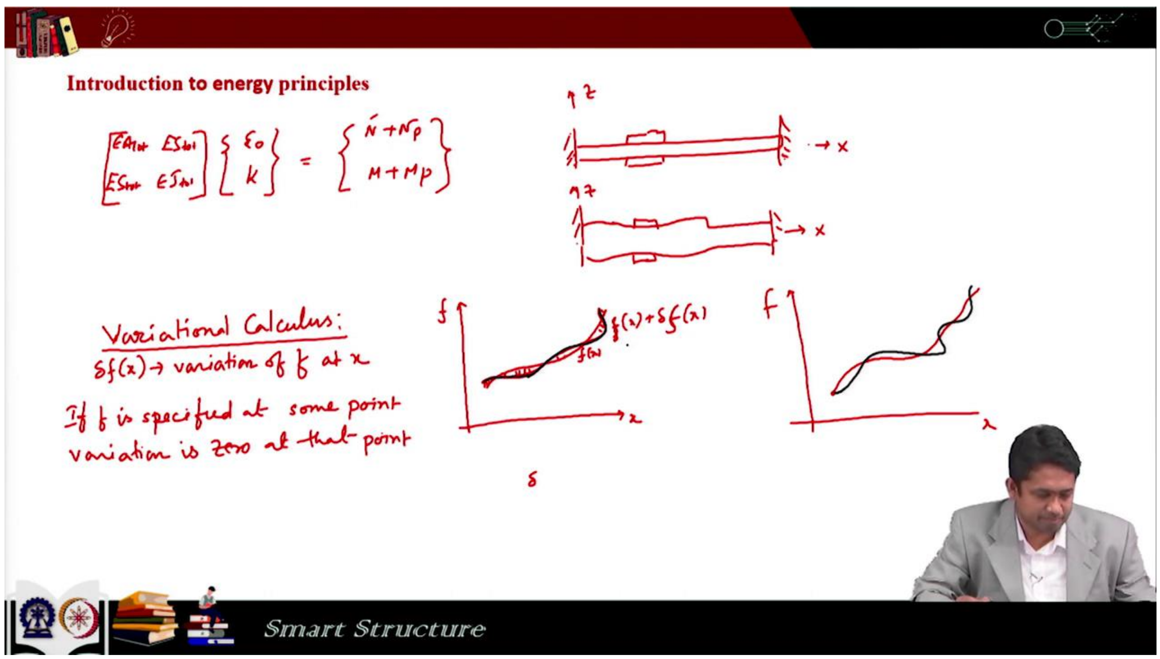
**Smart Structures**
**Professor Mohammed Rabius Sunny**
**Department of Aerospace Engineering**
**Indian Institute of Technology, Kharagpur**
**Week - 04**
**Lecture No - 19**
**Introduction to Energy Principles for Structural Analysis**

Today, we are going to start a new topic that is on energy based analysis of structures with piezoelectric patches.

(Refer Slide Time: 07:41)




Now, so far we have seen how to analyze these structures when we have the governing differential equation and we solve those equations. But most of the times such kind of solutions are very difficult because it is difficult to derive those equations in many times and even we can derive those equations, it is difficult to solve them. So, when I said difficult to derive the equations I mean deriving by balancing the forces and using that approach. And even if we derive it sometimes the solution becomes difficult. In fact, in very few cases it is easy to get the close form solution of those equations.

In most of the cases, it is difficult. For example, if we look at the equation that we got for our Euler Bernoulli assumption, so, there we had a matrix and this has $EA_{total}$ $ES_{total}$ and $EI_{total}$ and at the right hand side we have $(N + N_p)$ and $(M + M_p)$. Now, when the structure is statically determinant, it is easy to find this N and M. So, for example, we had a beam like this.

$$\begin{bmatrix} EA_{total} & ES_{total} \\ ES_{total} & EI_{total} \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ k \end{Bmatrix} = \begin{Bmatrix} N + N_p \\ M + M_p \end{Bmatrix}$$

In that kind of situations if I have some externally applied loads along x as well as z direction or some applied moment also we can find this N and M. But if the structure is statically indeterminate then the solution becomes rather complex. Now, take the fact that suppose we have lot of variation in the material or geometric property itself. Suppose the beam now looks like this or even it may have some jump in the properties and then suppose we put a piezo maybe somewhere here, in that kind of cases, getting a solution is not as straight forward as it was for this case when we did not have this extra fixed end. I mean when the problem was statically determinant.

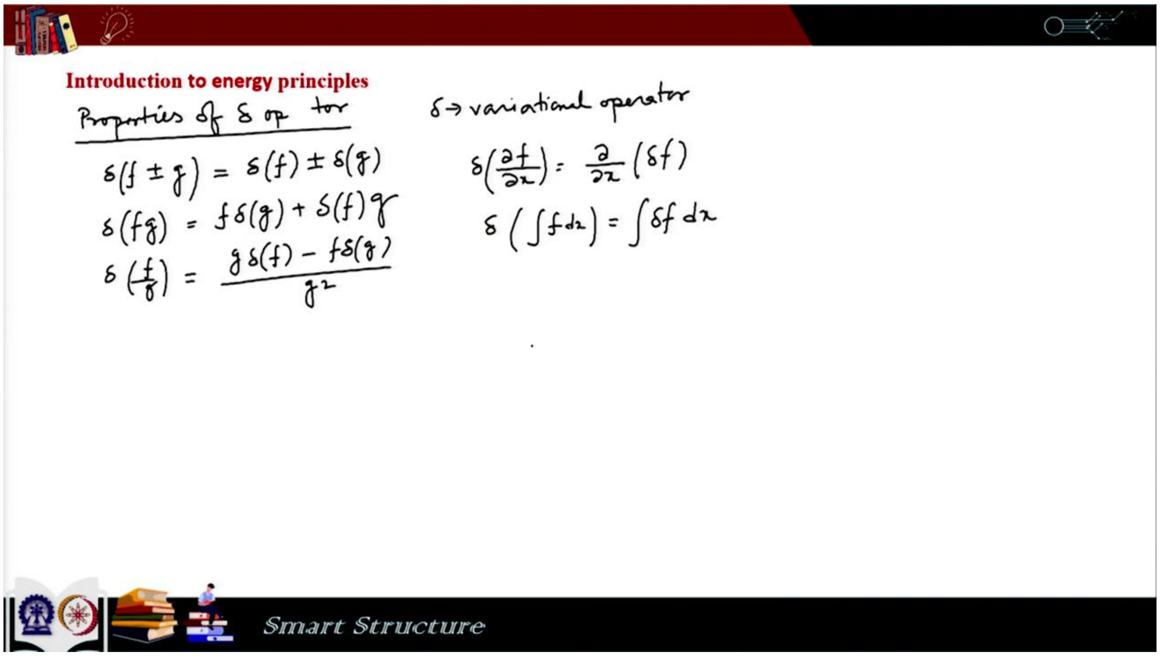
So, we generally use energy based principles for solution of this kind of problems. So, before discussing the energy principles and the related techniques, we would spend few minutes in discussing something called variational calculus because that is what we would be using extensively in our formulations. So, in variation calculus we have, suppose, a function f of x and suppose the function is defined like this. So, this is our f. Now, let us assume that we perturb the function little bit and get something another function maybe something like this.

So, it is a small perturbation. This new function which is this maybe I can put it different color here, so, this black function which is a perturbed version of the red function is called as, we may term it as, f (x) plus delta of f (x). So, it is the function f itself and then some added variation on top of it. So, this delta f (x) is variation of f at x. Now, delta f of x means this difference between this new function and the previous function f.

So, this is our delta f of x and that is variation of f at x. Now, if f is specified at some point variation is 0, that point, which means suppose, now I have a function f and this f is such that it has some specified value here, maybe it is a boundary condition. So, it says that the value of f has to be this. It has to be satisfied that is the boundary condition. In that case if that is specified there then at that point the variation is not possible.

So, the variation is 0 here. At other points, variation can be nonzero, but at that point we cannot have variation. So while solving our problems, we will see that our quantities of interest may have some boundary conditions. For example, when we want to solve these problems our solution, the displacement components can have boundary conditions here and here and we will see that at those points those variables cannot be varied; at other points they can be varied. And this variation of x this delta f is a small variation and for our purpose we will consider them to be arbitrary variation.

(Refer Slide Time: 10:11)




Now, these variations follow some properties. So, properties of delta so it is called a variational operator. So, delta is variational operator and they follow some properties. First of all if we have two functions f and g and if we add or subtract them and take the variation that means the result of the addition or subtraction then the result is addition or subtraction of the individual variations. If we multiply f and g and take the variation of the product then this is this.

The division f divided by g and then if I want to look at the variation it looks like this. Now, from this it can be shown that if I take the variation of the derivative of x that is equal to the derivative of the variation of f. So, if I take the variation of the derivative of f with respect to x and that is equal to the the derivative of the variation of f. Similarly, we can also say that if we integrate f and take the variation that is equal to this. So, these properties will be quite useful in our formulation that we do after that.

$$\delta(f \pm g) = \delta(f) \pm \partial(g)$$

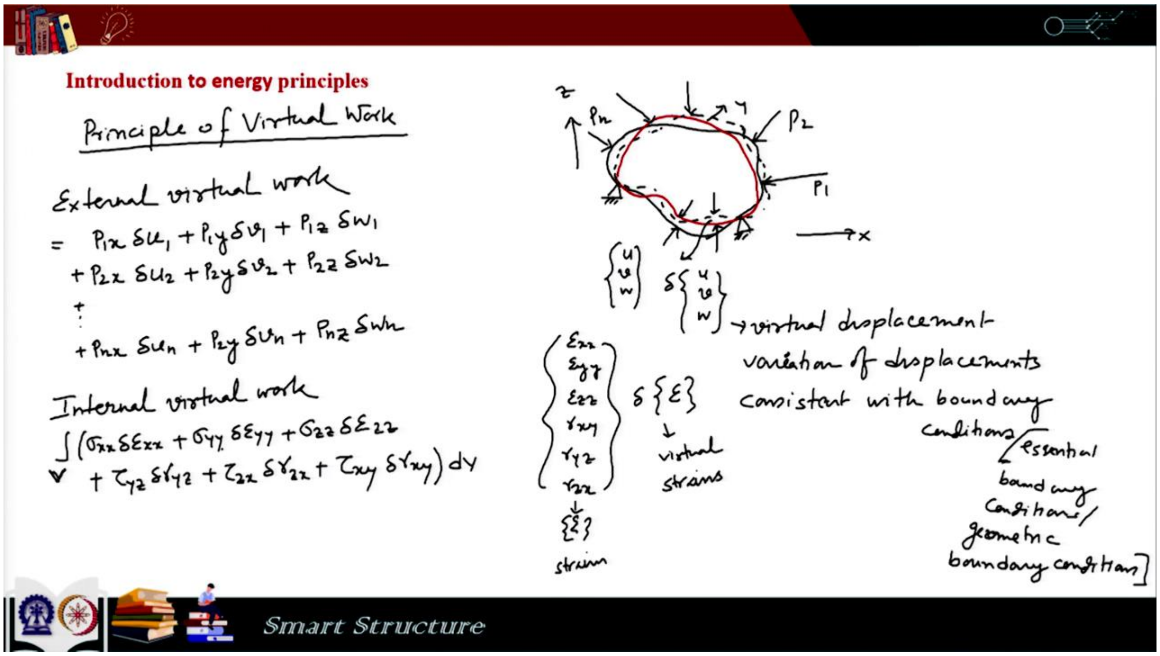
$$\partial(fg) = f\partial(g) + \partial(f)g$$

$$\partial\left(\frac{f}{g}\right) = \frac{g\partial(f) - f\partial(g)}{g^2}$$

$$\partial\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\partial f)$$

$$\partial(\int f dx) = \int \partial f dx$$

(Refer Slide Time: 21:40)




Now, we will talk about a very important principle that is principle of virtual work. So, let us assume that we have a three dimensional body. It is any structure. It is a very generic thing. This is X, we have Y and Z as the system and let us assume that it has some boundary conditions at some points.

Now, under some load it deforms and gets a new shape and suppose, the shape is this. Suppose this is the new shape. This is getting on the application of the load and if you want to show the load also suppose the loads are this. So, there are suppose n number of loads. So, there are n numbers of loads $P_1$, $P_2$ all the way up to $P_n$ and under that load this body deforms in this way.

So, in this deform configuration which is shown by this red lines this body is under stable equilibrium. Now, let us say that we give some perturbation to this deformed shape and suppose this dotted black lines show the perturbation. So, we have three kind of lines, one is firm black line that is the undeformed body, the red firm line that is the deformed body and the dotted black line that is the perturbed version of it. So, we can say that if I just show this difference may be difference between this and this, that is our delta of u v w where u v w are the displacement along X, Y and Z of any point in the body. Similarly, if I look at the difference between say two other points, the difference between the firm black line and the firm red line that is our u v and w.

So, it is at this point and these are other points. They are not shown in the same point. There are shown at different points. So, there is u v w that is the displacement that it experiences under the application of this load and on top of that deformed shape if I give some perturbation and then this is nothing, but a variation of this displacement and that is delta of this. So, these are called virtual displacement, which is nothing, but variation of displacement.

Now, as we said that at that point where our functions have specified values, the variation should be 0. So, we can see here that here we have some boundary conditions. So, our virtual displacement or the variation of this u v w, they are not violating the boundary conditions. So, they are 0 here. So, these are consistent with boundary conditions to be more specific we can say that they are consistent with essential boundary conditions or geometric boundary conditions.

So, they are consistent with essential boundary conditions which we also call as geometric boundary conditions. Now, during this virtual displacement from u v w to the delta of u v w. the external work can be considered to do some work. So, the work done by them is if I take the x component of $P_1$ and I multiply by the x component of the virtual displacement which is u. Similarly, if I take the y component of $P_1$ and multiply that by v and if I take the z component of $P_1$ and multiply that by w and if I sum them up that gives me the total work done by this $P_1$ and accordingly I can sum it up for all the forces and that is called external virtual work.

So, for the external virtual work is P1x, the component of P1 along x multiplied by del u at point 1 here. Please remember this u v w and delta of u v w we have written them at different points, but where I am specifically saying at which point they are specified I am taking a subscript that is 1 which means at this point plus P1y delta v1 plus P1z delta of w1. Similarly P2x delta of u2 plus P2y multiplied by delta of v2 plus P2z multiplied by delta of w2 and I can keep doing it all the way up to the nth force. So, that is our external virtual work. Now similarly there is something called internal virtual work.

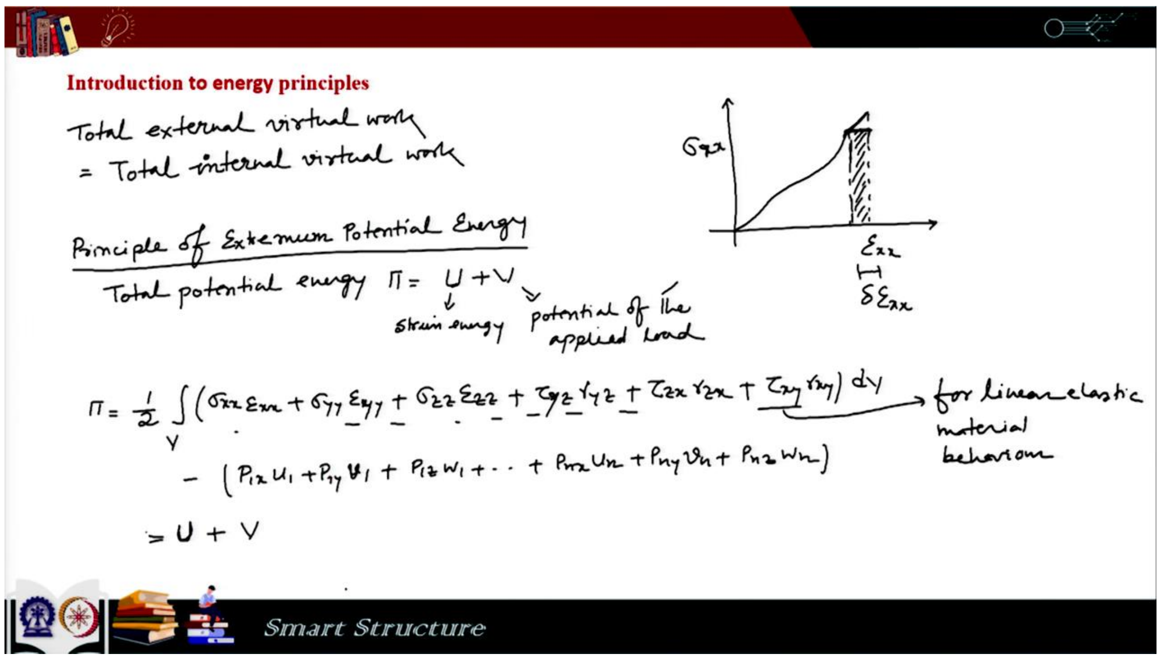
Internal virtual work means it has stress components sigma xx, sigma yy, sigma zz and all the shears and the corresponding strains. So, suppose that under this deformed configuration, this red line the stresses are sigma and when it goes from this red line to this dotted black line which means during this virtual displacement, it gets additional strain which is the virtual strains and let us denote them as delta of epsilon. So, if I denote these as displacements, the corresponding strains there are epsilon xx, epsilon yy, epsilon zz. Similarly we have gamma xy, gamma yz and gamma zx and accordingly we have, so, let us call it as epsilon and accordingly we can have the variation of the strains which are the virtual strains. So, these are strains which it has in this red configuration and these are virtual strains which come when it goes from the red configuration to the dotted black configuration.

So, the internal virtual work done is if I now stress-strain and they are all point functions. So, I have to define at point and then we have to integrate over the whole body. So, integrate over the whole volume V, here V is our volume integral of this quantity. So, we take normal stress along x and multiply with the corresponding virtual strain. Similarly, normal strain along y multiply with the corresponding virtual strain plus normal strain along z and multiply with the corresponding virtual strain and then the corresponding shears are to be consistent with what I wrote just now.

$$Internal\ virtual\ work = \int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\, dV$$

Let me call it zx it does not matter. Stress is symmetric for us. So, zx and xz are same. So, these are our internal virtual work.

(Refer Slide Time: 29:10)




Now, just to show it graphically, suppose, let us take one of the components, may be let us take a simple one dimensional case say we have only epsilon xx and sigma xx and let us suppose that the stress-strain variation is something like this. Now, this virtual work principle is applicable even when the virtual property is non-linear. So, just that is why to be more generic I drew a non-linear curve here. So, let us say that under the deformed configuration or deformed stable configuration this is the amount of strain stress and strain produced. Then if I perturb it further if I get the additional virtual displacement, it gets some additional displacement and because of this, the curve continues further may be like this.

So, suppose this additional part that we have is our virtual strain and at this strain may be the stress is this. So, the internal virtual work is this quantity. If I take sigma here which is our sigma at the stable equilibrium and then if we multiply that by delta of epsilon xx whatever we get here in this rectangle the area of this rectangle is our internal virtual work. And then we do it for all the stress and strain components and then we integrate over the volume and then we get the total internal virtual work. So, the principle here says that total external virtual work is equal to total internal virtual work and this holds when it is under equilibrium.

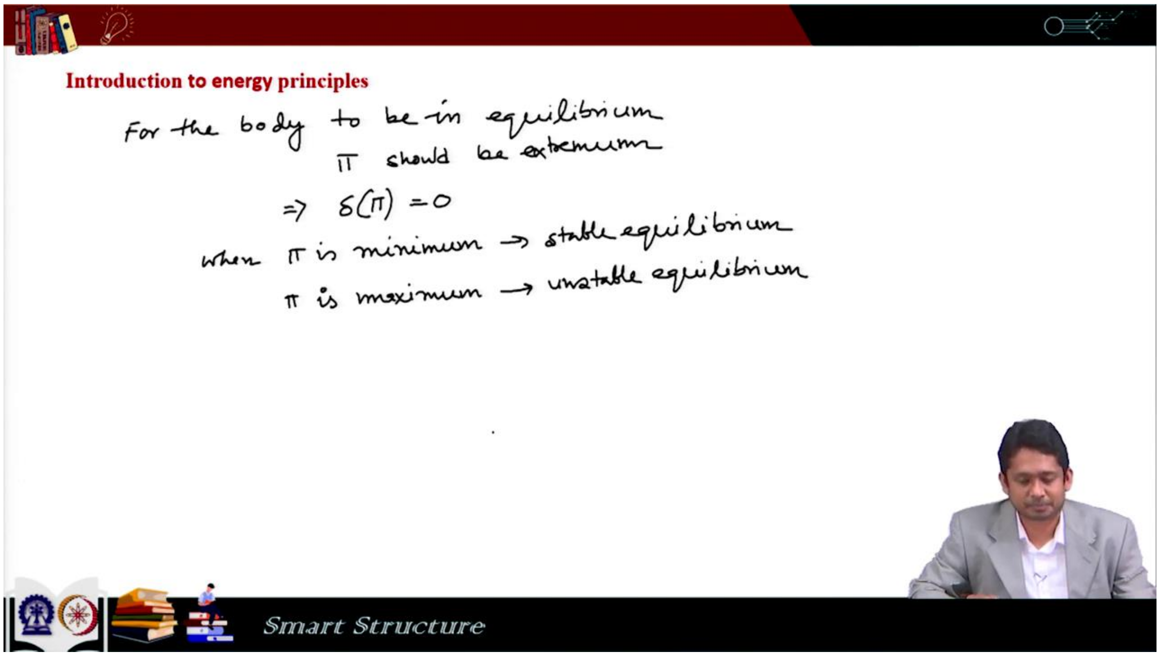
So, if some displacement condition satisfies this virtual work principle that is also an approximate solution of the governing differential equation. So, instead of solving directly the governing differential equation in many cases, we just satisfy the virtual work principle in an approximate way and that helps us get get a very close solution of the problem. Now, we will talk about another principle that is potential energy principle and the virtual work principle. They are very closely related. Here we define something called total potential energy, pi of the entire body and that has two components U and V where U is the strain energy and that is also total and this is potential of the applied load.

Now, for this problem if I want to look at the expression for the total potential energy, this comes to be this pi is equal to half integrated over the volume sigma xx multiplied by epsilon xx half into stress into strain for all the components. So, we have all the shear and normal components. So, this is my total potential energy U. Now please understand this expression can be written only when the behaviour is linear. For this kind of non-linear material behaviour, this we cannot write.

$$\pi = \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + \tau_{xy}\gamma_{xy})\, dV - \left(P_{1x}u_1 + P_{1y}v_1 + P_{1z}w_1 + \cdots \ldots \ldots + P_{nx}u_n + P_{ny}v_n + P_{nz}w_n\right) = U + V$$

So, the virtual work principle is more robust we can say and this we can write only for linear material behaviour. And for the potential of the applied load again we can write P1x multiplied by u1 plus P1y multiplied by v1, plus P1z multiplied by w1 and all the way up to Pnx multiplied by un plus Pny multiplied by vn plus Pnz multiplied by wn. So, the first term quantity is our u, the strain energy and the second quantity is our potential of the applied load. And again, this quantity is for linear elastic material behaviour.

(Refer Slide Time: 31.00)




Now the principle says that for this to be in equilibrium $\pi$ should be extremum which means either it should have minima or maxima.

And that means, the variation of pi should be 0 and u v and w should be such that the variation of pi is 0 which means it is an extremum. Now when pi is minima it means it is in stable equilibrium and when pi is maximum this means that it is in unstable equilibrium. So, we can enforce this virtual work condition or the energy condition, but this energy equation is limited to this linearly elastic material. So, we can impose this condition and we can find the solution where our structure is in equilibrium and stable or sometimes we want to find some unstable solution also. We would not do it here and that is also possible using this.

So, with that I would like to conclude this lecture here. I will see you in the next lecture.

Thank you.