Smart Structures
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Week 03
Lecture No: 16
Induced Strain Actuation – Static Analysis (Continued)
Part 04

Welcome to the fourth video of week 3.

We started with the block force method, and we looked at one of the cases using this method. Now, we will talk further about that case.

We saw that, the strain that we got in the beam. Now, here is the strain in the beam; because the strain is found out only in the beam, we are denoting it as epsilon b, but it has only epsilon zero component, which means only the axial component under the extensional case, and that is equal to EA_c by EA_c plus EA_b multiplied by epsilon p.

$$\varepsilon_b = \varepsilon_0 = \frac{EA_c}{EA_c + EA_b}\varepsilon_p$$

And if we recall, we got the same expression for the pure extension case in previously also when we were dealing with the Euler Bernoulli beam-based method.

So, the block force assumption does not alter the result here. Later on, we will see that when we talk about bending, the result gets changed. Now, before going to bending, one more thing: it is finding out the displacement because of this kind of strains. So, again, it is the same thing. If we find out the displacements because the strains are the same, the results would also be similar. u would be zero, when my x is less than x-zero. And u is going to be EA_b plus EA_c multiplied by x minus x-zero when it is seen. And u is going to be multiplied by l_C when x is greater than x-zero plus l_C.

$$u = 0 \qquad 0 < x < x_0$$

$$u = \frac{2Fbl}{EA_b + EA_c}(x - x_0) \qquad x_0 \le x \le x_0 + l_c$$

$$u = \frac{2Fbl}{EA_b + EA_c}l_c \qquad x > x_0 + l_c$$

So, here we are finding out the strains, and we are integrating the strain as we did before with the boundary conditions, and we get our displacements at different parts of the structure as this. Now, we will go to the next case, where we will see a bending effect.

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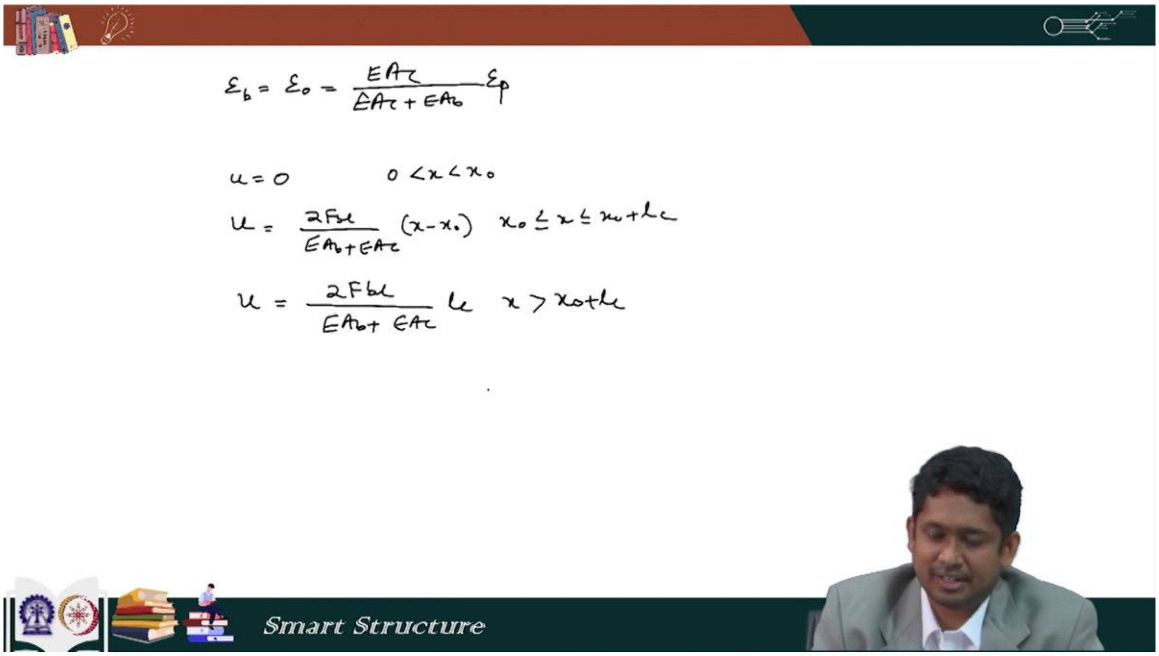

So, this is a pure bending case. And again, here, the actuation is the opposite: if it is plus here, it is minus here. If it is minus here, it is plus. So, if the top piezo tries to shorten in length, it will experience a tensile force, and the top part of the beam will experience a compressive force. On the other hand, because the actuation is opposite, the bottom part of the beam will experience a tensile force, and the bottom piezo would experience a compressive force and that would give to bending in this fashion. In the contrary, if the actuation is just the opposite, then the top piezo would experience a compressive force, and the top part of the beam would experience a tensile force, and the bottom part it would be the opposite. And in this case, it would result in this kind of bending. And everywhere, the force is just F.

So, needless to say, when these forces are opposite, the net force is zero, but it will induce a bending moment. So, it is a bending case. So, again, we will apply the same logic. We will find out the displacements here, and we will apply the compatibility condition that the displacement and the piezo and the displacement of the beam are the same. But here, because it is a bending case, the displacements of the beam along z are not the same. So, the displacement here and displacement here are different. So, the displacement here should be equal to the displacement here and the piezo. And similarly, the displacement here at the beam should be equal to the displacement here from the piezo side.

Let us assume that these forces are inducing a bending moment M. Now, because of that bending moment M, the strain here is M divided by I_b, the moment of inertia of the beam, multiplied by this distance: the distance from the middle to the top. So, it is t_b by two. So, if the entire thickness is t_b from z equal to zero to the top part of the beam, it is t_b by two. So, M by EI_b into t_b by two multiplied by one by E_b, that is our strain at the top with a negative sign, as per our convention. Now, bending moment: if the force here, the bending moment is F into t_b. So, our bending moment is F multiplied by t_b. So, we can replace M by F, and the expression becomes this. So, that is our strain at the top fiber of the beam.

$$\varepsilon_b^s = -\frac{M}{I_b}\left(\frac{t_b}{2}\right)\frac{1}{E_b} = -\frac{F}{E_b I_b}\left(\frac{t_b^2}{2}\right)$$

If the strain there is this, the change in length of this portion of the beam is delta l_b, which is equal to that strain multiplied by l_c. So, which is this. Now similarly, we have to consider the change in length of the piezoelectric actuator.

$$\Delta l_b = \varepsilon_b^s l_c = -\frac{F}{E_b l_b}\left(\frac{t_b^2}{2}\right) l_c$$

So, the piezoelectric actuator is under a force F. Again, applying the same procedure that we did before, we can find out delta l_c. And delta l_c, here is this.

$$\Delta l_c = \left(-\varepsilon_p + \frac{F}{E_c t_c E_c}\right) l_c = -\left(d_{31}\frac{V}{t_c} - \frac{F}{b_c t_c E_c}\right) l_c$$

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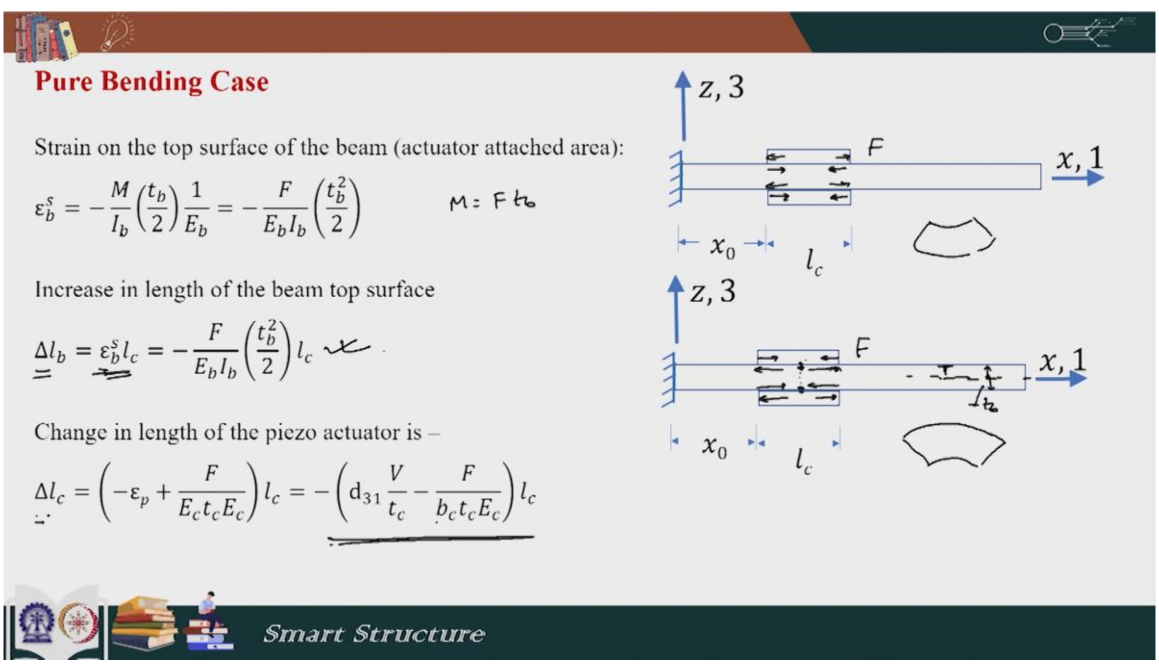

Now, again, we can equate these two, and after equating, we get our F as this. Solving, we get the force. The force is this.

$$F = \frac{d_{31}\frac{V}{t_c}}{\left(\frac{t_b^2/2}{E_b I_b} + \frac{1}{b_c t_c E_c}\right)} = \left(2d_{31}\frac{V}{t_b^2 t_c}\right)\frac{EI_b EI_c}{(EI_b + EI_c)} = F_{bl}\frac{EI_b}{(EI_b + EI_c)} = F_{bl}\frac{EA_b}{(EI_b + EI_c)}$$

Here, this expression is further compacted in terms of EI_b EI_c. EI_b is the bending stiffness of the beam itself, and EI_c is the combined bending stiffness of the two actuators. So, while calculating the bending stiffness of the actuators, we make some simplifications here. If this is our beam section with the two actuators, then while finding out the moment of inertia with respect to the midline of the beam, what we do is: we first find out the moment of inertia of the piezo patch with respect to its own centroid and then shift it here.

$$EI_b = E_b I_b = E_b A_b\left(\frac{t_b^2}{12}\right)$$

$$EI_c = 2(b_c t_c)\left(\frac{t_c}{2}\right)^2 E_c = EA_c\left(\frac{t_c}{2}\right)^2$$

Now, the moment of inertia with respect to its own centroid for a piezoelectric patch can be negligible when the patch thickness is quite small. So, under that assumption, we remove that term and retain only the term which we get by shifted to the midline of the beam by the parallel axis theorem. So, only we retain that term. And then again, it can be written as this. So, with these two, we can write the force in terms of the block force as this. And again, there is a relation between the EI_b and EA_c. And from that, we can again rewrite the expression in terms of EA_b and EA_c.

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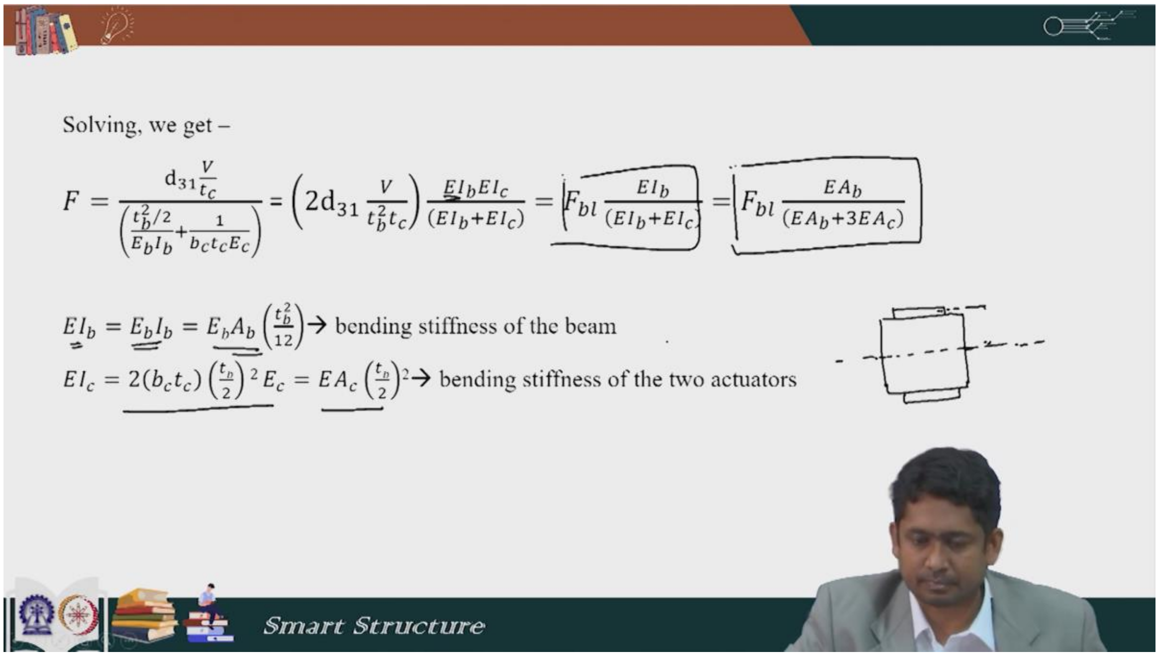

After that, we have to calculate the actuation moment. The actuation moment is: the actuation force multiplied by t_b and from that, we get this expression for the actuation moment.

So, this is our M_{bl} block moment because F_{bl} into t_b is our block moment. So, the moment is equal to force, F multiplied by the t_b, and M_{bl} is equal to F_{bl} multiplied by the t_b. These are block moments to say. So, this we can term as block moments analogous to block force. And again, we can replace EI_b and EI_c by EA_b and EA_c.

$$M = Ft_b = F_{bl}t_b\frac{EI_b}{(EI_b + EI_c)} = M_{bl}\frac{EI_b}{(EI_b + EI_c)} = M_{bl}\frac{EA_b}{(EA_b + EA_c)}$$

$$M_{bl} = F_{bl}t_b$$

Now, for pure bending actuation, the axial beam strains very linearly across the beam, and therefore, we can write the strain as this. So, this is our strain epsilon b and epsilon b as we know it is: we can find out from the standard formula moment divided by EI_b into z, and that gives us our strain at any z from these findings.

$$\varepsilon_b = -\frac{M}{EI_b}z = -\frac{M_{bl}}{EI_b + EI_c}z$$

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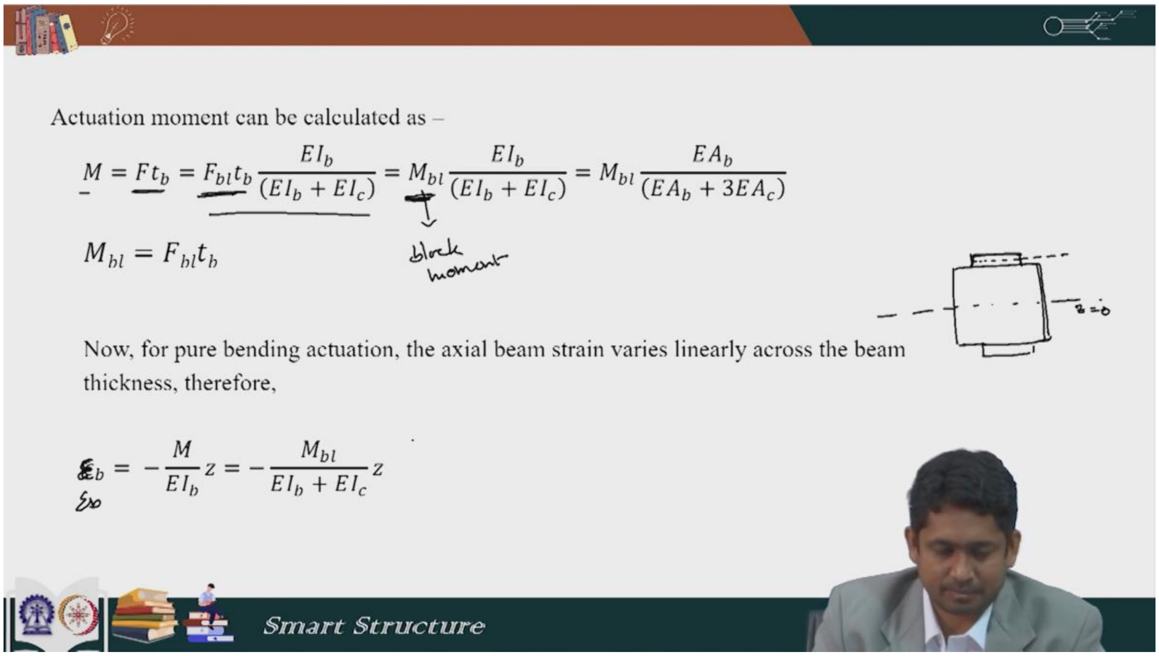

Now, from that, we can find out the moment at the top surface of the beam. So, if this is our beam, so, and here it is t_b by two, here it is t_b by two. So, this is the beam without showing the actuator. If I want to find out the strain here, we just replace z by minus t_b by two, and we get the moment here in terms of both EI_c and EA_b.

$$\varepsilon_b^s = -\frac{M_{bl}}{EI_b + EI_c}\left(\frac{t_b}{2}\right) = -\varepsilon_p \frac{EI_c}{EI_b + EI_c} = -\varepsilon_p \frac{3EA_b}{(EA_b + 3EA_c)}$$

Similarly, when we want to find the same thing at the bottom surface, we just replace z. Here, we replace z by t_b by two, and we get this. And for here, we replace z by minus t_b by two, and we get our strains at the bottom surface.

$$\varepsilon_b^{-s} = \varepsilon_p \frac{3EA_b}{(EA_b + 3EA_c)} = \varepsilon_p \frac{EI_b}{(EI_b + EI_c)}$$

So, strain here at the top surface, and strain here, strain here at the bottom surface, and we get the two strains at the two extreme surfaces.

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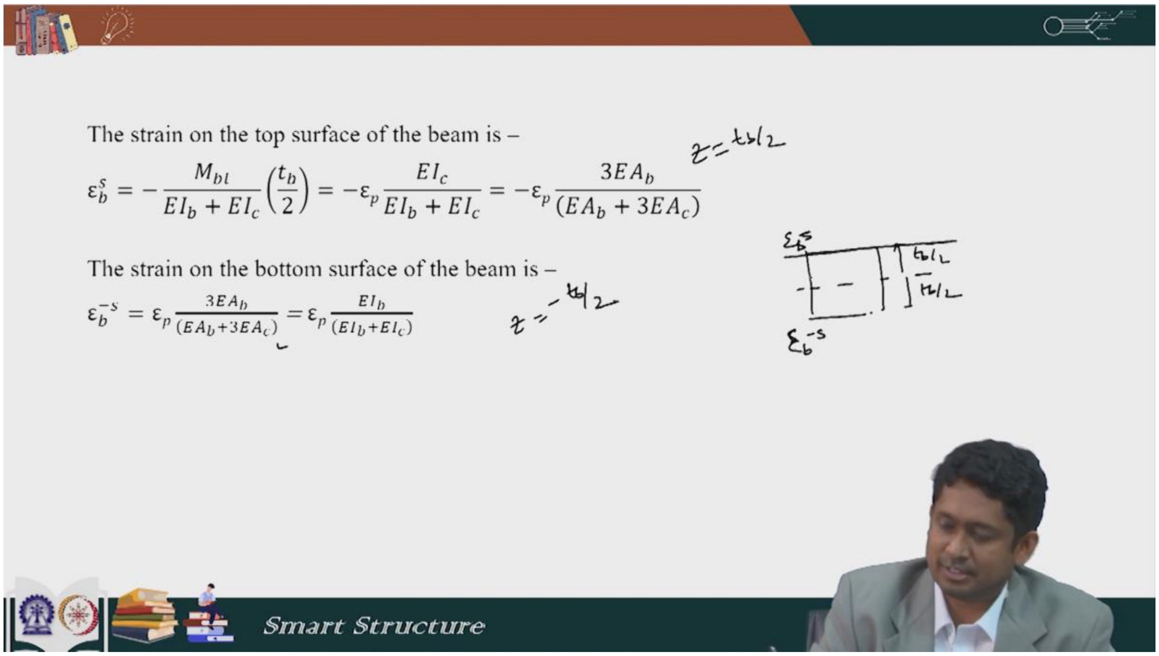

Again, for some extreme cases, we will see how it behaves. So, when we have our EI_c much greater than EI_b. In that case, our block moment is zero. So, if we look at this expression, this term becomes zero when EI_c is much bigger than EI_b. So, this term is almost zero. This term is almost zero means, the piezoelectric patches can expand or contract freely, and that we can see here. So, epsilon b at the top surface is minus epsilon p and epsilon b at the bottom surface is epsilon p.

When Flexural rigidity, $EI_c \gg EI_b$

$$M \approx 0 \qquad M = M_{bl}\frac{EI_b}{(EI_b + EI_c)}$$

$$\varepsilon_b^s \approx -\varepsilon_p \qquad \varepsilon_b^s = -\varepsilon_p \frac{EI_c}{(EI_b + EI_c)}$$

$$\varepsilon_b^{-s} \approx \varepsilon_p$$

When Flexural rigidity, $EI_c \ll EI_b$

$$M \approx M_{bl}$$

$$\varepsilon_b^s \approx 0$$

$$\varepsilon_b^{-s} \approx 0$$

In the reverse case, when EI_c is much less than EI_b, in that case, the movement of the piezoelectric patches are almost constrained. In that case, the moment is equal to almost the block moment, and the piezoelectric patches do not experience any strain.

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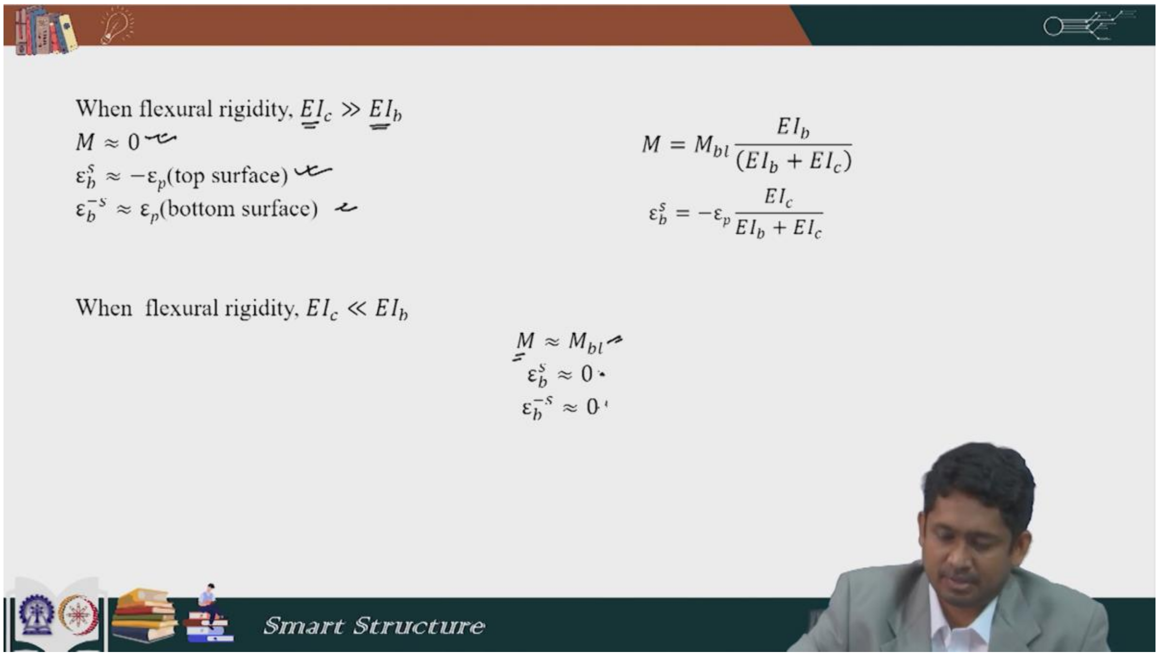

Now, once we know the moments here in terms of the block moments or the free strains, we can find out how the beam deforms. So, this is how it is. We already saw, how to solve these problems. And if we just follow the procedure, if we follow the governing differential equation that is EI del two W del x two is equal to M. Here we have to put the appropriate E. So, here we will put EI_b because we already know what moment the beam is experiencing and, in this method, we do all the calculations for the displacement considering EI_b. And M, we know in terms of the block moment. We can write it in terms of the block force or the free strain also, and then we can solve the differential equation. Put the corresponding boundary conditions and solve it. After solving it, we can see that in this region, the slope is zero. In this region, the slope is this, and in this region, the slope is this.

$$x < x_0 \qquad \frac{\partial w}{\partial x} = 0$$

$$x_0 < x < x_0 + l_c \qquad \frac{\partial w}{\partial x} = \left(\frac{M_{bl}}{EI_b + EI_c}\right)(x - x_0)$$

$$x > x_0 + l_c \qquad \frac{\partial w}{\partial x} = \left(\frac{M_{bl}}{EI_b + EI_c}\right) l_c$$

$$x < x_0 \qquad w = 0$$

$$x_0 < x < x_0 + l_c \qquad w = \frac{M_{bl}}{EI_b + EI_c} \frac{(x - x_0)^2}{2}$$

$$x > x_0 + l_c \qquad w = \frac{M_{bl}}{EI_b + EI_c}\left[\frac{l_c^2}{2} + l_c(x - x_0 - l_c)\right]$$

When finding out displacement, we see that in this region, displacement is zero. In this region, displacement is this, and in this region, displacement is this. So, before concluding the discussion on this beam problem, we would see that the result that we get for the beam problem using the block force method would turn out to be somewhat different from what we get by solving the Euler boundary beam-based technique. Because here, we are making an assumption, the assumption is that if we talk about this section only, it has a piezo at the top and at the bottom. So, when this bends, it bends in this fashion. And as per the Euler boundary beam, we saw that the top part of the piezo part also bends, which means the strain is changing in the piezo part as we go along the thickness of the piezo. However, in the block force method, our piezo strains are the same along the thickness. So, this variation, we are ignoring in our block force method.

So, that is why it gives some difference in the results. And in fact, if we see that: if we want to make our piezo behave in such a way that its strain is the same, that is not very much feasible physically. So, that is a big assumption in the block force base method. And that is why, although the result was the same in extension, because, in extension or contraction, it does not matter if this kind of bending does not come into the picture. But in the bending case, the results are different.

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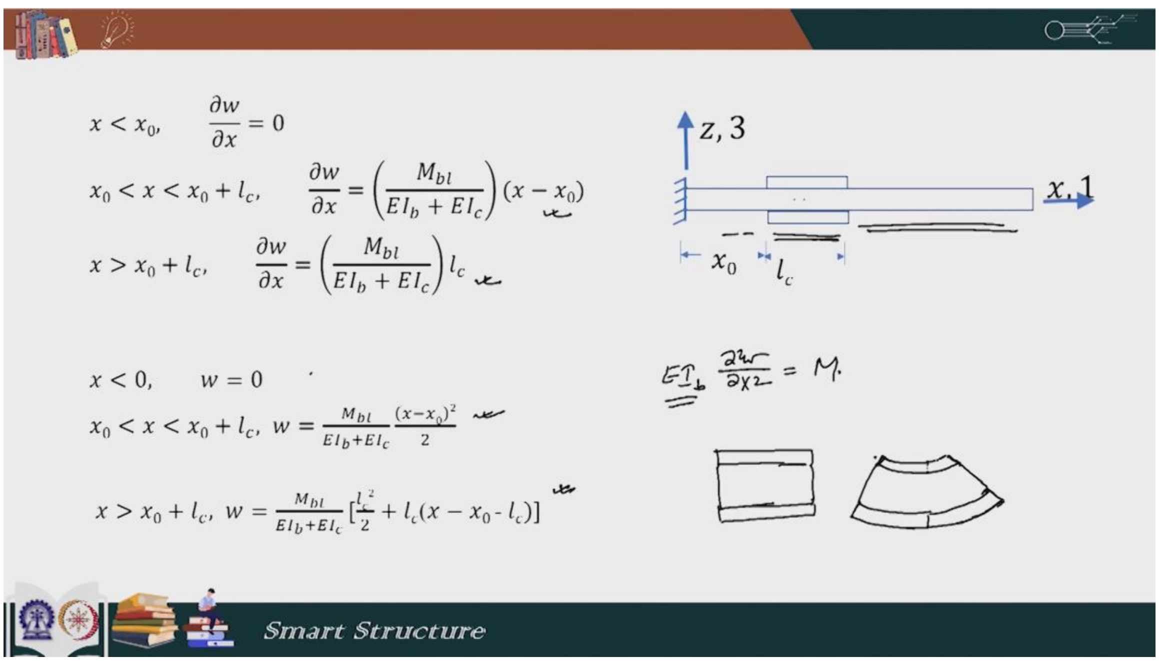

Now, here the axial strains are calculated in the same way the axial stresses can be calculated. So, in the first part of the beam where the actuator is not here, the stress is zero. In the mid part, I mean under the actuator part, there is some stress, and after that, again, the stress is zero. And similarly, the strain is also zero at the first part without the actuator, and then, there is some strain. And at the other part, which is free from the actuator, again, the strain is zero. And this is calculated here at the top part.

$$x < x_0 \qquad \sigma_b = 0$$
$$x_0 < x < x_0 + l_c \qquad \sigma_b = -\frac{M_{bl}E_b}{EI_b + EI_c}z$$
$$x > x_0 + l_c \qquad \sigma_b = 0$$

So, if you want to be generic and write it in terms of any z. So, it is M_{bl} by EI_b plus EI_c multiplied by z. And this is for the top surface. And this is for any z.

$$x < x_0 \qquad \varepsilon_b^s = 0$$
$$x_0 < x < x_0 + l_c \qquad \varepsilon_b^s = -\frac{M_{bl}}{EI_b + EI_c}\left(\frac{t_b}{2}\right) = -\frac{M_{bl}}{EI_b + EI_c}z$$
$$x > x_0 + l_c \qquad \varepsilon_b^s = 0$$

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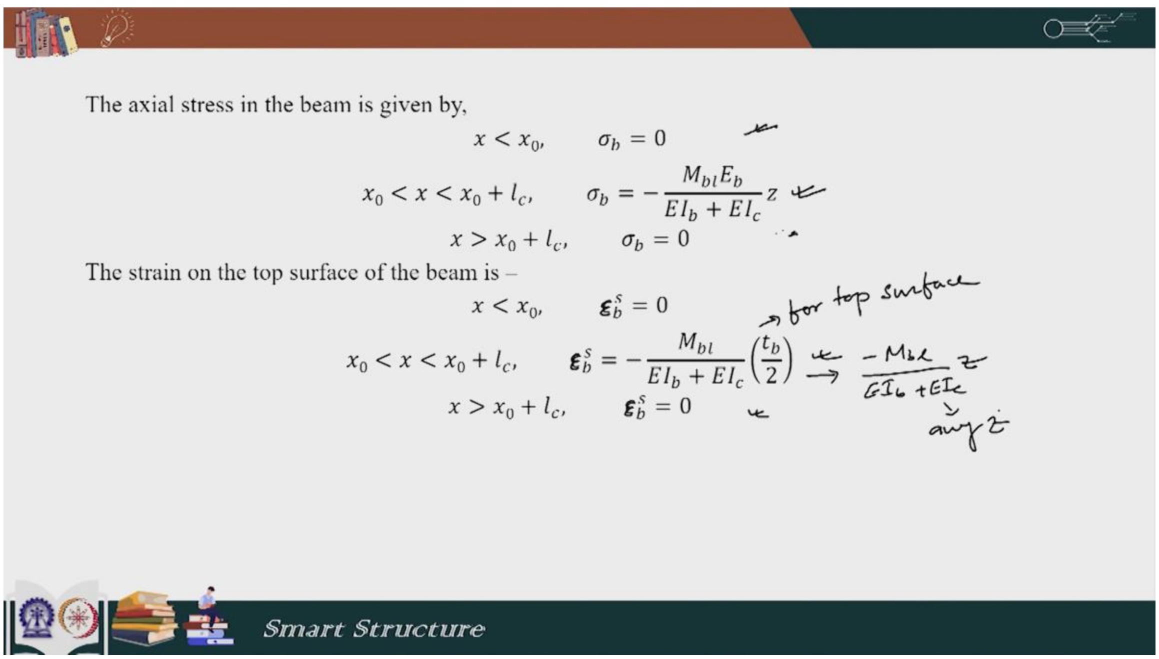

Now, similarly, we look into the case of unequal electric voltage. So, while dealing with the Euler Bernoulli beam, we saw the case when what to do when the electric voltages are not equal; the same situation comes here. The volt electric voltages may not be equal. In these two cases, in the same procedure, we decouple the voltage into V_{top} and V_{bottom}, sorry, V_1 and V_2. And V_1 and V_2 are in terms of V_{top} and V_{bottom}, are this.

$$V_1 - V_2 = V_{top}$$

$$V_1 + V_2 = V_{bottom}$$

$$V_1 = \frac{V_{bottom} + V_{top}}{2}$$

$$V_2 = \frac{V_{bottom} - V_{top}}{2}$$

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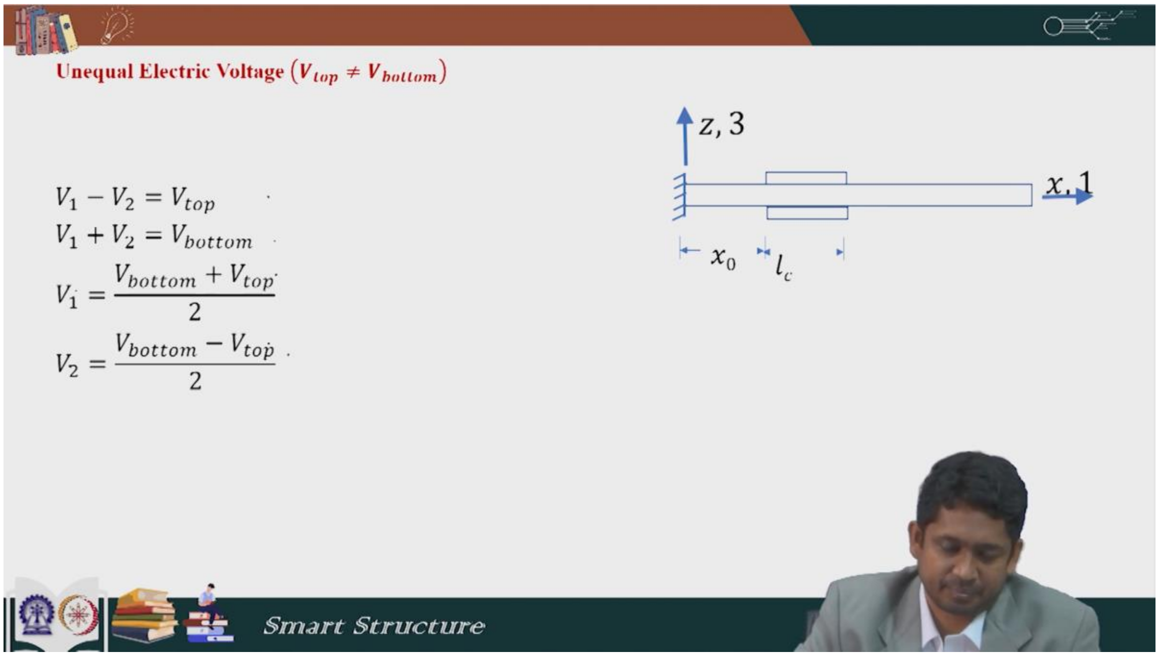

And then, for the V_1 part, this is the actuation force F^e. Here, E stands for extension. It can be an extension or contraction as well. And similarly, for the V_2 part, it results in bending. So, the corresponding force is F_b, and the corresponding moment is M. And then, from here, we find out, we just superimpose F^e and F^b, and we find out the force at the top F_{top}, and the force at the bottom F_{bottom}. So, if this is our beam, this is along the x-axis, and this is the z-axis, we have the piezo somewhere here, maybe.

$$F^e = F_{bl_1} \frac{EA_b}{EA_{c1} + EA_b} = F_{bl_1} \frac{EA_b}{EA_c + EA_b}$$

$$F_{bl_1} = E_c b_c t_c \varepsilon_{p_1} = EA_c \frac{\varepsilon_{p_1}}{2} = \frac{d_{31} V_1}{2t_c} EA_c$$

$$F^b = F_{bl_2} \frac{EI_b}{EI_b + EI_c} = F_{bl_2} \frac{EA_b}{EA_c + EA_c}$$

$$F_{bl_2} = \frac{d_{31} V_2}{2t_c} EA_c$$

$$M = M_{bl_2} \frac{EI_b}{EI_b + EI_c} = \frac{2d_{31} V_2}{t_b t_c} \frac{EI_b EI_c}{EI_b + EI_c}$$

Total force on the top surface, $F_{top} = F^e - F^b$

Total force on the bottom surface, $F_{bottom} = F^e + F^b$

Let us assume that the forces are in this form, but they are different. So, this can be written as F_{top}. This can be written as F_{bottom} or vice versa. The sign also may change. And F_{top} and F_{bottom}, they are not same. They can be different in terms of their directions also and magnitude wise also they can be different.

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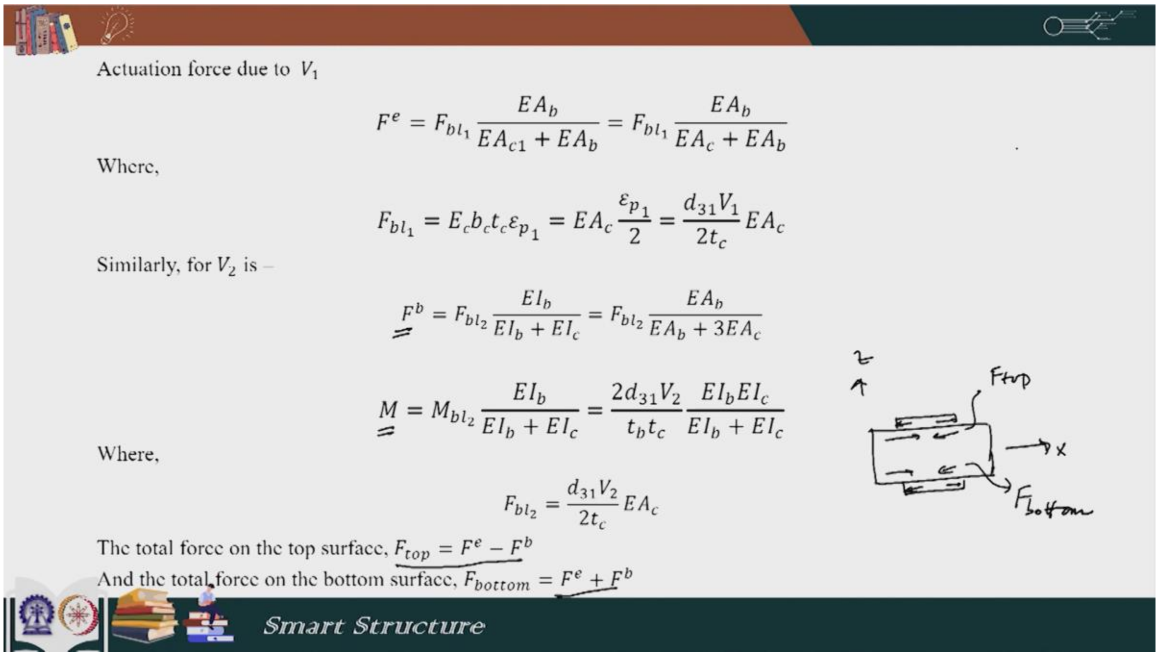

So, with that, I would like to finish this lecture here.

In the next lecture, we will look at the other cases using the block force method.

Thank you.