Smart Structures
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Week - 03
Lecture No - 14
Induced Strain Actuation-Static Analysis (continued)

In the last lecture we started discussing on the Euler Bernoulli beam based formulation and then we looked into a specific case which was a symmetric actuation case where same voltage is applied at top and bottom. Here by same voltage we mean the relative direction of the voltage and the polarization. So, in this case it produces either a contraction or a tension depending on the relative direction of the polarization and voltage and the D D 3 1 coefficient. Now, we got this relation E A total epsilon 0 is equal to N plus N p and this equation is sufficient to solve it because the other equation does not come into picture because this is symmetric case. So, E A total is 0. Now, we will look into the displacement that is caused here.

$$EA_{tot}\varepsilon_0 = N + N_P$$

$$EA_b \frac{\partial u_0}{\partial x} = 0, 0 < x < x_0$$

$$(EA_b + EA_c)\frac{\partial u_0}{\partial x} = N_P, x_0 \le x \le x_0 + l_c$$

$$EA_b \frac{\partial u_0}{\partial x} = 0, x > x_0 + l_c$$

$$u_0 = c_1$$

$$u_0 = \frac{N_P x}{EA_b + EA_c} + c_2, x_0 \le x \le x_0 + l_c$$

$$u_0 = c_3, x > x_0 + l_c$$

$$u|_{x=0} = 0 \Rightarrow c_1 = 0$$

$$\frac{N_P x_0}{EA_b + EA_c} + c_2 = 0 \Rightarrow c_2 = -\frac{N_P x_0}{EA_b + EA_c}$$

$$\Rightarrow u_0 = \frac{N_P}{EA_b + EA_c}(x - x_0), x_0 \le x \le x_0 + l_c$$

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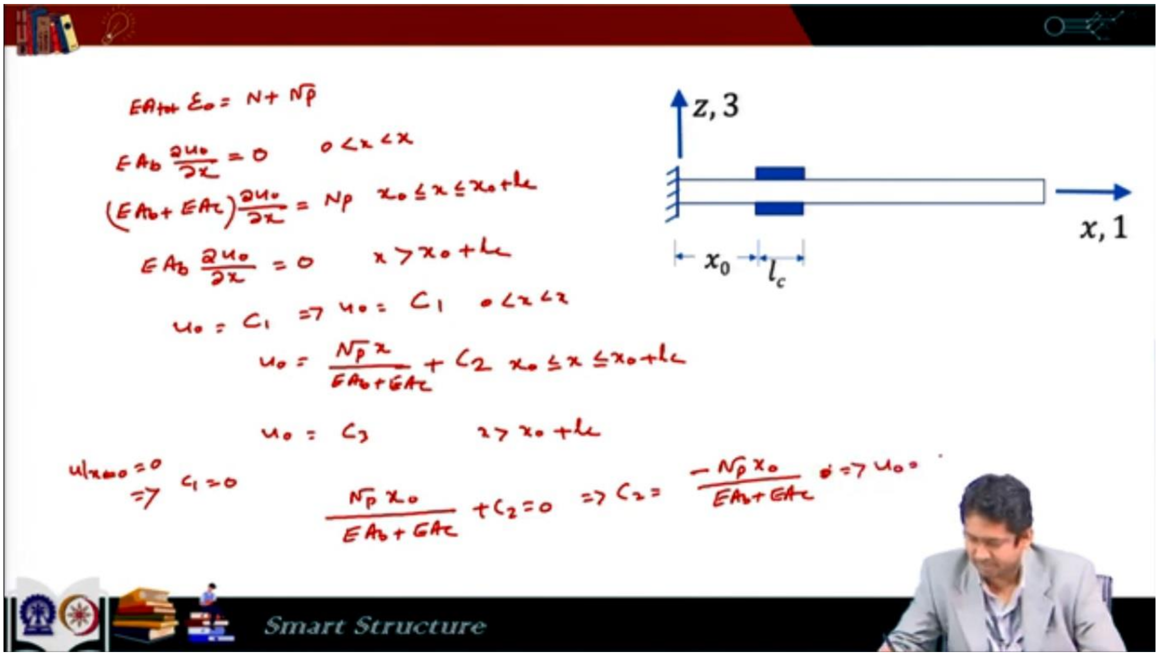

So, let us assume that there is no externally applied load. So, N is 0 if N p is there then how the how do the beam deforms how does the beam deforms that is what we want to see now. So, E A b del u 0 del x is equal to 0 that is the equation which we can write in this zone. Now, in the piezo zone we can say E Ab plus E Ac del u 0 by del x is equal to N p and here we have x 0 else x is between x 0 and x 0 plus lc.

Now, please understand here E Ac is the has the contribution from both the piezo zone and then again beyond that piezo zone we have E Ab del 0 by del x is equal to 0 here x is more than x 0 plus Lc. So, if you integrate the first equation we get E Ab u 0 is equal to C 1. If we integrate the second equation that gives me E Ab plus E Ac u 0 is equal to N p x it is better that we bring the denominator this side. So, then we can write E Ab plus E Ac N p x by E Ab plus E Ac and there is a constant C 2. Here it does not matter because anyway it is 0 and then in the other side or this here also better observe the constant in this side and call u 0 as just C 1.

So, C 1 by E Ab that is one constant let us call it C 1. And then we have again for the last case u 0 is equal to C 3. So, this is when we have x between 0 and x this is when we have x between x 0 and x 0 plus L C and this is when we have x beyond x 0 plus L C. Now, we have to apply the boundary conditions and the continuity conditions. So, here we have two three constants C 1, C 2, C 3 and we have three conditions the displacement is 0 here the displacement is continuous here.

So, whatever displacement we get from this side it should be same from this side similarly whatever displacement we get from this side it will be same as what we get from this side. So, if you put these conditions it gives me C 1 is equal to 0. So, C 1 is equal to 0 because u at x equal to 0 is equal to 0 that gives me C 1 equal to 0. And then if we apply the condition that displacement is same here and here that gives me that N p x 0 divided by E Ab plus E a c plus C 2 is equal to 0. So, C 2 becomes minus N p x 0 by E A b plus E Ac.

So, this tells me that u 0 is N p divided by E Ab plus E Ac x minus x 0 when x is between x 0 and x 0 plus L c. Similarly when we put the other condition that the displacement is 0 at this side at this side that tells me that u is equal to C 3 is equal to N p by E Ab plus E Ac multiplied by L c when x is beyond x 0 plus L c. So, if we draw the graph now it would look like this if I draw the displacement graph. So, initially the displacement remains 0 then the displacement varies linearly and then the displacement remains 0. So, the value here between this is N p by E Ab plus E Ac multiplied by L c.

$$u = c_3 = \frac{N_P}{EA_b + EA_c} l_c, x > x_0 + l_c$$

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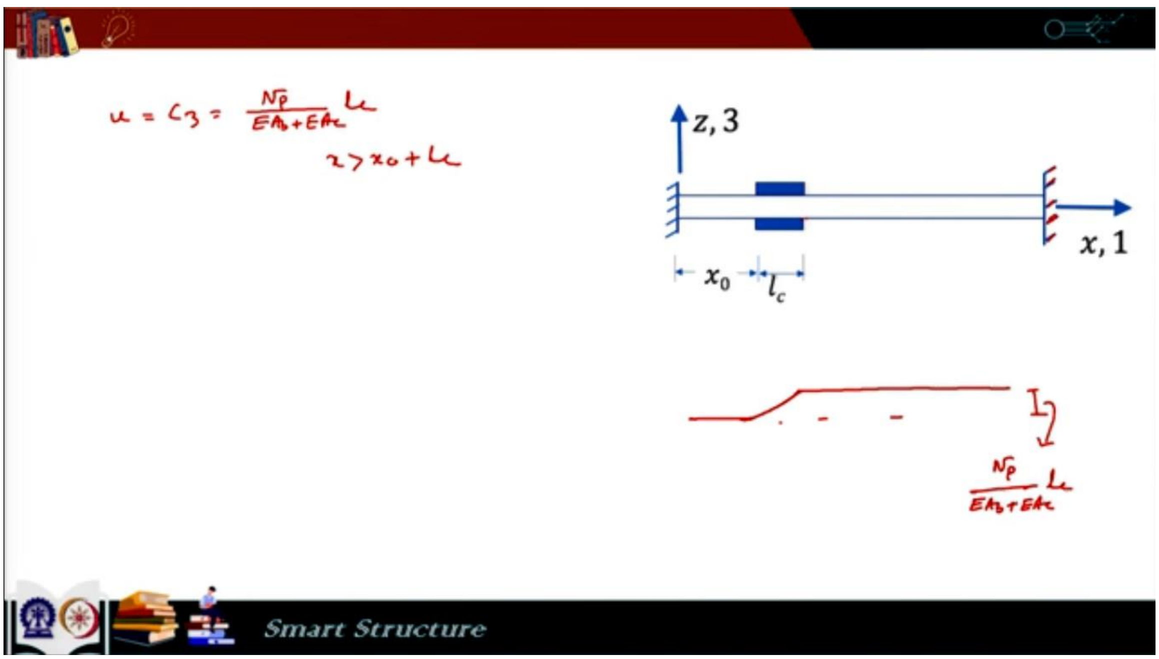

Now if we solve this problem so, where I have another boundary condition at x equal to L if you want to solve this problem it becomes a statically indeterminate problem because we cannot find out the reaction here. So, it is a statically indeterminate problem by degree 1. Now if you want to solve this then what we can do is we can decompose this problem into two sub problems. In the first case let us assume that it is a it is only fixed at one side and we have only this and then in the second case let us assume that it does not have the piezo, but it has the reaction from the support and let us call it R. I mean sorry it has the piezos, but the voltage is not applied here.

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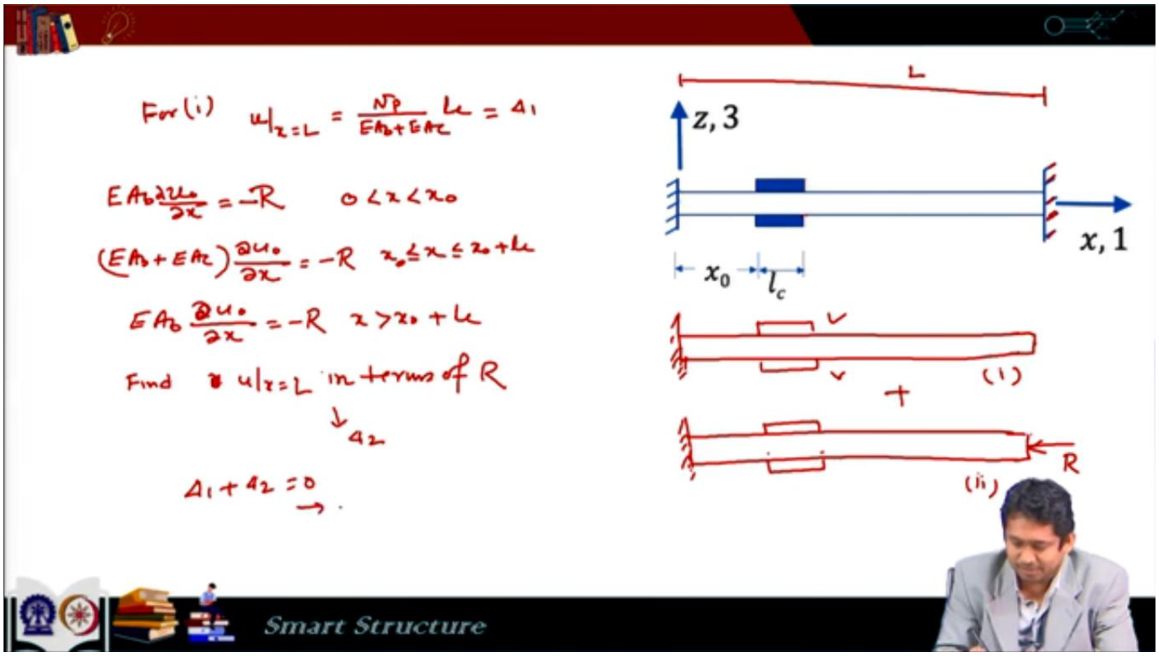

$$u|_{x=L} = \frac{N_P}{EA_b + EA_c} l_c = \triangle_1$$

$$EA_b \frac{\partial u_0}{\partial x} = -R, 0 < x < x_0$$

$$(EA_b + EA_c)\frac{\partial u_0}{\partial x} = -R, x_0 \leq x \leq x_0 + l_c$$

$$EA_b \frac{\partial u_0}{\partial x} = -R, x \rangle x_0 + l_c$$

$$\varDelta_1 + \varDelta_2 = 0$$

So, here we have V V and here we do not have any voltage and then if we superimpose these two results we can find out the displacement here in terms of this R and then we can say that the displacement is 0 that will give me the reaction R once we get the reaction R we can find out all the other things. So, let us call it 1, let us call it 2. So, for 1 we have u at x equal to L. So, let us assume that the beam has a length L. So, u at x equal to L is N P by E Ab plus E Ac multiplied by L C.

We have already solved this problem. Now for this we need to look into the differential equation where we have E Ab del u 0 by del x is equal to minus R because because of this everywhere the normal force is R when it is this and then E Ab plus E Ac del u 0 by del x is equal to minus R when we have x 0 plus L C and again we have E Ab del u 0 by del x it is del is equal to minus R when we have x greater than x 0 plus L C. Again we can solve it problem we can integrate these three separately it will give us three constants we can specify the boundary condition here displacement is 0 we can say the displacement is continuous here we can say the displacement is continuous here that will help us give us that will help us get all the constants after we get all the constants we can find out the displacement here alright. So, let us give it a name. So, find u x L in terms of R and let us give it a name delta 2 and let us call it delta 1.

Now the total displacement should be 0 which means if I combine the displacement that we get from this case plus this case. So, delta 1 plus delta 2 should be 0 actually because we have a support and from here we can find out R and once we can find out R we can solve the problem we can find out the displacement at any point in this bar. Now we will go to another case that is also that is a kind of anti symmetric actuation it is the beam is symmetric in terms of the properties, but in terms of the actuation we can call it anti symmetric. So, if we give a voltage V here we are going to give a voltage minus. Now we will look into another problem where the actuation is anti-symmetric.

Now the beam is symmetric with respect to the properties it has the piezo placed at the same place at the two opposite ends of same properties, but the voltage is actuation is opposite. So, if we give a voltage V here we give a voltage minus V here. Now here we are not concerned whether this V is causing tension or compression here or this minus V is causing compression or tension here whatever it is the formulation remains same only thing is that they are opposite to each other. Opposite means if here the voltage is aligned to the polarization in the same direction here the polarization and voltage in the opposite direction if here the voltage and polarization is in the opposite direction here they are in the same direction. So, needless to say if I get a free strain epsilon p here we are going to get a free strain epsilon minus epsilon p here.

$$EA_{tot}\varepsilon_0 + ES_{tot}\kappa = N + N_P$$

$$ES_{tot} + EI_{tot} = M + M_P$$

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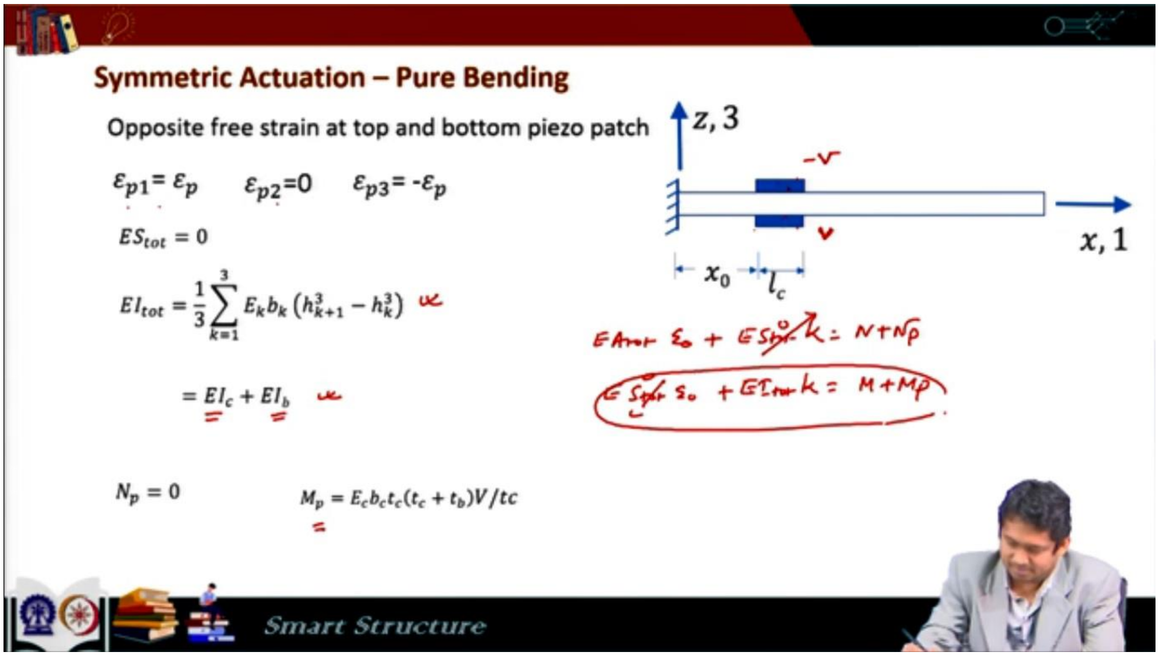

So, epsilon p 1 which is the epsilon p here is equal to epsilon p here we do not have any free strain it is inert here we have opposite free strain minus epsilon p. Again the problem is symmetric with respect to geometry and material properties. So, we can say that E s total is 0 and so among these equations E A total epsilon 0 plus E S total kappa is equal to N plus N_P and E s total epsilon 0 plus E I total kappa is equal to M plus M_P because this is 0 this is 0 these two equations are uncoupled and here because the actuation is anti-symmetric it is not going to cause anything. So, we can solve this problem separately. Now so E a total is not needed here to solve this problem we need E i total and we already know how to find out E i total we have three layers.

So, we just apply the formula and we get this. So, whatever the contribution comes from these two piezoes that is absorbed in E i c and the contribution to E i total from the beam is absorbed in E i b and then m p is m p is just this E c b c t c t c plus t b V by t c. So, here T is a it is a it is a sub subscript c. Now in the form of two parameters psi and theta b psi is the external stiffness ratio E c B v E b B v T b by E c B c T c and theta b is the beam beam thickness ratio. So, beam thickness by thickness ratio beam thickness by actuator thickness T b by T c.

So, this expression is written in a more I would say parameterized form parameterized in terms of theta b and psi as this. We can parameterize as we need for anything and then many a times when the actuator thickness is quite small as compared to the beam thickness. In that case also terms like t c cube appearing are 0 that kind of approximations are also made several times when t c is much smaller than t b. Now we will study the deflection of the beam for this kind of problems. So, here we have the governing different the equation that governs is E i b delta W by del x 2 is equal to 0 when our x is greater than 0 and less than x 0 and we have E i b plus E i c again E i c has contribution from both the piezoes it is del 2 W by del x 2 is equal to M p.

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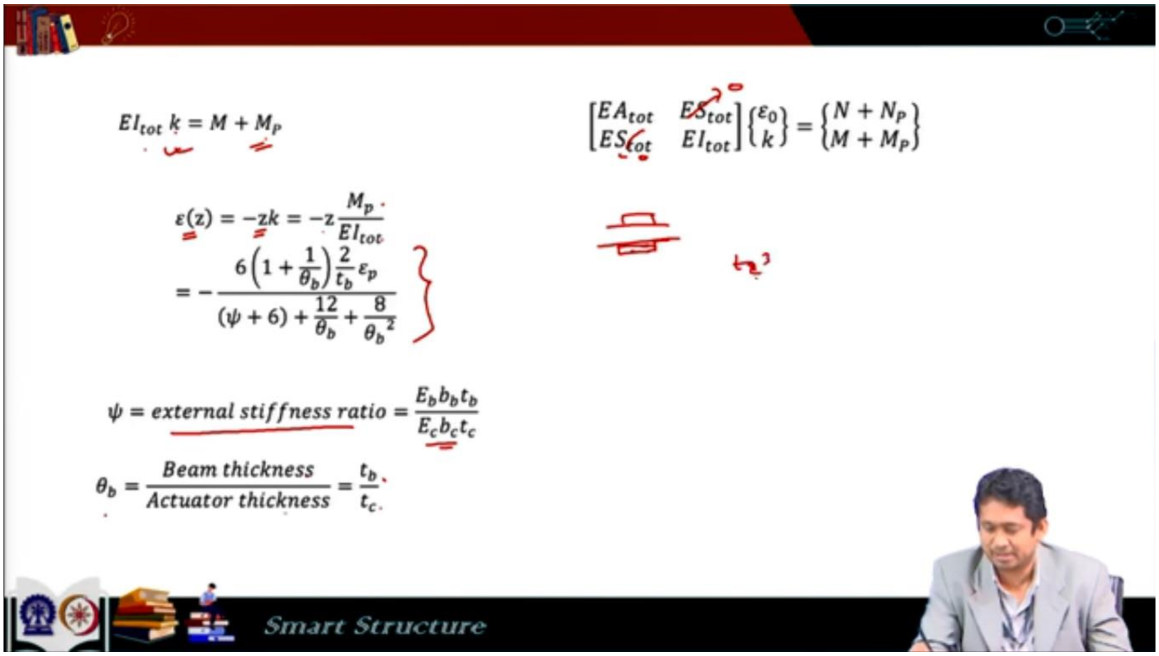

$$t_c{}^3 = 0$$

Let us say that our M p is 0 and this is valid when x is between x 0 and x 0 plus L c and then we have E i b del 2 W by del x 2 is equal to 0 when x is greater than x 0 plus L c. So, if we integrate the first equation it gives me again we do not need to have this E i b in the left hand side. So, we have del W by del x is equal to c 1 then for the second equation we have del W by del x is equal to M p by E i b plus E i c x plus c 1 and for the third equation we have del W by del x is equal to c 3 it is a c 2 constant here for the second case for the third case it is c 3. And now we can again integrate it once more. So, this is when 0 this is when x 0 is between x and x 0 plus L c and this is when this sorry this is last one is when x is beyond x 0 plus L c.

If we further integrate we get c 1 x plus c 2 here we get M p 2 E i b plus E i c x 0 2 x 0 x square by 2 plus c 2 x plus we have already used up to c 4 c 3 c 4 and then we have c 5 and then here we have c 3 x plus c 6 again this is between 0 and x 0 this is between x 0 and x 0 plus L c and this is beyond x 0 plus L c. Now, again we have to apply the boundary and continuity conditions here the boundary condition is slope is 0 here slope should be continuous here slope should be continuous here displacement is 0 here displacement should be continuous here displacement should be continuous here. So, I get 6 equations and I have 6 unknowns to solve c 1 c 2 c 3 and then c 4 c 5 c 6. So, after we satisfy this 6 equations then we this 6 unknowns are determined and after doing that the solution that we get is this. So, delta w by delta x we can write after applying the boundary and continuity conditions.

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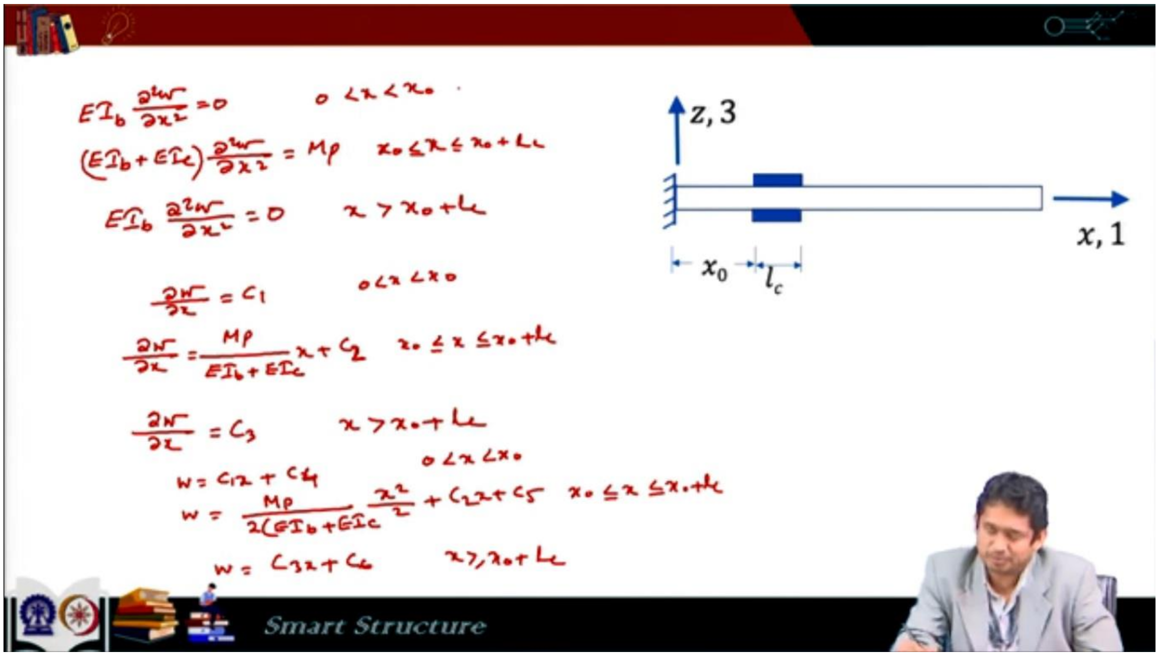

$$EI_b \frac{\partial^2 w}{\partial x^2} = 0, 0 < x < x_0$$

$$(EI_b + EI_c)\frac{\partial^2 w}{\partial x^2} = M_P, x_0 \le x \le x_0 + l_c$$

$$EI_b \frac{\partial^2 w}{\partial x^2} = 0, x > x_0 + l_c$$

$$\frac{\partial w}{\partial x} = c_1, 0 < x < x_0$$

$$\frac{\partial w}{\partial x} = \frac{M_P}{EI_b + EI_c} x + c_2, x_0 \le x \le x_0 + l_c$$

$$\frac{\partial w}{\partial x} = c_3, x > x_0 + l_c$$

$$w = c_1 x + c_4, 0 < x < x_0$$

$$w = \frac{M_P}{2(EI_b + EI_c)} \frac{x^2}{2} + c_2 x + c_5, x_0 \le x \le x_0 + l_c$$

$$w = c_3 x + c_6, x \ge x_0 + l_c$$

So, we get delta w by del x is equal to 0 there is no slope when my x is between 0 and x 0 then we have delta w by del x is equal to M p multiplied by x minus x 0 by E i b plus E i c when my x is between x 0 and x 0 plus L c and then we have del w by del x is equal to M p L c by E i b plus E i c when my x is greater than x 0 plus L c. As far as the displacements are concerned there is no displacement at the beginning then we have x minus x 0 square 2 E i b plus E i c plus M p L c by E i b plus E i c plus m p multiplied by x minus x 0 minus L c. So, if we draw the slope graph it would look like this. So, it is 0 then slope varies linearly and then the slope remains constant and if we draw the displacement graph.

So, it is slope. So, it is I could I would say del w by del x graph and if we draw the displacement graph it would look like this. Initially displacement is 0 and then there is a curvature. So, it we have. So, we would see some curvature here and then this slope and whatever the slope here is this would continue. So, this part is curved part it has a curvature.

So, this is w. So, in this zone from here to here del 2 w by del x 2 is equal to 0 and del w by del x is equal to constant and within this zone del 2 w by del x 2 is equal to constant. So, here I have a constant curvature and it gets some slope here that same slope continues. So, it is a straight line after that. Now we will go to another problem where we have asymmetric actuation and this asymmetric comes because we have put one actuator just at one side. So, we do not have any another any other actuator here.

So, that is why the problem is asymmetric. Now when we have just one actuator this is going to induce both bending and in plane displacement. So, it will have both response and we have to solve it. Now this problem can be solved in two ways. In the first approach what we do is we consider the mid part of the beam which is this this mid part of the beam.

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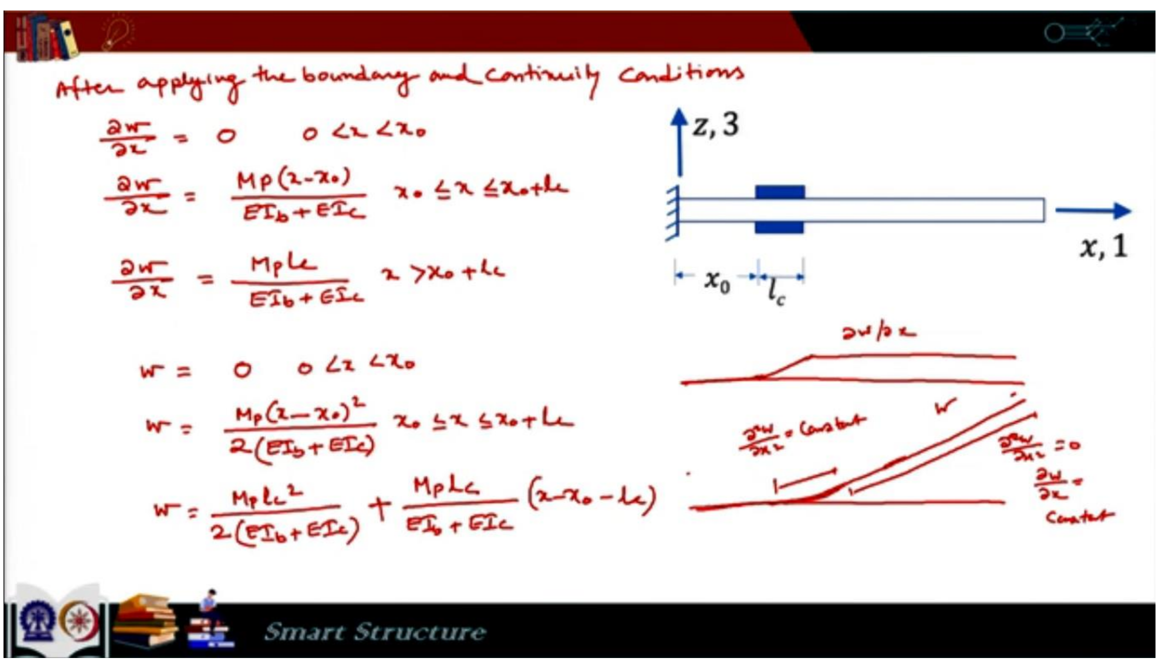

$$\frac{\partial w}{\partial x} = 0, 0 < x < x_0$$

$$\frac{\partial w}{\partial x} = \frac{M_P(x - x_0)}{EI_b + EI_c}, x_0 \leq x \leq x_0 + l_c$$

$$\frac{\partial w}{\partial x} = \frac{M_P l_c}{EI_b + EI_c}, x > x_0 + l_c$$

$$w = 0, 0 < x < x_0$$

$$w = \frac{M_P(x - x_0)^2}{2(EI_b + EI_c)}, x_0 \leq x \leq x_0 + l_c$$

$$w = \frac{M_P {l_c}^2}{2(EI_b + EI_c)} + \frac{M_P l_c}{EI_b + EI_c}(x - x_0 - l_c)$$

Let us consider that as our z equal to 0 line. Now with respect to this point if I find out my E s it would be non zero. E a total as we can find out E b T b E b B b T b. So, area of this multiplied by the corresponding young modulus plus B C T C and multiplied by the corresponding modulus it is remain same. Only thing is that here the contribution is from only one side E s total is this.

So, E s total is this and then we have E i total and N p is this M p is this. So, the free strain in the in the beam is here epsilon p. Then we can write the combined equation which is this we have both N p and M p induced and E s total are non zero. So, we cannot solve the equation separately. We have to solve the coupled equations and after solving the coupled equations we will get epsilon 0 and k both and that will give our displacement.

So, we will have both non zero displacement along axial direction and non zero displacement along the vertical direction. Now let us talk about approach 2. In approach 2 we look at the actual physical neutral axis of the beam. Now because this beam is asymmetric. So, if we draw a cross section here the cross section might look like this.

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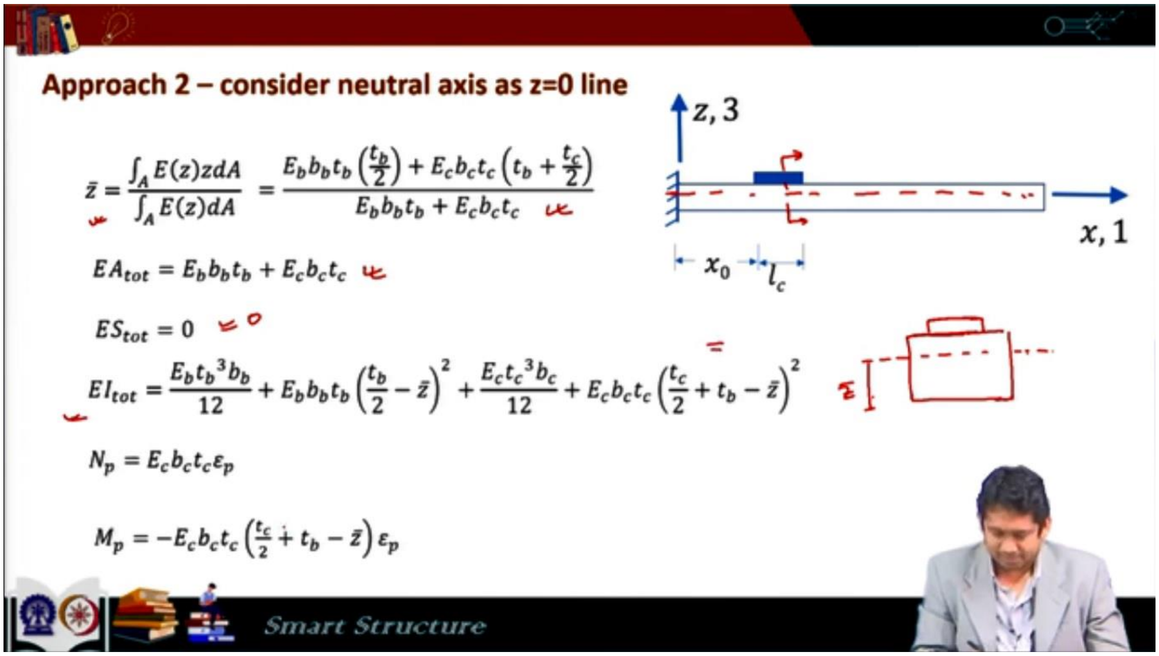

Let us consider that the width of the beam and the piezo are not same just to be more generic in this case the neutral axis shifts up. So, we consider this neutral axis as our z equal to 0 line. Now how I found out the neutral axis? To find out the neutral axis all we do is just evaluate this integral. So, it is we know how to find out the centroid if we forget about this E rest of it is if we put E is equal to 1 rest of it is just the formula for finding out the centroid. Now to find out the neutral axis all we are doing is we are putting a weight and this weight is just the Young modulus.

So, instead of having z d A we have E as a function of z into z d A integrated and here just having d A integrated we have E d A integrated. So, that gives us the position of the neutral axis which is z bar and then accordingly we can find out E A total then we if we take this as our z equal to 0 and then if we find out our E A total it would come to be 0 and then we can find out our E I total and after this we can find out N P and M M P. Now the problem has become like this. So, E s total has become 0 because we have found found out it with respect to the physical neutral axis.

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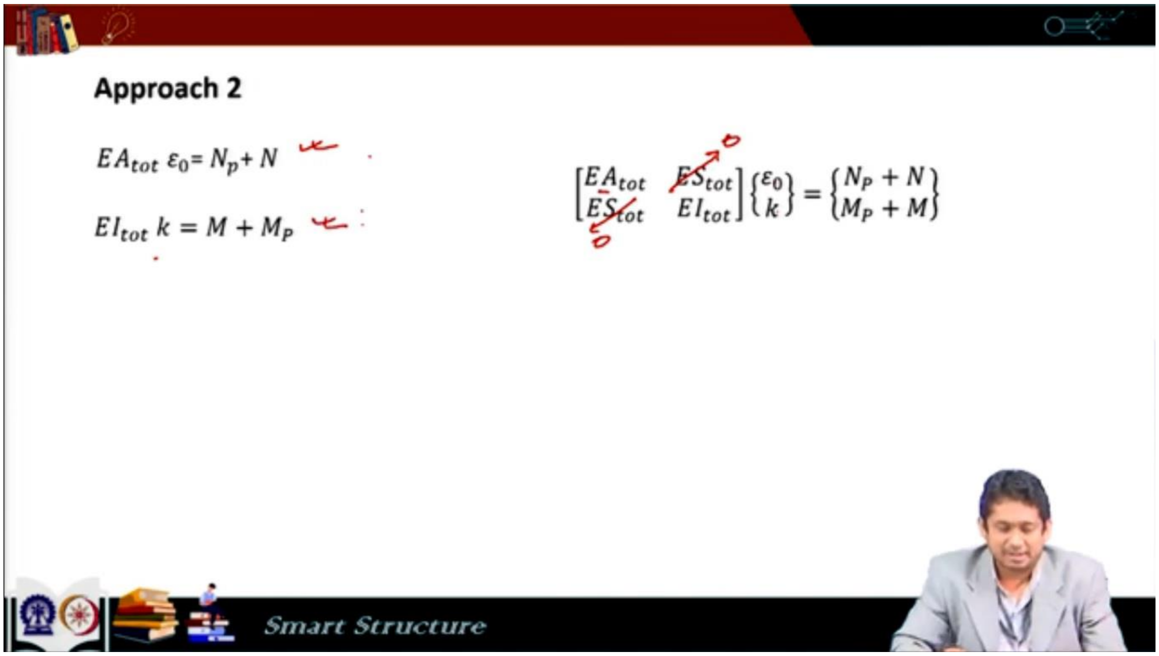

So, E s total is 0. So, this equation E A total multiplied by epsilon 0 plus epsilon 0 is equal to N P plus N and we have another equation E I total multiplied by kappa is equal to M plus Mp. Now these two equations are can be solved separately and the results can be superimposed. So, again results should be same. So, we will find out we will have both axial and vertical displacement, but the advantage is that we do not have a coupled equation. So, solving these two individually individual equations are easier than solving a coupled equation.

So, with that we would conclude this lecture here. We will look into more cases in the future lectures.

Thank you.