Smart Structures

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Week - 03

Lecture No - 13

Induced Strain Actuation-Static Analysis

We are going to start a new topic that is Static Analysis under Strain Induced Actuation. So, we will be dealing with structures like this here. So, we have a beam and the beam will be fitted with piezoelectric patches So, these are the piezo patches and they can be there can be two patches at the two surface opposite surface or there can be one patch and these are all one dimensional problems. So, extension along direction one is possible or a bending with respect to y axis here y axis means a plane perpendicular to this x z plane that is also possible other deformations will not will not be there. So, we will neglect any strain or stress along the y direction our stresses only stresses are along x direction our only stresses are along x direction and the deformations are along. So, stresses are along x direction.

So, this is our direction x which we call direction one also strains are also along the direction x and displacements the beam displaces deforms are along direction x and z. So, we look into this one dimensional phenomena. So, under the when this piezoelectric patches are under some voltage due to the reverse piezoelectric effect it would try to expand our contract accordingly the beam will deform and we will study that. Now, before studying before start starting that let us look into one important concept which we will be using several times and which we have also covered.

So, this is one isolated actuator and as we know that if we have electric field and so, these our direction x or one direction z or three. So, if there is an electric field in direction three then this patch tries to expand or contract depending on where along which direction it is polarized positive three or negative three and what is the electric field direction and what is it d 3 1 it will try to expand or contract. If it is allowed to expand fully then it develops no stress because under that electric field whatever the whatever is its natural length it has been allowed to assume that, but because it is fitted to a structure this structure has some stiffness. So, because of that it will not be able to expand or contract fully. So, depending on the stiffness of the piezo and stiffness of the structure it would come somewhere in between maybe here.

$$(\varepsilon - \varepsilon_p)E = \sigma$$

$$\varepsilon = \frac{1}{E}\sigma + d_{31}E_3$$

$$\Rightarrow \sigma = E(\varepsilon - d_{31}E_3)$$

$$= E(\varepsilon - \varepsilon_p)$$

So, this much of the strain difference that remains the strain difference between the free strain and the strain that is possible. Free strain is what it could have achieved under the free condition when it was not bonded to the structure that we call free strain and epsilon is the actual strain which has been possible which it has been allowed to achieve. So, if I subtract epsilon p from epsilon and multiply it with its Young's modulus that tells me how much is the stress in the in the in the actuator now and this is obvious from the constitutive relations also. We wrote the constitutive relation in this form epsilon. So, now, here because we have only one component of stress and one component of strain we are not dealing with any several components of stress and strains we have only one component.

So, we are not putting any subscript now only subscript p means its free strain other than that there is no subscript strain means it's in x direction normal stress, stress means its in x direction strain means its in x direction normal strain and stress means its in x direction normal stress. So, if I write the constitutive relation in a simplified way for one dimension it would look like this which we have been doing and on solving for sigma we get and this d 3 1 multiplied by E 3 is the amount of strain that is that would have been here if the actuator was free and that is the free strain minus epsilon. So, this is what it is. So, we will be using this several times in our analysis in this one dimensional form. Now, while doing this analysis we will be using two approaches one is Euler Bernoulli beam based approach and another is Bloch force based approach.

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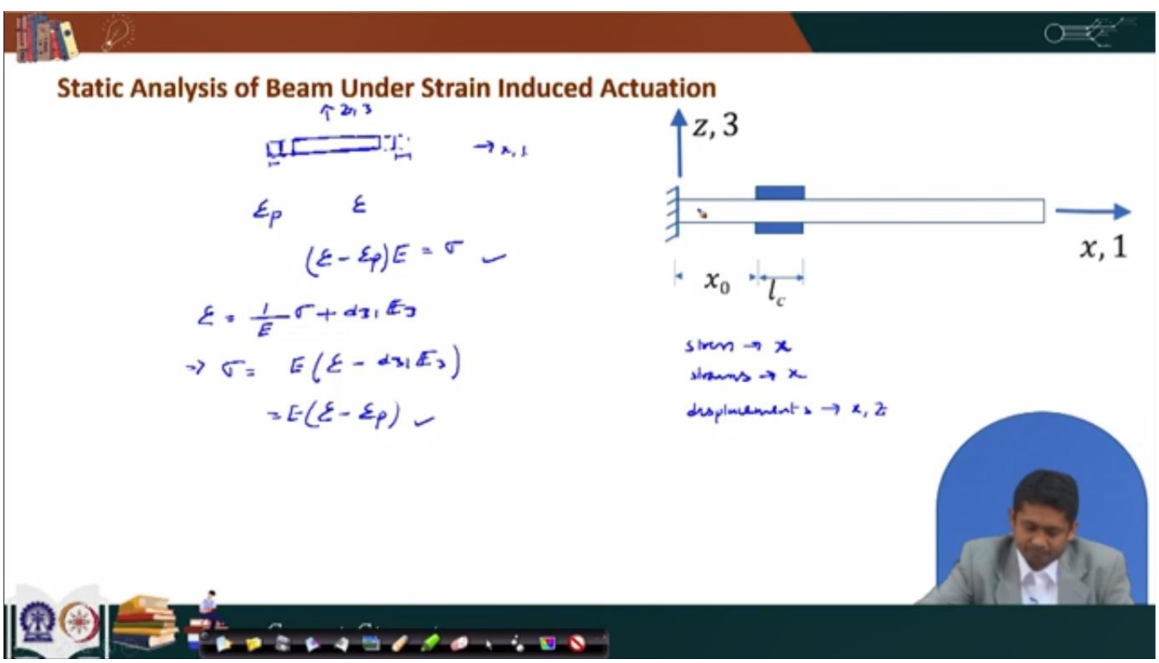

So, let us start with the Euler Bernoulli beam based approach in the deformation the effect of shear is not incorporated the assumption is that the entire deformation is due to bending only. Now, if we look at it we can see that the top fiber here after bending it has compressed. So, its under compression and the bottom fiber here it's under tension because it has expanded. So, the top fiber is under extreme compression and the bottom fiber is under extreme tension and also the model the theory says that the thickness of the beam remains same. I mean the cross section of the beam if I look at the cross section from this side it does not distort and also the plane section remains plane before and after bending.

So, it is plane here. So, this also remains plane it is not a curve it is a plane and this is also a straight line. So, we can see that there is a maximum amount of shear here sorry maximum amount of compression here. So, which we can draw here on the undeformed beam. So, we can see that it is a compression and then we can see that there is a max at the bottom the tension is maximum.

So, it is tension in between them because it is a linear variation the strain variation has to be linear and accordingly it has to vary the variation is linear. Now, there is a point here which does not experience any strain and that we call neutral axis. So, if I see the neutral axis here it is here. So, that length here should be equal to length here. So, it is not under any strain and if the section is symmetric both in terms of material and geometry the neutral axis here is just at the middle.

So, if I call this as x axis. So, my x is here and then if I denote as x if I from this if I measure z upward then at any z the displacement along x is if I denote this as u at any z the displacement along x is u is equal to minus of z into theta where this is my theta. Why minus because as we move up u is negative as we move as we go down when z is negative u is positive. Now, the question is what is this theta? Now, let us see that this deformed curve it resembles the deformation along z axis. So, if I say that w is my displacement along z and then if I draw a tangent to this curve and if I look at the slope here that slope is just del w by del x because this curve is nothing, but the w curve it is signifying displacement along z direction and then if I take a slope and if I take a tangent and measure the slope that is del w by del x.

Now, if this angles remain 90 degrees then we can say that this theta is equal to this angle. So, this is also theta because it is 90 degree. So, because we have neglected shear we can say it is 90 degree and in that I mean in the deformed beam and that is why it is theta is equal this angle and this angle are same and this angle is nothing, but del w by del x. So, we can say that theta is equal to del w by del x. So, we can write u is equal to minus z multiplied by del w by del x.

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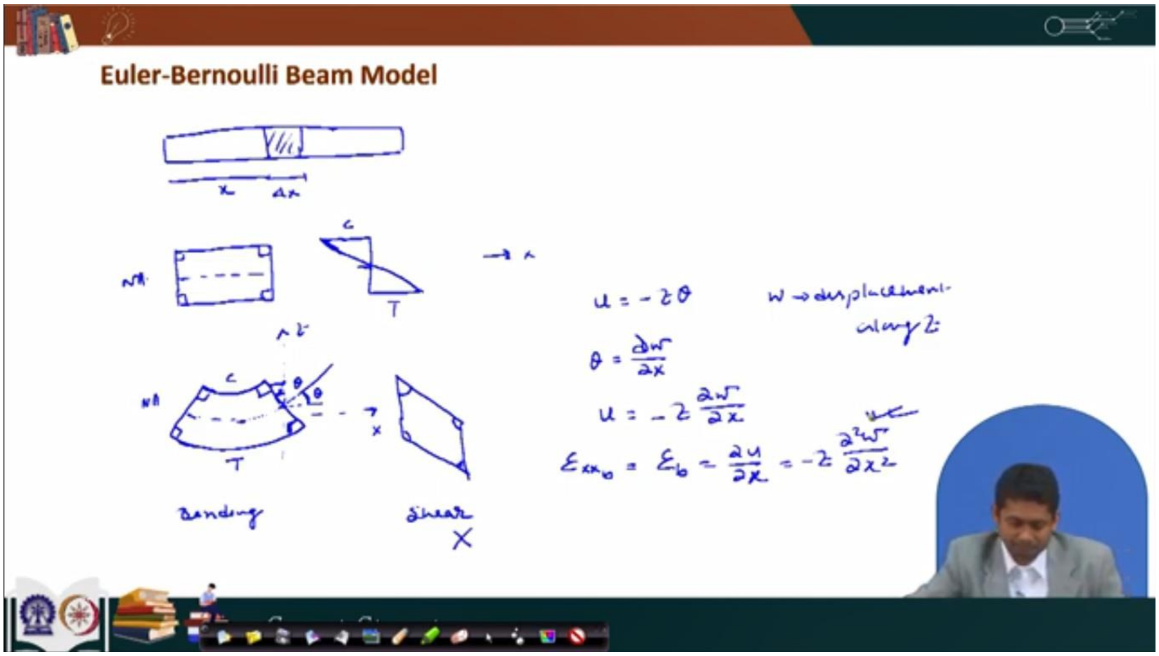

So, naturally we can say that strain along x direction which we can say epsilon xx due to bending now we have only one strain components we are now getting rid of this xx. So, you call it epsilon just b, b is because of bending is equal to del u by del x and that is equal to minus of z into del 2 by del x 2. So, using the Euler Bernoulli beam we get this relation and this relation we would use in our formulation. Now, if the beam apart from bending has some axial deformation also. So, the initial beam was this delta x and it has some axial displacement.

So, it can be a contraction or it can be an expansion, but it is a pure axial displacement in that case the displacement along z. So, if it is x if it is z displacement along z along x along z at all the points along z remain same. So, now, u is just u 0 and it is just a function of x. So, this point can have a separate displacement from this point, but this about along z the displacement u 0 remains same. So, u is just a function of x which is and we call it u 0.

$$u = -z\theta$$

$$\theta = \frac{\partial w}{\partial x}$$

$$u = -z\frac{\partial w}{\partial x}$$

$$\varepsilon_{xx_b} = \varepsilon_b = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

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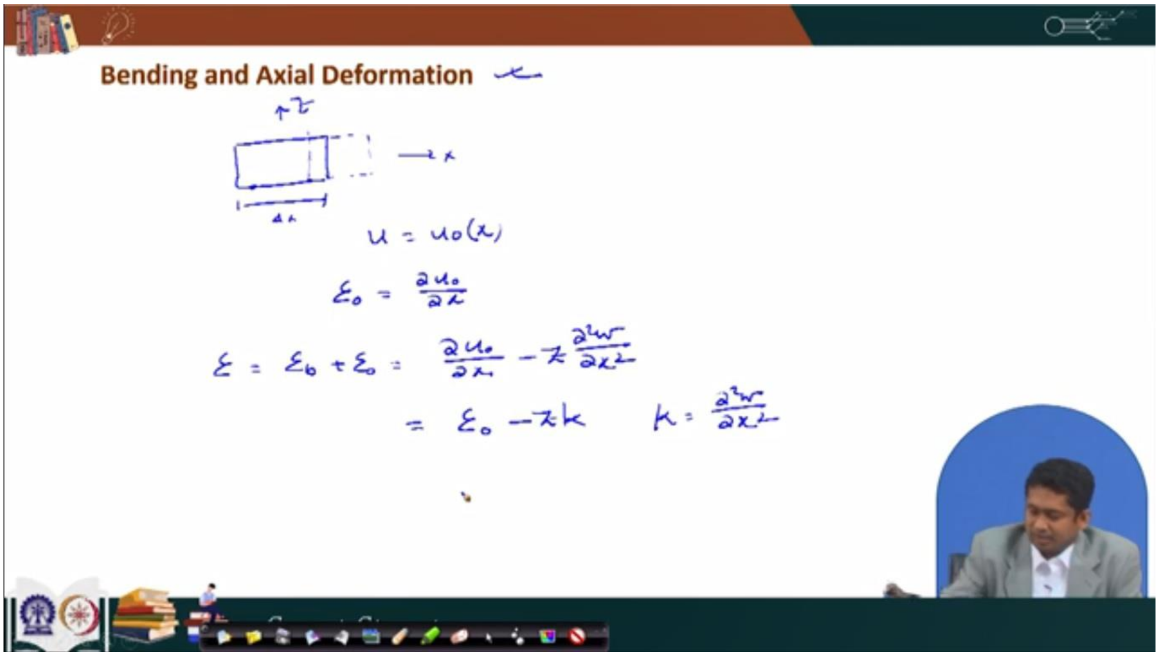

So, normal strain epsilon 0 due to only expansion is del u 0 by del x which is when there is only expansion or contraction a pure axial displacement. So, for bending and axial displacement combined we have epsilon is equal to epsilon b plus epsilon 0 and epsilon 0 is del u 0 by del x and epsilon b is minus of z into del 2 w by del x 2 and that we can writes in a simplified form as epsilon 0 minus z into kappa where we say kappa as del 2 w by del x 2. Now, this expression we will be using several times in our formulation. So, we will start with the Euler Bernoulli beam based formulation. So, this is the basic formulation we have seen that strain is epsilon strain is epsilon is equal to epsilon 0 minus z into kappa and if the material is active material if it is a piezoelectric material then we subtract the free strain from the strain and if I multiply this quantity with the elastic modulus that gives us the stress and this free strain is nothing, but electric field which is v by tc.

$$u = u_0(x)$$

$$\varepsilon_0 = \frac{\partial u_0}{\partial x}$$

$$\varepsilon = \varepsilon_b + \varepsilon_0 = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$
$$= \varepsilon_0 - z\kappa$$

$$\text{where } \kappa = \frac{\partial^2 w}{\partial x^2}$$

So, E 3 is equal to v by t c as we saw and if I multiply this quantity with the d 3 1 that gives us the free strain and we get the stress here. Now, along the cross section if I integrate the stress that gives us the normal force at the cross section. Now, stress is this we just put it here and after integrating over the cross sectional area this can be written as this E a total multiplied by epsilon 0 plus E s total multiplied by kappa minus N p total. Now, let us look into this integral E multiplied by epsilon 0 over dA. Now, epsilon 0 is not a function of the cross section along the cross section no distortion occurs.

$$\varepsilon = \varepsilon_0 - z\kappa$$

$$E_3 = \frac{v}{tc}$$

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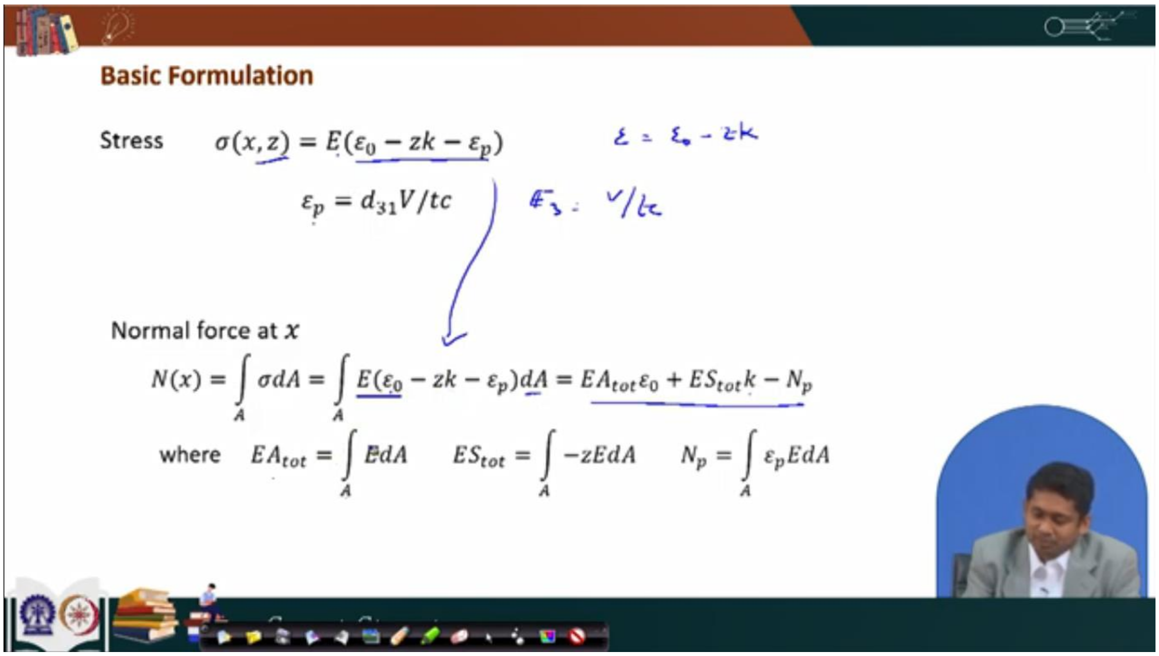

So, strain is function of x only. So, it can be taken out of the cross out of this area integral then we are remaining with E dA integrated over the area A. So, E dA integrated over the area A gives me E a total. Similarly, here we have E multiplied by minus zk integrated over dA here also kappa can come out of the integral. So, we have minus minus Z e integrated over area and that is gives that gives us E S total.

So, we carry a minus sign in the definition of E S total and in this integral we have epsilon p E dA and that is our N p that means, the normal force due to the free strain. So, this is one equation and then we find out the bending moment. To find out bending moment we multiply sigma with minus Z and integrated over integrate over the area which can be seen here we just we are just expanding sigma in terms of its constituents and we can write it as E S total multiplied by epsilon 0 plus E I total kappa minus M p. E S total we already defined and if I look at this quantity minus Z kappa minus Z e then we have Z square e integrated over the area that gives us E I total and we define minus Z e epsilon p integrated over the cross sectional area as M p bending moment corresponding to the free strain. So, if I combine these two equations then we can write the combined equation in this form.

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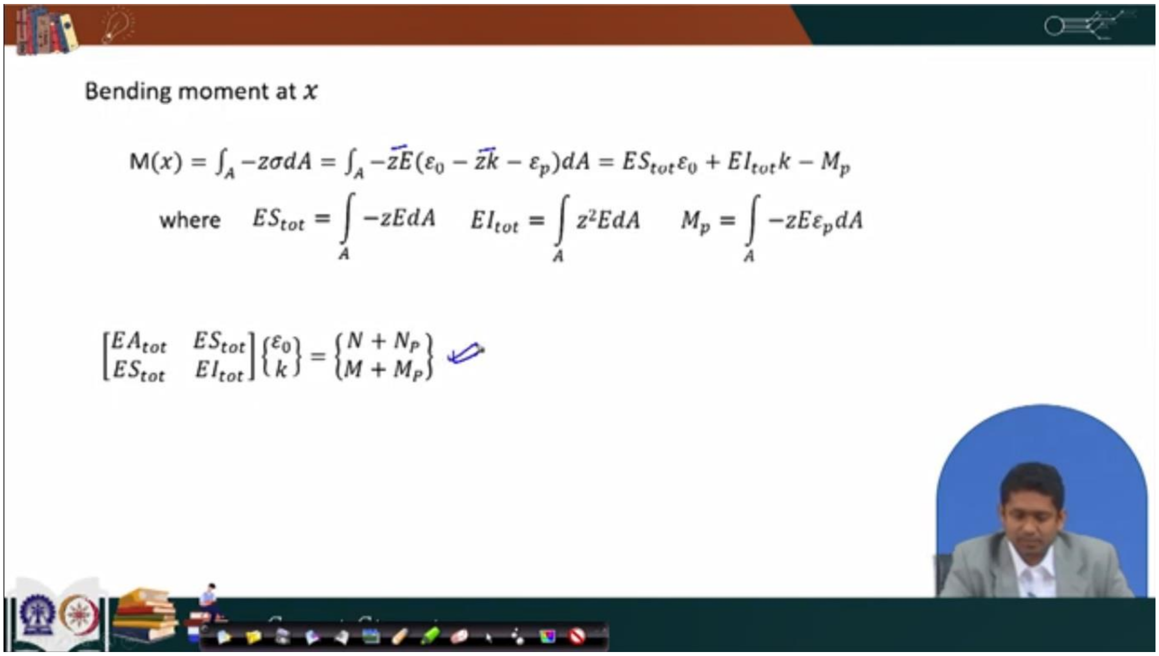

So, this equation we will use again and again for different sub cases that we deal with. Now, that we have the basic formulation we will look into different cases. Before we will go to the cases let us see what if our material what if our beam consists of several layers. So, here is a beam that we have and it consists of n number of layers. So, this we can call layer 1, layer 2 all the way up to layer n and the Z coordinate of it this we may call h 1 the Z coordinate of it is h 2 and this goes on till h n plus 1.

Now, here within each layer the material and geometric properties in a layer does not vary. So, it does not change. Two layers can have different properties. So, with that if we integrate we can break down the integral into several pieces. So, if I want to find out E A total.

So, it is E T a over the entire cross-sectional area then we can find out the we can integrate it over each layer because each layer has same property and we can just add up. So, we divide it into n number of layers. So, summation k goes from 1 to n and within each layer the integral becomes this E k B k multiplied by h k plus 1 minus h k. So, at the kth layer the at kth layer b is width and E k is elastic modulus. Similarly, we can find out E S total and after being integrated it gives me this expression and E i total gives me this expression after the integration.

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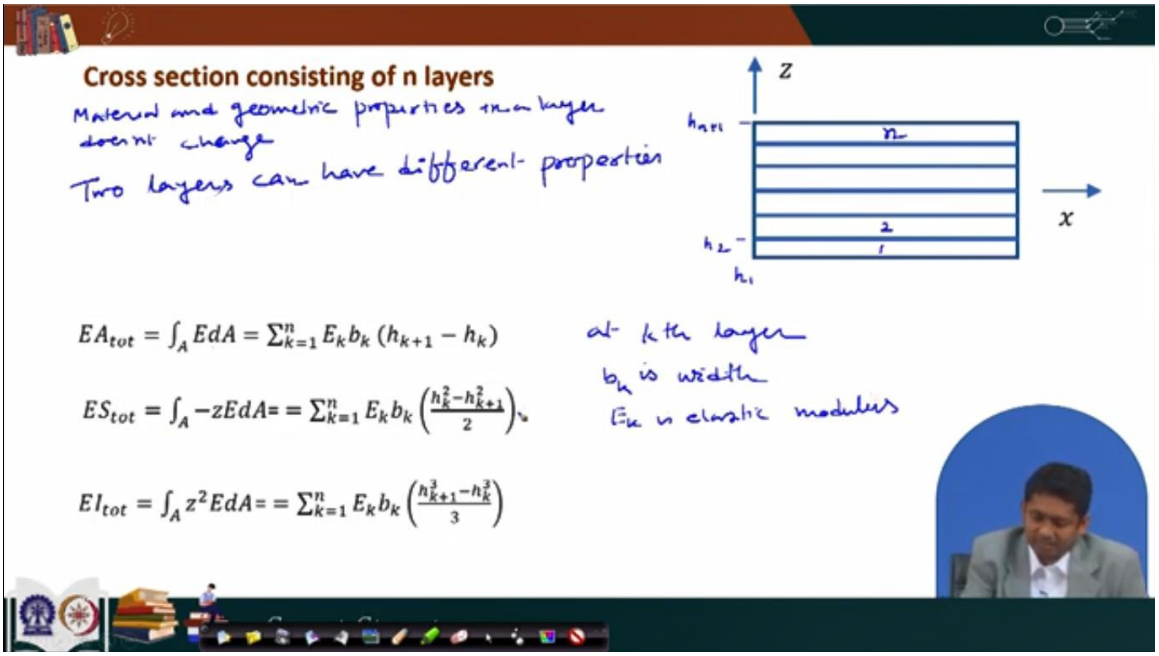

Now, let us look into one case which is a symmetric actuation case. Now, in symmetric actuation means we have two piezoelectric patches and under these two patches are identical patches and they are placed as just the two opposite surface and under this symmetrical actuation we have a sub case called pure extension or it can be contraction also. So, it is basically pure extension or contraction or we can say pure axial effect. Now, this is possible when these two piezoelectric patches are actuated in a similar manner. Now, what do I mean by similar manner to understand this let us see here.

So, suppose we have a piezoelectric patch here these our dimension 1, these our direction 3. Now, it is polarized in direction 3. So, suppose it is polarized in this direction. Now, to have this kind of polarization during polarization the poling voltage must have been applied in this form then only the polarization direction can be this. Now, say while doing the actuation we are also giving the voltage in this fashion we are keeping this side as positive then it is we will term this as a positive voltage.

So, positive voltage means the voltage that we are giving now is similar in nature as the poling voltage. Poling voltage means the voltage that was given to it while poling which resulted in this kind of polarization. If it is reverse for example, if it is polarized in this direction then these our positive voltage. On the other hand if it is polarized in this direction and if this side is negative then these are negative voltage or if it is polarized in this direction and if this side is negative then it is a negative voltage. Now, when these two piezoelectric patches are placed at the two sides suppose it is polarized in this direction it is polarized in this direction and that is how they are placed then if I put plus here positive voltage here that means, it is a positive symmetric actuation I mean the actuation is such that it causes a pure axial displacement pure axial effect or if it is negative and if it is negative our applied voltage then also it is going to cause a pure axial effect.

So, we will do our actuation in this form we will keep either both side under positive voltage or both side under negative voltage. Now, whether it is resulting in an in an expansion or contraction that depends on its D 3 1 and that, but that is not going to change our mathematics behind it. So, only thing that we need to know are that the both side is under same type of actuation. So, when it is the case the free strain here and the free strain here are going to same. So, epsilon p 1 which is free strain here is epsilon p this part is the beam which is the inert part.

So, we have a host this beam this we also call as a host structure because this is hosting the piezoelectric patches and they are not active like piezoelectric material. So, they are inert inert not active. So, it does not have any free strain. So, the free strain is 0 here and epsilon p epsilon p 3 means this material it has the same free strain epsilon p and the free strain is d 31 V by tc. Now, whether the V is positive or negative we are not going to go there it is just a V and it is same.

So, epsilon so, we have defined the what it what it means by actuating in the same fashion to cause a pure extension or contraction and it results in this kind of free strain distribution. Now, it has 3 layers. So, h 1 which means the z coordinate of the bottom most interface that is h 1 is here. So, which is this now in this configuration we say t b as the dimension I mean depth of the beam t c is the depth of the piezo and t c is the depth of the piezo both the piezo are same.

So, they have same T c. So, if I measure our z is equal to 0 here this lines are z equal to 0 which means this the bottom most the extreme bottom end of this bottom piezo is minus t b by 2 plus t c this interface is minus T e by 2 which is h2 h3 is this t b by 2 h4 is t b by 2 plus t c the upper end of the upper piezo. Now, using the formula that we have already discussed we can find out E A total E A total is just the area of summation of the area of this constituents and that gives us 2 E c tc B c plus E b tb B b. Now, this comes from the piezoelectric part 2 piezoelectric patches we call it E a c and this comes with the beams we call it E a b. So, this should be actually E suffix b. So, we should write this as E Asuffix b and similarly here also it should have been suffix t.

Now, N p is epsilon p E integrate over area and that also gives us the same thing. So, these are all suffix. So, it can be written in a better form as this 2 E c bc tc v by tc and M p is 0 because epsilon p is same here epsilon p is same here if I integrate minus sigma z over d A we will see that the integral becomes 0 here. So, M p is 0. So, finally, because M p is 0 and we will also see that E a s total comes to be 0 because the section is symmetric.

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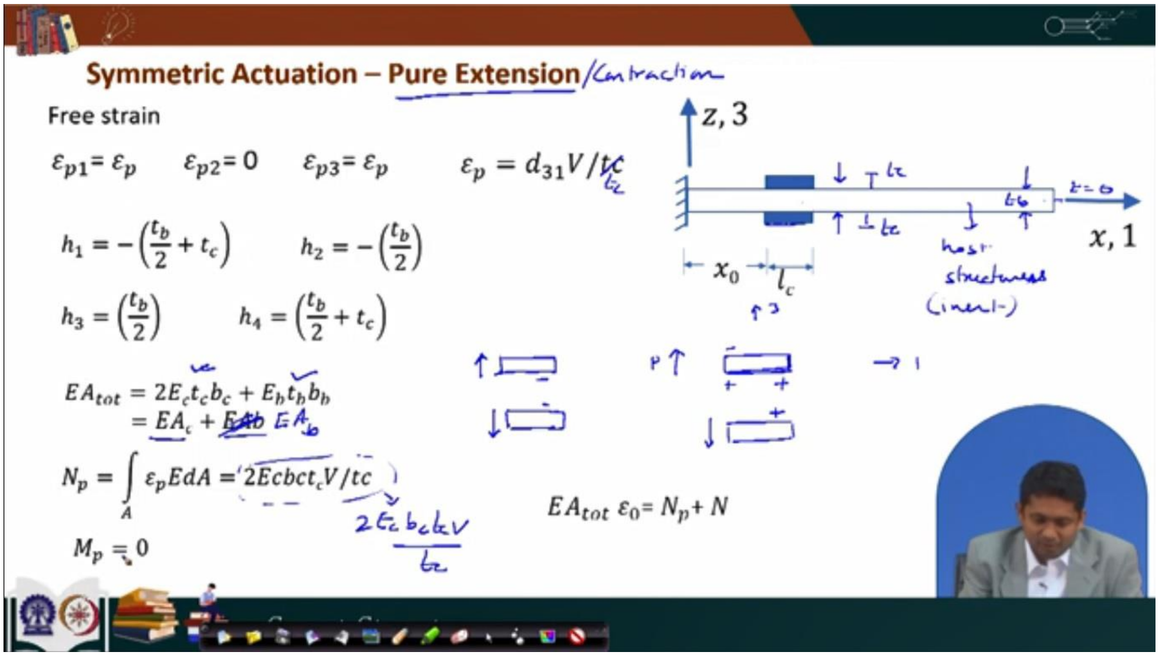

So, it would result in E S total to be 0 if I apply the same formula and if I evaluate the E S total it will be 0. Now, E a s total is 0 and M p is 0. So, if I look back to this equation when this term becomes 0 and when this quantity becomes 0 we have just one equation E a total multiplied by epsilon is equal to N plus N p and this equation can be independently solved. So, that results in only one equation for this case for the pure extension or contraction case which is this and then after solving this we can find out our epsilon 0. Now, N p is the force corresponding to the free strain and N is the force corresponding to any applied load externally applied load.

Now, these phenomena can be represented graphically also. So, if I look at this piezo and structure combination and the free strain is here these are the free strains at the top and bottom piezo's the beam does not have any free strain. So, this is epsilon p and suppose the actual strain is this which is same everywhere then if I subtract epsilon p from epsilon 0 this results in this. So, this is epsilon 0 minus epsilon p.

So, we are subtracting this graph from this graph. So, now, if we multiply this with the corresponding young moduli of the constituents it gives us a stress distribution which looks like this. So, sigma is equal to young modulus multiplied by epsilon 0 minus epsilon p and because there is no externally applied load. So, we will see that if I find out the force corresponding to this and this. So, and if I find out the force corresponding to this on being added it becomes 0 because the direction because the direction of these two forces is opposite to this force. So, if I integrate see if I find out the force here corresponding to this sigma if I find out the force corresponding to this sigma if I add it up that should be equal to the force corresponding to this sigma when there is no externally applied force.

$$\sigma = E(\varepsilon_0 - \varepsilon_p)$$

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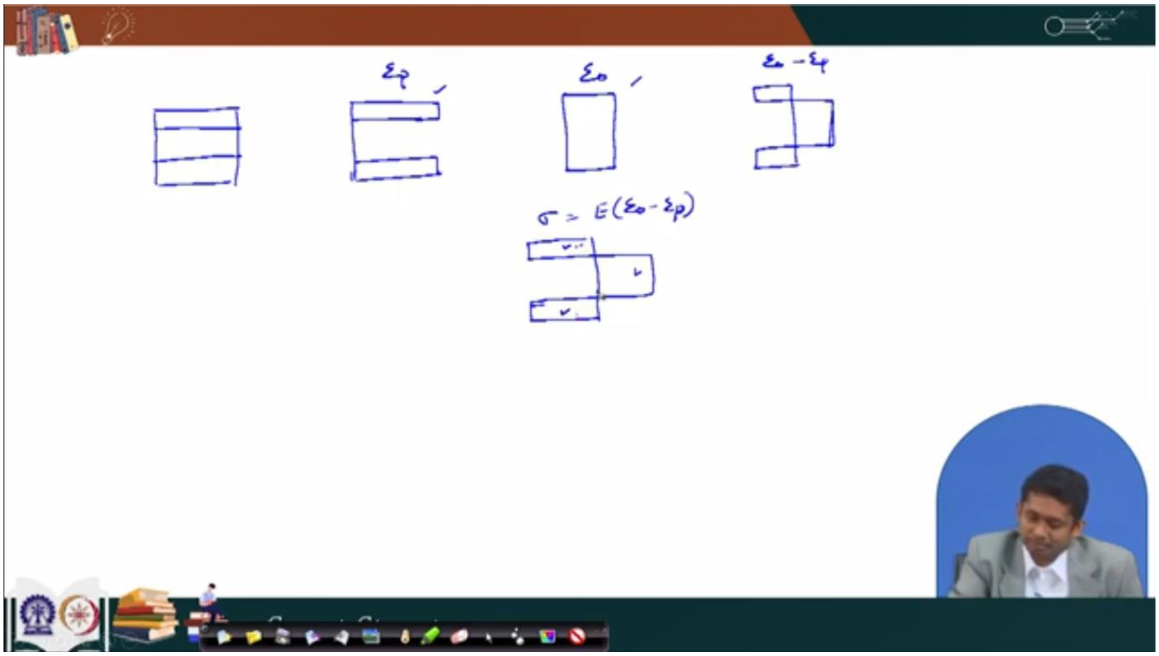

So, with that I would like to end this lecture here in the in the next in the next lecture we will start from here and move forward and then we will gradually move into other cases as well.

Thank you.