Smart Structures
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Week - 02
Lecture No - 12
Numerical Problems and Solutions

So, far we have talked about constitutive relations of piezoelectric materials and related items. We started with basic electrostatics and then we looked into the conservation laws and from there we derived the constitutive relations of the piezoelectric materials. And we looked into various forms of the constitutive relations and their inter relations. And then we talked about the sensors and actuators in a simplified way.

Today we will solve some numerical problems from these topics.

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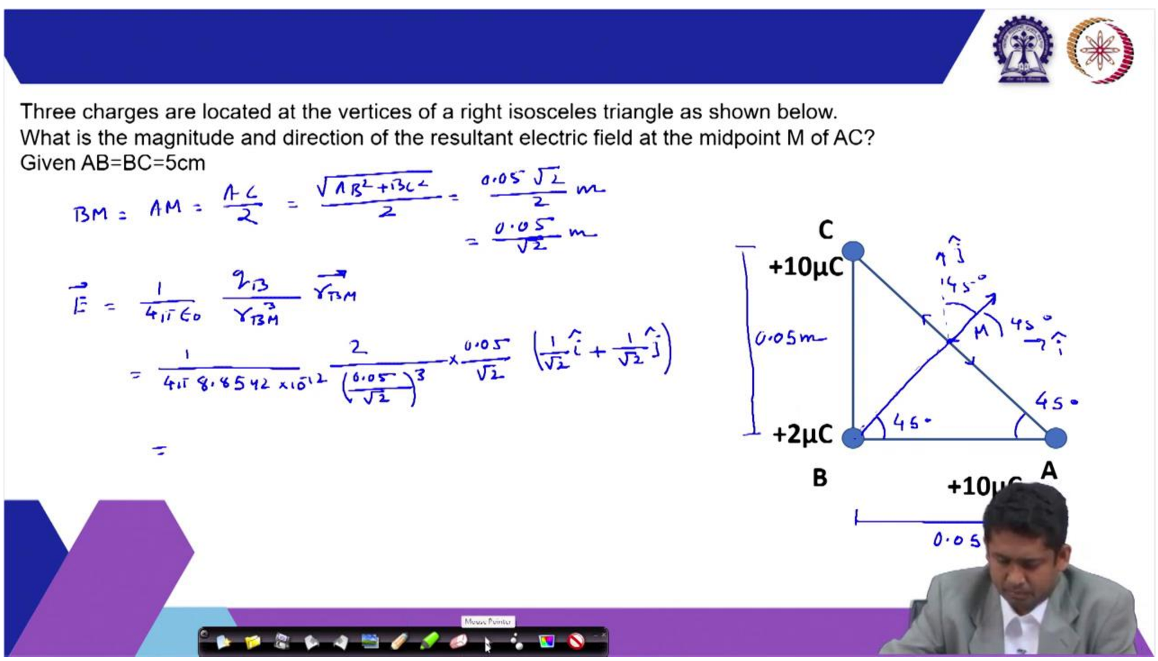

And these problems will be from basic electrostatics and some problems where the constitutive relations of the piezoelectric materials will be used. Here is the first problem. It says three charges are located at the vertices of a right isoscale triangle as shown below. What is the magnitude and direction of the resultant electric field at the midpoint M of AC. Given AB and AC are 5 centimeters.

So, here these dimensions are 5 centimeter and we have to find out the electric field here at the mid of AC and that point is designated as M. Now, this is the midpoint. So, the distance from here from this M to A and M to C are same. And if we look at the electric field at M cause due to A, it is directed towards this direction and if I look at the electric field at M cause due to C, it is directed towards just the opposite direction and their values are same. So, they will cancel each other. So, the net result of C and A at M is 0. So, the only thing that would contribute is our charge at point B and that would be directed towards this direction. So, if I join B and M and if we extend it, that extended line is our direction of electric field. Now, this is an isosceles triangle. So, we have 45-degree angle here and we have 45-degree angle here because our ABC is a right angle.

So, we can say that BM is equal to MA. So, our BM is equal to AM which is equal to AC by 2 and AC can be found out from AB and AC which is AB square plus BC square root over by 2 and that gives me root over 2 by 2 meter or we can again write 0.05 by root over 2 meters. So, electric field E is equal to 1 by 4 pi epsilon 0 multiplied by qB the charge at B divided by r BM. If I want to denote this as a vector then we have to write r BM cube and multiplied by r BM as a vector and that would give us 4 pi epsilon as you know 8.8542 multiplied by 10 to the power minus 12. So, 2 by 0.05 divided by root 2 cube then multiplied by 0.05 divided by root 2 and we have to write the unit vectors. So, this line makes an angle 45 degree here and a line 45 degree here. So, the unit vectors are 1 by root 2 i plus 1 by root 2 j where i is our unit vector along this direction and j is our unit vector along this direction. Now we can simplify this entire expression and find out our E as a electric field as a vector.

$$BM = AM = \frac{AC}{2} = \frac{\sqrt{AB^2 + BC^2}}{2} = \frac{0.05\sqrt{2}}{2} = \frac{0.05}{\sqrt{2}} m$$

$$\vec{\mathbb{E}} = \frac{1}{4\pi\varepsilon_0} \frac{q_B}{{\gamma_{BM}}^3} \vec{\gamma}_{BM} = \frac{1}{4\pi \times 8.8542 \times 10^{-12}} \frac{2}{(\frac{0.05}{\sqrt{2}})^3} \times \frac{0.05}{\sqrt{2}} (\frac{1}{\sqrt{2}} \hat{\imath} + \frac{1}{\sqrt{2}} \hat{\jmath})$$

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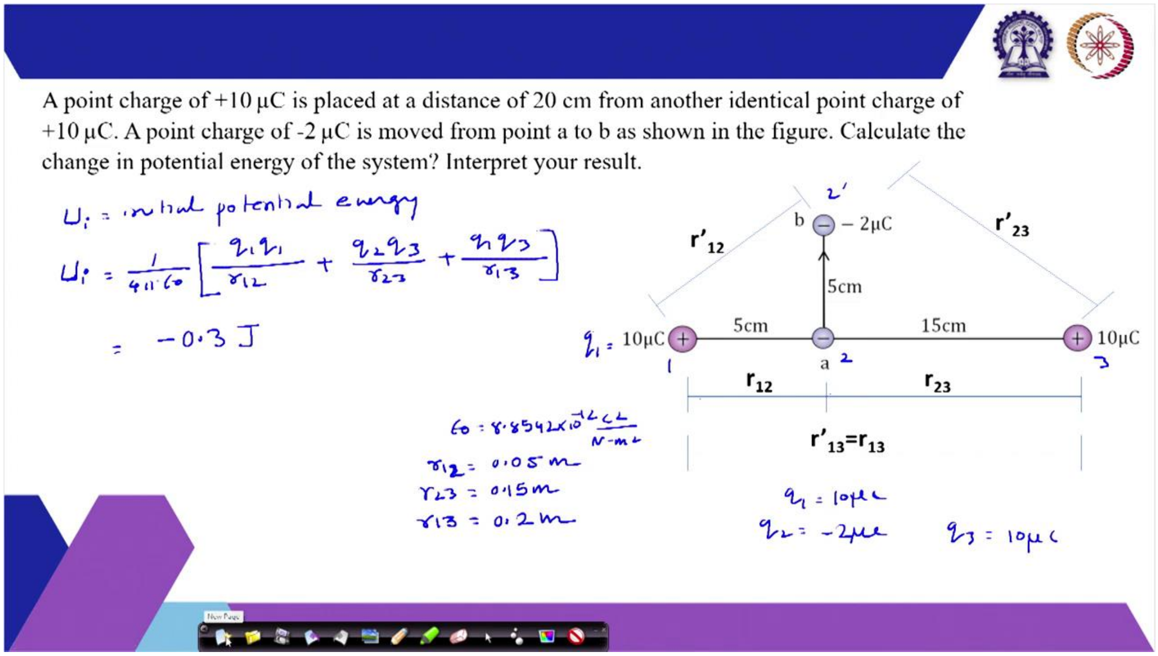

Now we will look into the next problem. The next problem says a point charge of 10 micro coulomb is placed at a distance of 20 centimeters from another identical point charge of 10 micro coulomb. A point charge of minus 2 micro coulomb is moved from a point A to B as shown in the figure. Calculate the change in potential energy of the system and interpret your result. Now to do this, we need to designate some distances here. So, let us call this point as 1, 2 and 3 and the distance from point 1 to point 2 is r12 and point 2 is point 3 is r 23 and point 1 to 3 is r 13. When it is moved to this point let us call it 22 prime. So, it is r 12 prime and it is r 23 prime and this since in the deformed system, these two points does not move.

So, r 13 is also r 13 prime. Now we can find out the potential energy at the beginning as Ui we say initial potential energy and that can be found as if I take 1 by 4 pi epsilon outside then we have q1 q2 divided by r 12 plus q2 q3 divided by r 23 plus q1 q3 divided by r 13. Now q1 is discharge, q2 is this. So, q2 is minus 2 micro coulomb which was initially here and then it moves here and q3 is 10 micro coulombs, q1 as we said 10 micro coulomb r 12 we know 5 centimeter r 23 shown here 15 centimeter. So, r 13 is 0.05-meter, r 23 is 0.15 meter and it is r 12 is 0.05-meter, r 23 is 0.15 meter and r 13 is 0.2 meter. So, once we put all these values and epsilon 0 as we know it is 8.8542 into 10 to the power minus 12 coulomb square by Newton meter square. So, once we put all these values in this expression finally our potential energy comes to be minus 0.3 Joule. Now we will find the same quantities in the deformed configuration.

$$U_i = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right] = -0.3J$$

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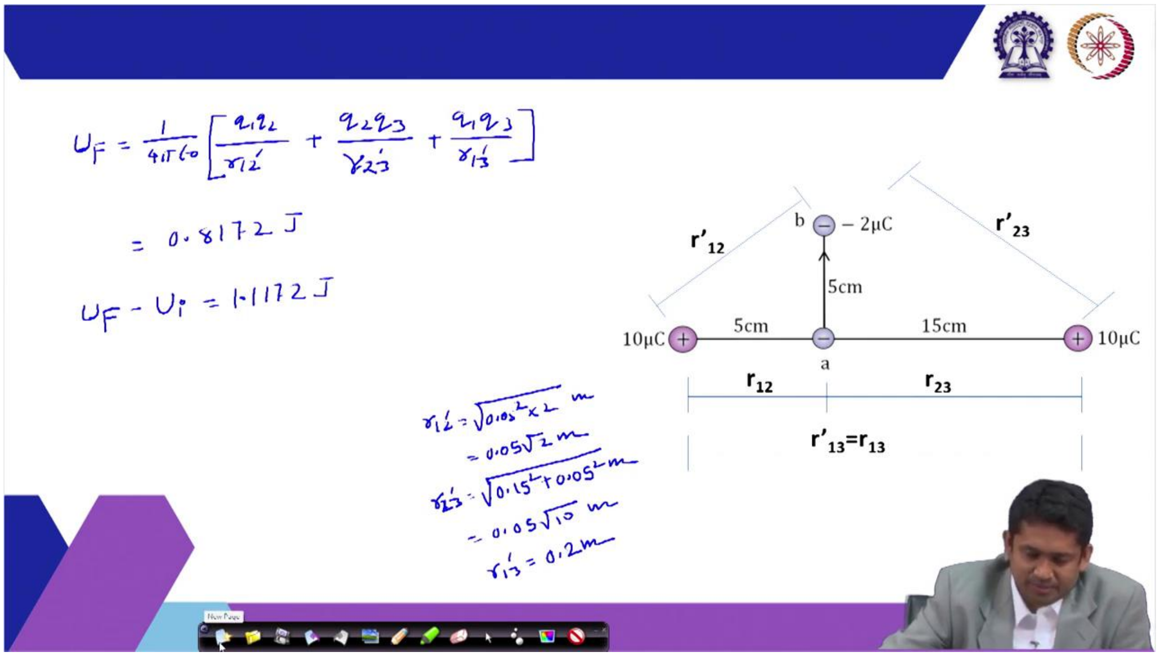

To do this, we write a similar expression Uf f, f means final, 1 by 4 pi epsilon 0 and then by r 12 prime plus q2 q3 by r 23 prime plus q1 q3 by r 13 prime. Now r 12 prime is multiplied by 2 meter which gives us 0.05-meter, r 23 prime is this. So, it is square of this plus square of this under root. And this gives a value of root over 10 meter and r 13 prime as we know it is 0.2 meter. Once we put all these values here q1 q2 q3 we already know then Uf comes to be 0.8172 Joule. Now Uf minus Ui if we subtract Ui from Uf, the value comes to be 1.1172 Joule. So, Uf minus Ui is a positive quantity which means to move this charge from here to here, we need to do some work. So, this concludes the second problem.

$$U_i = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right] = 0.8172\,J$$

$$U_F = U_i = 1.1172\,J$$

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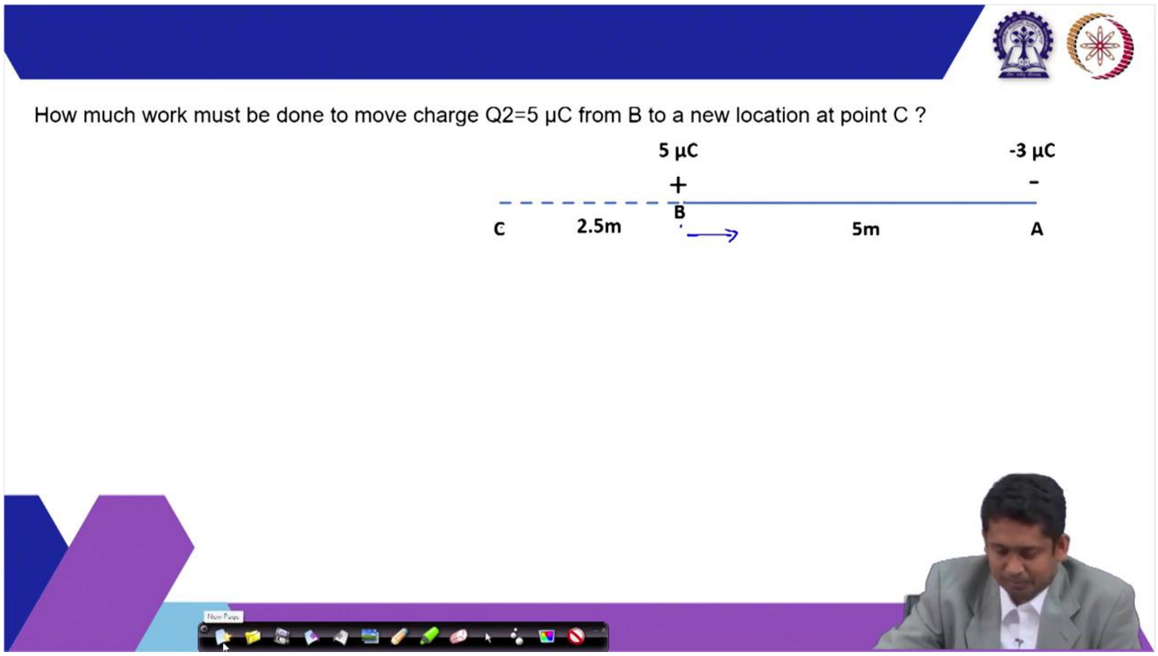

Now let us go to another problem. How much work must be done to move a charge q2 is equal to 5 micro coulomb from B to a new location at point C. So, the charge is supposed to be moved from this point to this point and how much work is to be done. Again we can follow the same approach we can find out the potential energy here and we can find out the potential energy here and we can take the difference and that would tell us how much work is to be done.

Now here we can see that here is a negative charge here and this is attracting this positive charge. So, this charge you try to bring it towards this side. However, we want to bring it from here to here which means we need to do some work. So, at this position the potential energy is higher. At this position, the potential energy is lower and if we subtract them we will get a positive quantity and that is the amount of work we have to do to move this charge from here to here. So, this can be solved using the similar approach that we adopted in the last problem.

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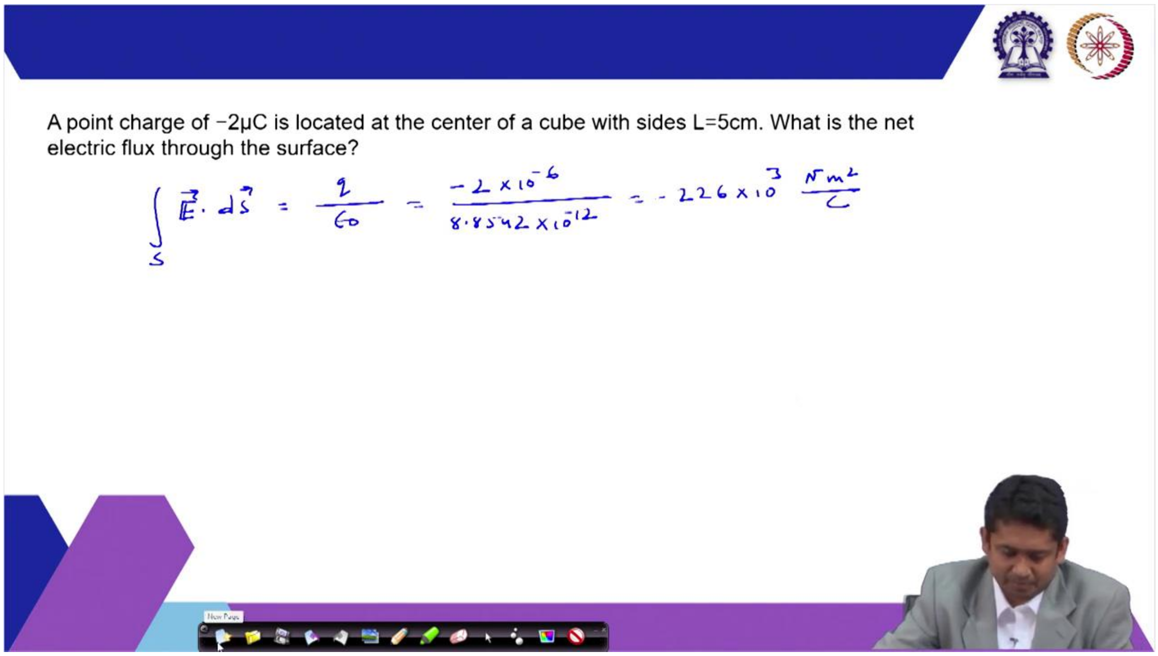

A point charge of minus 2 micro coulomb is located at the center of a cube with sides L is equal to 5 centimeters. What is the net electric flux through the surface? Now to solve this our net electric flux is surface integral of electric field ds and that gives us the net electric charge. Now from the Gauss law this can be written as q which means the total free charge divided by epsilon 0 because if I take this epsilon 0 here that gives me the electrical displacement vector and electrical displacement vector dotted with ds and integrated over surface is equal to the total free charge q that is the Gauss law and let us assume that it is in the free space.

So, permittivity is 1. So, we have epsilon 0 and this quantity q we know because we know there is one point charge which is minus 2 micro coulombs. So, q is equal to minus 2 into 10 to the power minus 6 and epsilon 0 is 8.8542 multiplied by 10 to the power minus 12 and that gives me the flux and it should come to be minus 226 multiplied by 10 to the power 3 Newton meter square per coulomb.

$$\int_S \vec{\mathbb{E}}.d\vec{S} = \frac{q}{\varepsilon_0} = \frac{-2\times10^{-6}}{8.8542\times10^{-12}} = -226\times10^3\,\frac{Nm^2}{C}$$

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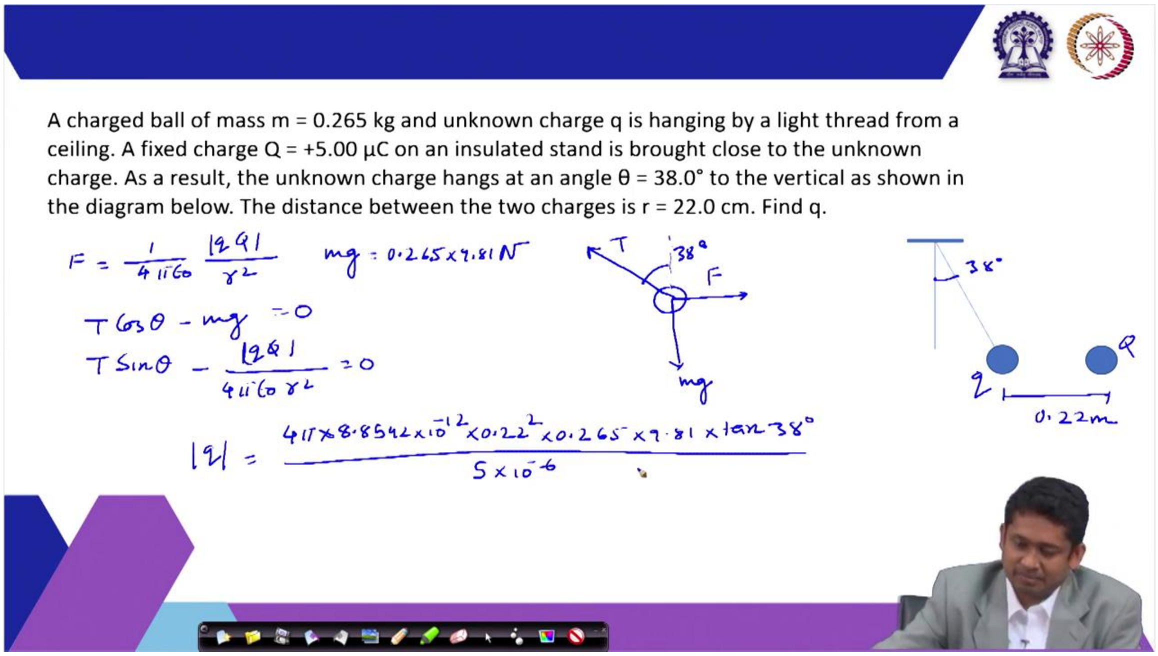

A charged ball of mass m is equal to 0.265 kilogram and unknown charge q is hanging by a light thread from a ceiling. So, this is the charge ball. A fixed charge Q is equal to 5 micro coulomb or an isolated stand is brought close to the unknown charge. So, this is brought close to this and as a result the unknown charge hangs at an angle theta is equal to 38 degree to the vertical as shown in the diagram below. So, because of that, this assumes this inclination and the angle becomes 38 degrees. The distance between the two charges is 22 centimeters. So, the distance between these two is 0.22 meter. We have to find Q. So, this is q and this is capital Q. First of all, because of this charge, this charge has come closer. So, now, Q is positive. So, small q should be negative then only it will come closer to it. So, small q is a negative charge and at this position, we have to find out the force here.

So, if I draw the free body diagram of this small q, it looks like this. So, it experiences tension due to T, it has its own weight the one weight is mg, mass into mass into acceleration due to gravity and here I have force F. Now, the force F as we know the magnitude of the force F is 4 pi epsilon 0 q Q by r square. Here we are concerned with the magnitude only. We all know all these values epsilon 0 we know and we know q Q r. T is not known to us. Mass is given to us. So, mg is 0.265 kg multiplied by 9.81. That gives us a gives us the mass in Newton.

Now from this free body diagram this angle is 38 degree. So, we can write T cosine theta minus mg is equal to 0 if I sum up all the forces along the vertical direction and T sin theta minus by 4 pi epsilon 0 r square is equal to 0. Then from these two equations we can eliminate T and find out our charge magnitude q and that comes to be 4 pi multiplied by 8.8542 multiplied by 10 to the power minus 12 into 0.22 square multiplied by 0.265 multiplied by 9.81 multiplied by tan 38 degree divided by 5 multiplied by 10 to the power minus 6. So, we can calculate it and find out the magnitude of the charge q and we know that the there is a negative charge. So, that gives me full information about this charge.

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|qQ|}{r^2}$$

$$T\cos\theta - mg = 0$$

$$T\sin\theta - \frac{|qQ|}{4\pi\varepsilon_0 r^2} = 0$$

$$|q| = \frac{4\pi \times 8.8542 \times 10^{-12} \times 0.22^2 \times 0.265 \times 9.81 \times \tan 38°}{5 \times 10^{-6}}$$

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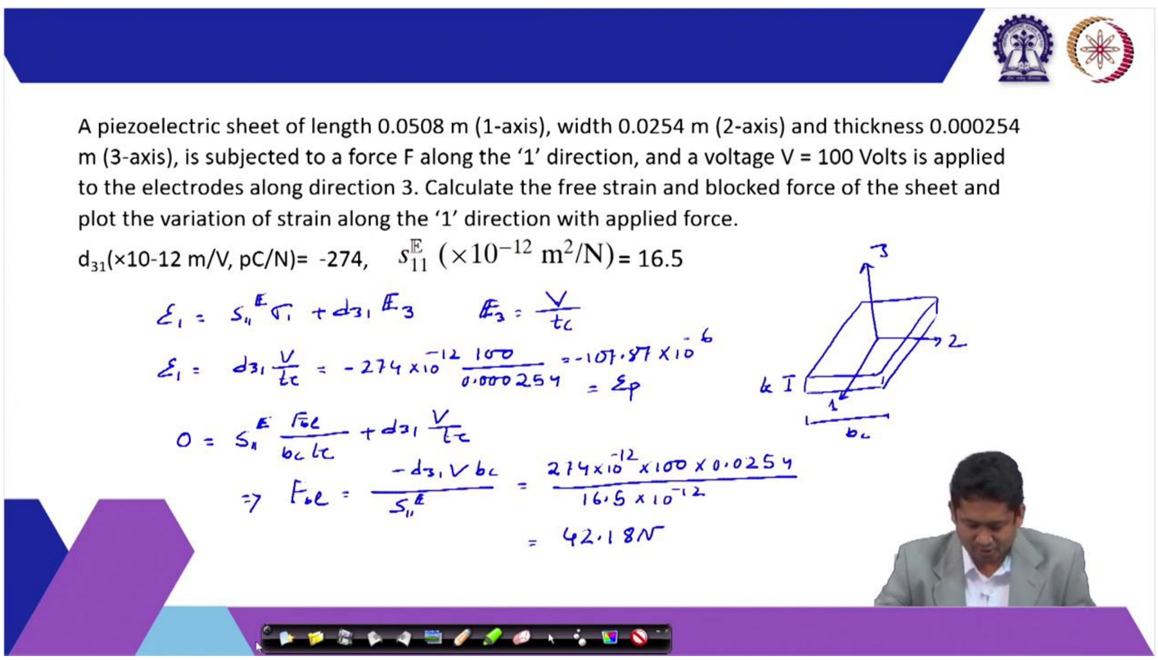

Now let us look into this problem. So, this says a piezoelectric sheet of length 0.0508 meter which is the dimension along one axis, width 0.0254 meter which is the dimension along two axis and thickness 0.000254 meter which is the dimension along three axis is subjected to a force F along direction 1 and a voltage V is equal to 100 volts is applied to the electrode along direction 3. So, the piezoelectric patch would look like this and as we have seen we have direction 1, 2 3.

If we call this as direction 1, if we call this as direction 2 and this as direction 3. Now the dimensions are given along all the directions and it has electrodes along direction 3 at this surface and at the opposite surface and the voltage 100 volt is applied along that. So, we have to calculate the free strain and block force and we have to find out the variation of the strain along direction 1 with the applied force. So, to find out the free strain, so, here we find this equation useful S11 E multiplied by sigma 1 plus D31 multiplied by the electric field E3. This can help us finding out the free strain as well as the block force.

Now electric field E3 can be related to the voltage V by V by tc where tc is this dimension thickness of the piezoelectric patch and this we can call bc. Now so, tc we know that the dimension is given 0.000254 meter. So, to find out the free strain, free strain means when the piezo is not constraints, it can expand freely. So, the stress is 0. So, in that condition the equation reduces to S11 is equal to just d31 V by tc. d31 is given to us minus 274 multiplied by 10 to the power minus 12. Voltage is 100 volt given to us and the thickness is 0.000254 and this gives us the amount of strain that we get.

So, the strain that we get is minus 107.87 into 10 to the power minus 6. Now we have to calculate the block force. Block force means when the piezo is not allowed to expand or contract, in that case the full stress or full force is developed it. So, strain is 0. In the same equation if I put strain is equal to 0 then the equation can be written as S11 electric E sigma 1 means the amount of force, the block force that is generated.

So, we call it Fbl divided by this area. So, it is bc by tc and then we have plus d31 V by tc. Now if we find out the block force from this expression, it gives us minus d31 V divided by S11 E multiplied by bc. So, if we put the values 274 multiplied by 10 to the power minus 12 multiplied by 100 volt multiplied by the dimension 0.0254 and then we have to divide by S11 electric field which is given here 16.5 into 10 to the power minus 12 and after simplification, the force comes to be 42.18 Newton. So, we know the free strain. So, this is our free strains we can denote this as epsilon p. So, these are free strain, these are block force. With that we can draw a diagram of force versus strain.

$$\varepsilon_1 = S_{11}{}^{\mathbb{E}}\sigma_1 + d_{31}\mathbb{E}_3$$

$$\mathbb{E}_3 = \frac{V}{t_c}$$

$$\varepsilon_1 = d_{31}\frac{V}{t_c} = -274 \times 10^{-12}\frac{100}{0.000254} = -107.87 \times 10^{-6} = \varepsilon_p$$

$$O = S_{11}{}^{\mathbb{E}}\frac{F_{bl}}{b_c t_c} + d_{31}\frac{V}{t_c}$$

$$F_{bl} = \frac{d_{31}V b_c}{S_{11}{}^{\mathbb{E}}} = \frac{274 \times 10^{-12} \times 100 \times 0.0254}{16.5 \times 10^{-12}} = 42.18\ N$$

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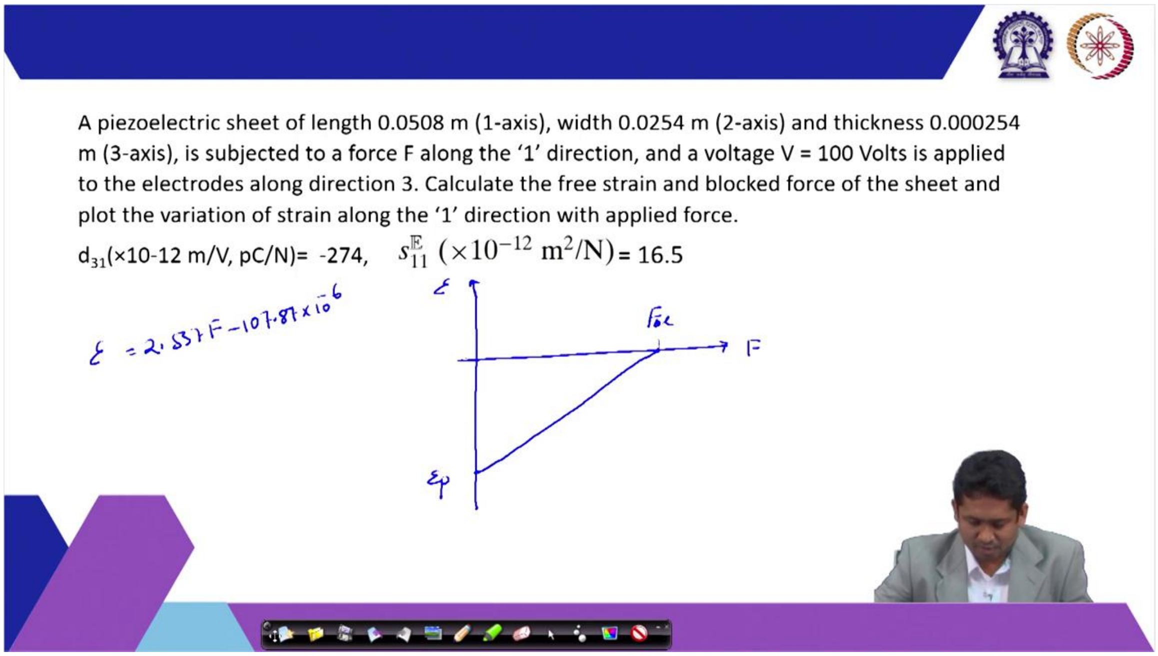

So, to do that if we plot the strains along these directions, so, let us plot force along this direction and strain along this. So, this is strain. The free strain is a negative quantity. So, it should be here. It is my free strain and the block force is here Fbl. Actually when we have the block force known and when we have the free strain known as we saw before, we can for this kind of actuators which is linear we can fit a straight line to it.

The straight-line equation would look like this. In this case if we follow the procedure that we saw previously while discussing actuator load line if you follow that, this becomes all straight line for this case. Now, if we join these two, it becomes our straight line showing the variation of strain with the force. So, this is our free strain that we got. This is the block force that we got. So, with that this problem ends.

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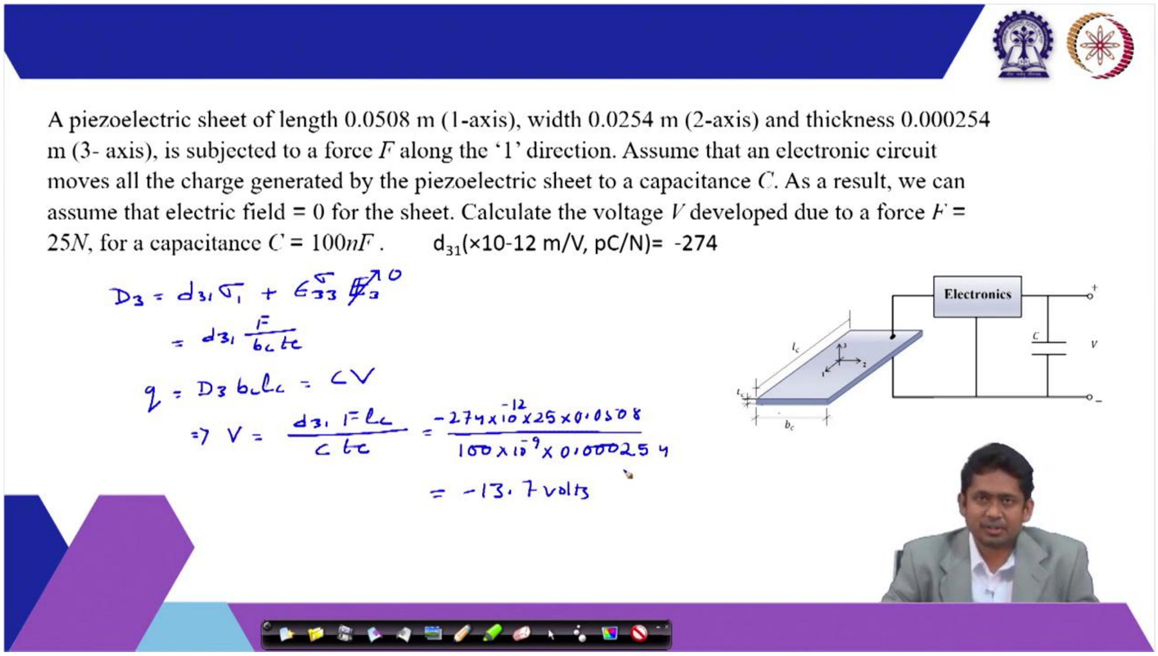

Now we should go to the next problem. Now, let us look into this problem. A piezoelectric sheet of length 0.0508 meter the dimension along one axis with 0.0254 meter dimensional on two axis and thickness 0.000254 meter dimensional along three axis is subjected to a force F along one direction. So, it is subjected to a force along this direction. Assume that an electronic circuit moves all the charges generated by the piezoelectric sheet to a capacitance C.

So, there is a circuit. This circuit moves all the charges here. As a result, we can assume that the electric field is 0 for the sheet. So, the electric field across this is 0. Calculate the voltage V developed due to a force F is equal to 25 Newton for a capacitance C is equal to 100 nano Farad. So, the value of the force is given to us and the capacitance is given to us. We have to find out the voltage here and the property of the piezo is given to us. So, we can write the equation, the relevant part of the constitutive equation d31 multiplied by sigma 1 plus epsilon 33 sigma multiplied by E 3. Now, E 3 is 0, it is stated in the problem, sigma is the stress. If I know the force the stress is force by area. So, I have to divide this area.

So, it becomes d 31 force by bc into tc. Now, the charge if we know the electrical displacement d 3, the charge can be calculated as D 3 multiplied by bc into lc and that is equal to our capacitance multiplied by the voltage capacitance of this. So, we can simplify this and write voltage is equal to d 31 multiplied by F lc divided by C tc. So, now, if we put the values minus 274 multiplied by minus 12 multiplied by 25 the force in Newton multiplied by 2 into 0.0254 multiplied by 2. So, so this is basically lc. So, lc is just this value we can just write it as just 0.0508 then divided by C the capacitance 100 multiplied by 10 to the power minus 9 and then this has to be multiplied by the thickness 0.000254 and that is equal to our voltage that is minus 13.7 volts. So, we can find out the voltage in this fashion and that completes this problem.

$$D_3 = d_{31}\sigma_1 + \varepsilon_{33}{}^{\sigma}\mathbb{E}_3 = d_{31}\frac{F}{b_c t_c}$$

$$q = D_3 b_c l_c = CV$$

$$V = \frac{d_{31} F l_c}{C t_c} = \frac{-274 \times 10^{-12} \times 25 \times 0.0508}{100 \times 10^{-9} \times 0.000254} = -13.7\, Volts$$

So, with this problem we will end this lecture here. I will see you in the next lecture.

Thank you.