Smart Structures
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Week - 02
Lecture No - 10

3D Constitutive Modeling of Piezoelectric Materials -4

Welcome to the fourth lecture of Constitutive Modeling of Piezoelectric Materials.

So, in the last lecture we saw three different forms of these constitutive models. In this lecture we will see one more form and then we will see interrelation of these different forms. So, we saw that the various forms are obtained through various thermodynamic potentials. So, today we will talk about one more thermodynamic potential and that is Gibbs free energy. Now we define Gibbs free energy as G equal to U minus E d minus sigma into epsilon.

$$G = U - ED - \sigma\varepsilon$$

$$dG = \sigma d\varepsilon + EdD - EdD - DdE - \sigma d\varepsilon - \varepsilon d\sigma = -DdE - \varepsilon d\sigma$$

$$dG = \left(\frac{\partial G}{\partial \sigma}\right)^{E} + \left(\frac{\partial G}{\partial E}\right)^{\sigma}, \frac{\partial G}{\partial \sigma} = -E, \frac{\partial G}{\partial E} = -\varepsilon$$

$$d\varepsilon = \left(\frac{\partial \varepsilon}{\partial \sigma}\right)^{E} d\sigma + \left(\frac{\partial \varepsilon}{\partial E}\right)^{\sigma} dE = -\left(\frac{\partial^2 G}{\partial \sigma^2}\right) d\sigma - \left(\frac{\partial^2 G}{\partial \sigma\, \partial E}\right) dE$$

$$dD = \left(\frac{\partial D}{\partial \sigma}\right) d\sigma + \left(\frac{\partial D}{\partial E}\right) dE = -\left(\frac{\partial^2 G}{\partial \sigma\, \partial E}\right) dE$$

$$\begin{Bmatrix} \{\varepsilon\}_{6\times1} \\ \{D\}_{3\times1} \end{Bmatrix} = \begin{bmatrix} [S^{\varepsilon}]_{6\times6} & [d^T]_{6\times3} \\ [d]_{3\times6} & [\varepsilon^{\sigma}]_{3\times3} \end{bmatrix} \begin{Bmatrix} \{\sigma\}_{6\times1} \\ \{E\}_{3\times1} \end{Bmatrix}$$

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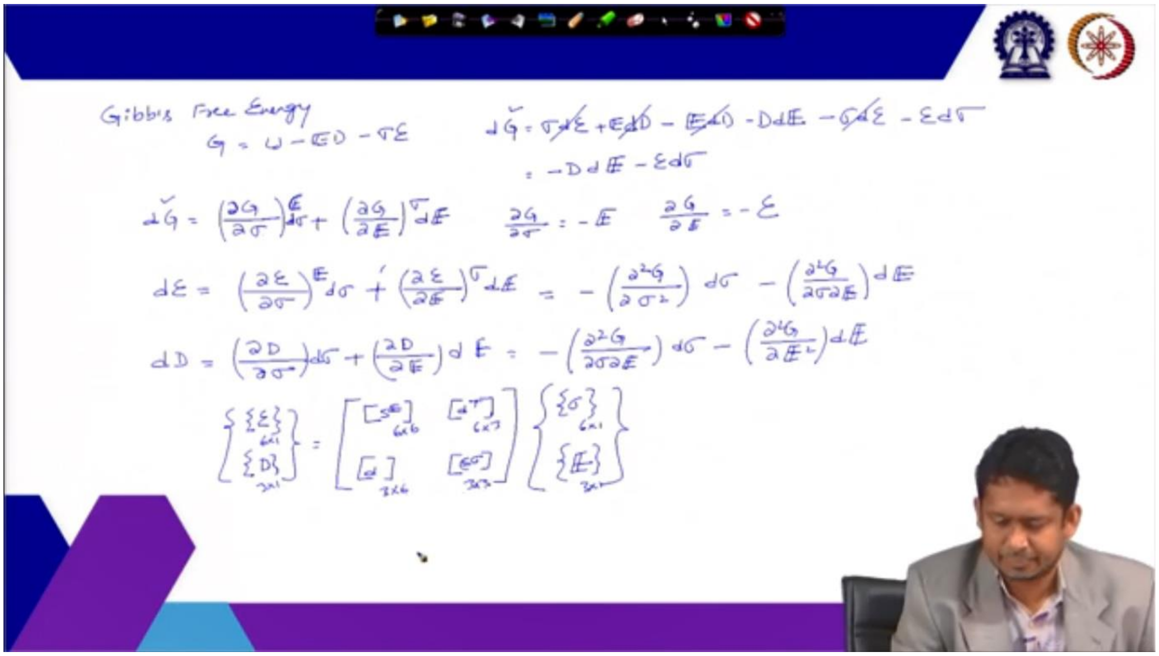

So, accordingly in the differential form we can write this as sigma d E plus E d d minus of. So, here these are our electric fields. So, I should put a double struck E minus E dD minus d d E minus sigma d E minus sigma. So, sigma d E sigma d E they cancel here E dD and E dD they cancel here is equal to minus d epsilon E minus epsilon d sigma.

So, these our Gibbs free energy in the differential form. Now we can also write this as dG is equal to dG is equal to delta G by delta of electric field keeping our sigma constant we can start with maybe sigma the other variable. So, delta G by delta sigma keeping E constant plus delta G by delta of electric field keeping sigma constant. So, if I compare this expression with this expression we can write that delta G by delta sigma is equal to minus of electric field and similarly delta G by delta of electric field is equal to minus of epsilon strain. Now we can define a differential quantity d epsilon and d epsilon we can define as minus of del epsilon by sorry del epsilon by del sigma keeping our other state constant multiplied by, we have to multiplied by d sigma here plus del epsilon by del E keeping sigma constant multiplied by our electric field E.

Here one thing we also need to multiplied by the d sigma's otherwise the expression is not complete. Now we already know that our epsilon is this and electric field is this. So, accordingly we can write this as minus of del 2 G by del sigma 2 multiplied by d sigma minus del 2 G by del sigma del E multiplied by d of electric field. Similarly, we can write the electric field quantity as del partial derivative of electric field by sigma. So, we can write the quantity d electrical displacement as partial derivative of that quantity with respect to sigma multiplied by d sigma plus partial derivative   of d with respect to electric field multiplied by d of electric field.

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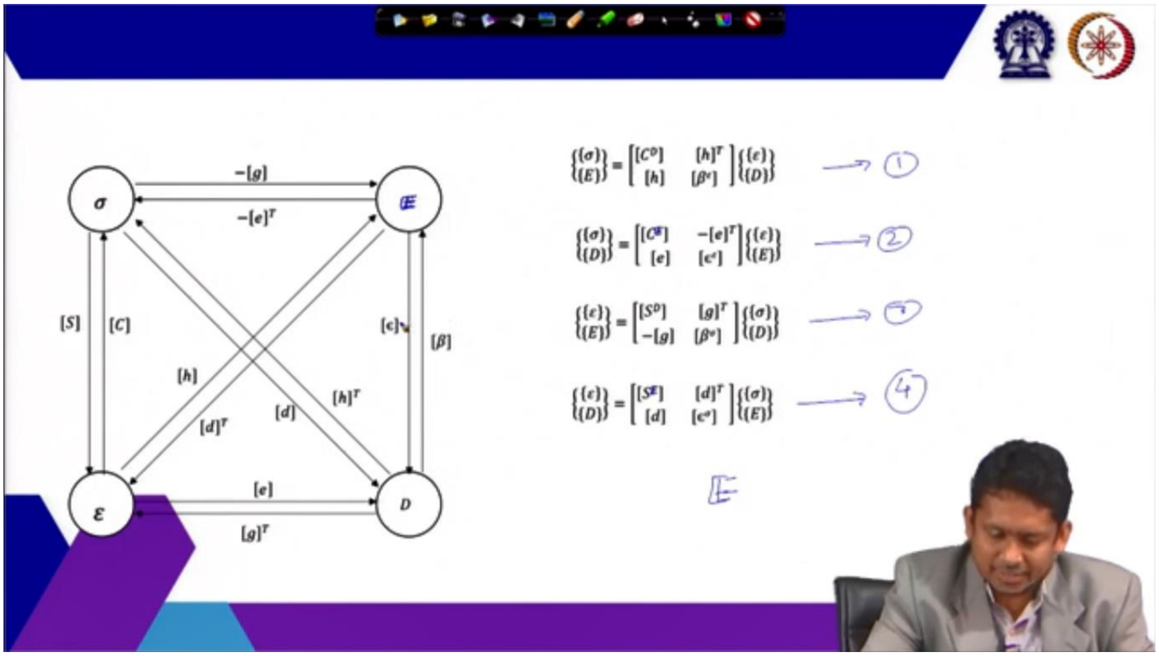

Now, this quantity can be written as minus of del 2 G del sigma del E multiplied by d sigma minus of double derivative of G with respect to electric field multiplied by d of electric field. Now combining these two we can define another constitutive relation and the relation can be written as this. So, here the state vectors are sigma and electric field. So, they are 6 by 1 and 3 by 1 and we get here strain and electrical displacement they are also 6 by 1 and 3 by 1. So, these are 6 by 6 matrix this is a 6 by 3 matrix this is 3 by 6 and this is 3 by 3.

So, with that we finished our all the four forms of constitutive relations. Now we will see them together. So, yeah so, here we can see them together. So, these are the four forms of the relations. So, this we can call 1, this we can call to be 2, this we can call to be 3 and this we can call to be 4.

Now here please understand this E's denote electric field. So, they have to be double struck E for here also and for here also these are all double struck E's of this form. Now this chart summarizes what is happening here. So, sigma if I want to get sigma from the epsilon then I have to multiply a C matrix which we can see here. So, C D is multiplied with epsilon and to get sigma similarly here also C E is multiplied with epsilon to get sigma.

Now depending on the other state vector this C D and C E they are different, but here we are not differentiating those because we do not have the scope. So, this C can be C D or can be sigma depending on what is the other state vector. Similarly if I want to get the strain from sigma I have to multiply s with sigma which we can see here we have S D and S electric field. And similarly, if you want to get electrical displacement from the electric field again it is a it is a double struck E because it is electric field. So, if I want to get that then we have to multiply this epsilon with the electric field and again this epsilon has two forms epsilon sigma and epsilon, but we have not differentiated here.

Similarly, if I want to get electric field from the D, we have to multiply beta. Now these quantities H D G and E they are all coupling terms they relate the electrical variables with mechanical variables which we see here this h,e, g and D. Now we will spend some time on finding the inter relation between this constitutive relations. So, to do that all right before doing that let us revisit the assumptions which we made so far in getting these equations. So, our first assumption was P is proportional to or we can write as the magnitude the polarization P is proportional to E and that is only possible when the electric field is weak, but in reality, this does not hold I mean it holds as long as the electric field is weak, but in reality, when the electric field is stronger this relation is not linear.

$$P \propto E$$

$$\tau_{ij} = \tau_{ji}$$

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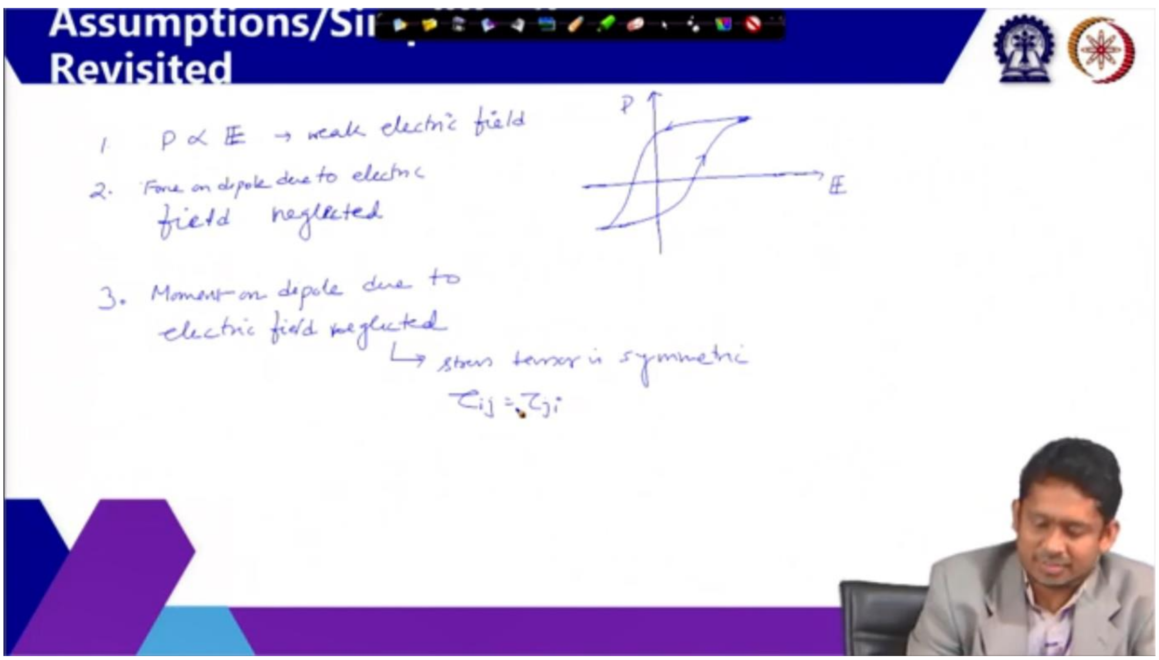

If an in fact, at that time the relationship becomes a hysteretic relationship. So, if I want to plot electric field along this line and polarization along this line this looks something like this. So, it is a path dependent phenomena if I increase the electric field it takes this path when I reduce it takes this path. So, it is not such a simple relation when the electric field is strong. So, we neglected force on dipole due to electric field.

If the electric field is uniform this can be neglected and we made those kinds of assumptions so, we neglected this. So, neglected and because of that we got rid of the body force term in the linear conservation of linear moment of equation. And also we moment on dipole due to electric field that was also neglected. So, because of this the body couple that we have in the conservation of non-linear conservation of angular momentum that got that got neglected. And this gives us that stress tensor is stress tensor is symmetric because of which we can assume that.

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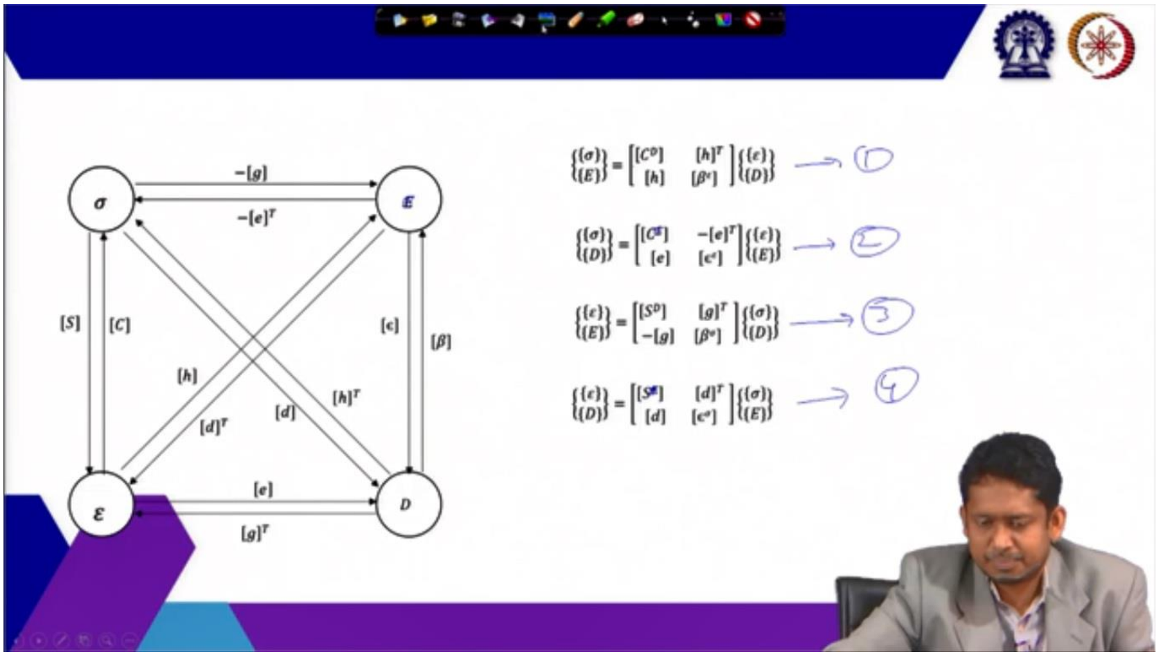

So, tau ij becomes tau ji with this assumption. Now, if somebody is interested in looking into something which helps in understanding how these non-linear effects can be incorporated in the constitutive in the mathematical modeling of these materials these papers can be referred. These papers treat these non-linearities and they show you how to how the model looks when these non-linearities incorporated. So, they are what look. Now, we will talk about the relationship between the different constants.

So, this we already saw and please understand that these is they have to be double stroke e because these are all electric fields again, I can do that. And we call it equation 1, we call it equation 2, we call it equation 3 and we call it equation 4. So, with that now we will start our looking into the relationship between these forms of constitutive relations. CE, SE and through this we can find out the relation between e and d also. So, from equation 4 we can write epsilon equal to SE multiplied by sigma plus D t multiplied by the electric field.

$$\{\varepsilon\} = [S^E]\{\sigma\} + [d^T]\{E\}, \{\sigma\} = [S^E]^{-1}\{\varepsilon\} - [S^E]^{-1}[d]^T\{E\}$$

$$\{\sigma\} = [C^E]\{\varepsilon\} - [e]^T\{E\}$$

$$[C^E] = [S^E]^{-1}, [e]^T = [S^E]^{-1}[d]^T, [e] = [d][S^E]^{-1}$$

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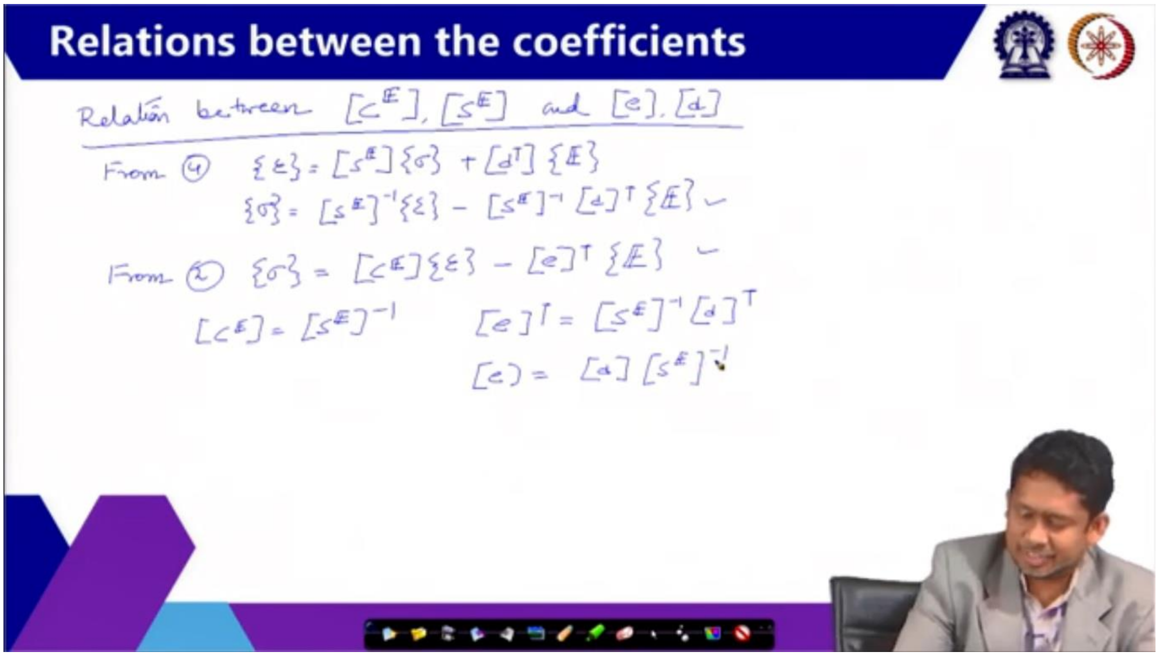

So, if we multiply both side by S inverse then we can write simplify the expression I mean not simplify we can write the expression where stress is found out in terms of the strain and the electric field and the expression looks like this. Now from 2 we can write sigma and that is equal to epsilon t multiplied by electric field. Now if I compare this equation with this equation we see that CE is equal to SE inverse and similarly we see that e t is equal to SE inverse d t. So, we can write e is equal to d multiplied by SE inverse. So, if I take the transpose of SE inverse it becomes SE inverse.

$$\{\varepsilon\} = [S^E]\{\sigma\} + [d]^T\{E\}, \{D\} = [d]\{\sigma\} + [\varepsilon^\sigma]\{E\}$$

$$\{E\} = -[\varepsilon^\sigma]^{-1}[d]\{\sigma\} + [\varepsilon^\sigma]^{-1}\{D\}$$

$$\{\varepsilon\} = [S^E]\{\sigma\} - [d]^T[\varepsilon^\sigma]^{-1}[d]\{\sigma\} + [d]^T[\varepsilon^\sigma]^{-1}\{D\}$$

$$\{\varepsilon\} = [S^D]\{\sigma\} + [g]^T\{D\}$$

$$[S^D] = [S^E] - [d]^T[\varepsilon^\sigma]^{-1}[d]$$

$$[g]^T = [d]^T[\varepsilon^\sigma]^{-1} \Rightarrow [g] = [\varepsilon^\sigma]^{-1}[d]$$

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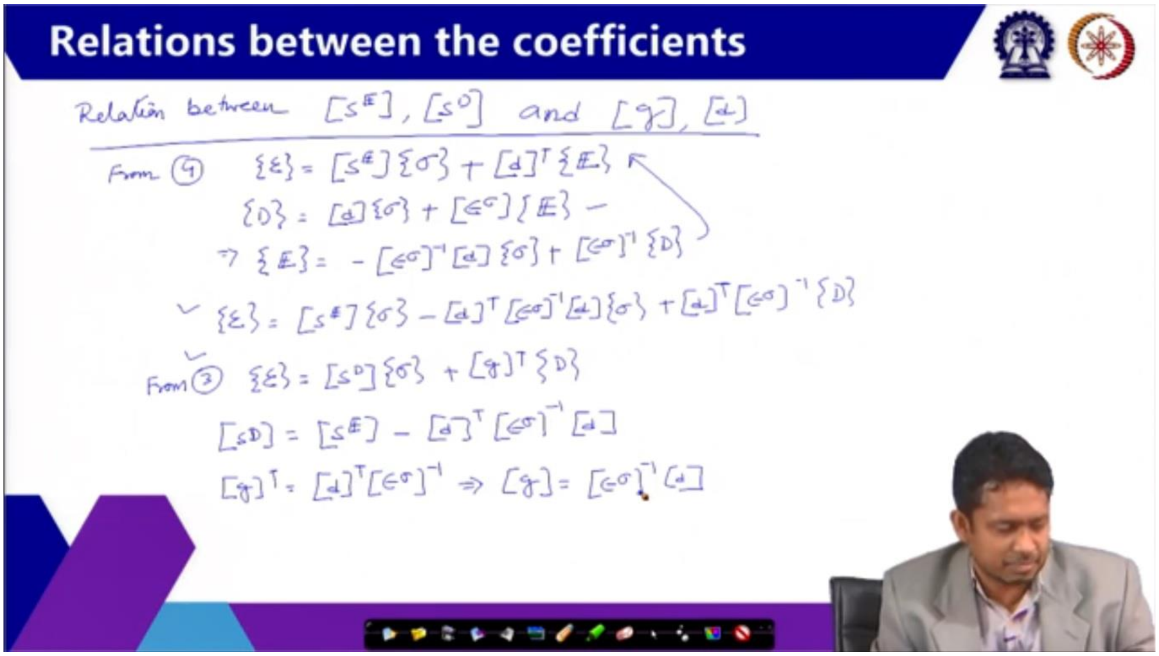

So, that is why we are writing this all right. Now we will look into the relationship between relation between SE and SD and Now, g and d again to do this from equation 4 we know epsilon strain is equal to SE multiplied by sigma plus d multiplied by epsilon E and then we also have another equation which is D is equal to d multiplied by sigma plus epsilon sigma multiplied by electric field. Now if I multiply both side of this equation by this epsilon sigma inverse we get this expression of electric field and the expression is minus epsilon sigma inverse multiplied by d multiplied by sigma plus epsilon sigma inverse multiplied by D. So, now that we need we know that the expression for electric field is this we can take it back here and then we can rewrite the expression for epsilon as S with superscript E multiplied by sigma minus d T epsilon sigma inverse multiplied by d plus d T multiplied by epsilon sigma inverse into D. Now from equation 3 we have epsilon equal to SD multiplied by sigma plus g T multiplied by D.

$$\{\varepsilon\} = [S^D]\{\sigma\} + [g]^T\{D\} \Rightarrow \{\sigma\} = [S^D]^{-1}\{\varepsilon\} - [S^D]^{-1}[g]^T\{D\}$$

$$\{\sigma\} = [C^D]\{\varepsilon\} + [n]^T\{D\}, [C^D] = [S^D]^{-1}, [n] = -[g][S^D]^{-1}$$

$$[C^D] = [C^E] + [e]^T[\varepsilon^E]^{-1}[e], [n] = -[\varepsilon^E]^{-1}[e]$$

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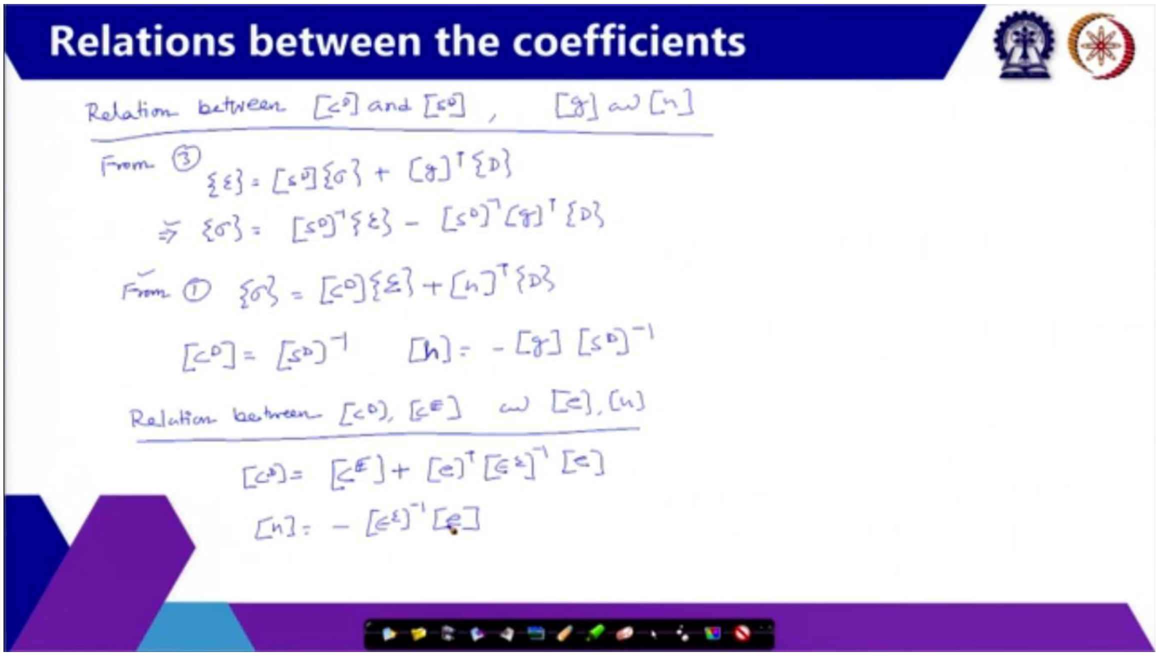

Now if we compare this expression with this expression, we get that SD is equal to SE minus dT multiplied by epsilon sigma inverse multiplied by d and similarly we have g T is equal to dT multiplied by epsilon sigma inverse which tells us that g is equal to epsilon sigma inverse multiplied by d. Now we will find out few more relations. So, now we will talk about relationship between CD and SD as well as g and h. From equation 3 we can write epsilon equal to SD multiplied by sigma plus gT multiplied by D. So, again if I multiply SD inverse at the both sides, we get sigma is equal to SD inverse and that multiplied by epsilon minus SD inverse gT into D and then from 1 we get CD multiplied by epsilon plus h T multiplied by D.

Now if I compare this with this, we get CD equal to SD inverse and similarly we get h equal to minus of g into SD inverse. Now similarly if I want to find out relation between CD and CE as well as e and h then following a similar approach we can see that our CD comes to be CE plus e transpose multiplied by epsilon inverse multiplied by e and h comes to be minus of epsilon epsilon inverse multiplied by e. So, here we related the constants that relates one mechanical quantity to others and while doing that the coupling terms also got related. Now we relate the terms which the constants which relates the electrical quantities. So, we will do some examples and rest would be just an extension of that relation between say beta epsilon and epsilon epsilon and h, e. Now we can write from equation 1 that electric field is equal to h multiplied by epsilon plus beta epsilon D and then if we multiply both side by beta beta epsilon inverse then we get D is equal to minus beta epsilon inverse H into epsilon plus beta epsilon inverse multiplied by the electric field E. And then from 2 we can get from equation 2 we get E equal to D equal to E multiplied by epsilon plus epsilon epsilon multiplied by the electric field vector E. Again if I compare this with this it gives me that E epsilon epsilon is equal to beta epsilon inverse and similarly e the coupling constant is equal to minus of beta epsilon inverse multiplied by h. Now while relating C D and C E we saw that h came to be equal to minus of epsilon epsilon inverse e, but these relations are same.

$$\{E\} = [n]\{\varepsilon\} + [\beta^{\varepsilon}]\{D\}$$

$$\{D\} = -[\beta^{\varepsilon}]^{-1}[n]\{\varepsilon\} + [\beta^{\varepsilon}]^{T}\{E\}$$

$$\{D\} = [e]\{\varepsilon\} + [\varepsilon^{\varepsilon}]\{E\}$$

$$[\varepsilon^{E}] = [\beta^{\varepsilon}]^{-1}, [e] = -[\beta^{\varepsilon}]^{-1}[n], \qquad [n] = -[\varepsilon^{E}]^{-1}[e]$$

$$[\varepsilon^{\varepsilon}] = [\beta^{\varepsilon}]^{-1}$$

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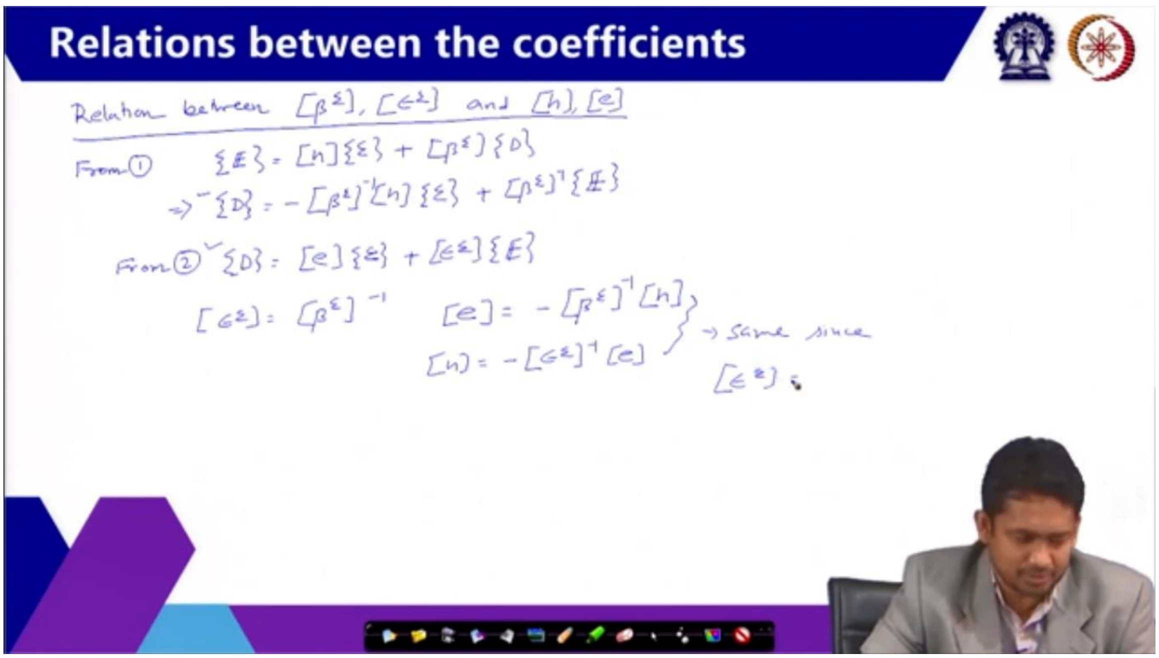

So, these relations are same because we know that epsilon epsilon epsilon is equal to beta epsilon inverse. So, we wanted to relate the electrical constants and that again related the coupling constants. So, whatever the relation between the coupling constants that we got from here the same relation we obtained before also while relating the mechanical constants. So, that is a check that our coupling constants are related properly. Now rest of the relations are just an extension of this using the same procedure.

So, for example, if we want to relate beta epsilon and beta sigma the relationship comes to be beta sigma is equal to beta epsilon minus h multiplied by C D inverse h T and this also gives us the relation that g is equal to minus of h into C D inverse and previously, we got h is equal to minus of g into S D inverse and they are same. Since C D inverse is equal to

S D inverse. Now we can write we can relate between we can find the relation between epsilon epsilon and epsilon sigma and again that will help us relate E and D also. So, using 4 using equation 2 and equation 4 we can find out the relation and the relation comes to be epsilon sigma is equal to epsilon plus e multiplied by C E c epsilon inverse multiplied by e T and also, we get d equal to e multiplied by C E inverse and again previously we got e equal to d multiplied by S E inverse again they are same because S E inverse is equal to C E. Now we will do the last one relation between relation between beta sigma and epsilon sigma and using equation 3 and 4 the relationship comes to be epsilon sigma is equal to beta sigma inverse and also, we get d is equal to beta sigma inverse multiplied by g and previously we got g is equal to epsilon sigma inverse multiplied by D again they are same relation because our epsilon sigma inverse is equal to beta sigma.

$$[\beta^\sigma] = [\beta^\varepsilon] - [n][C^D]^{-1}[n]^T$$

$$[g] = -[n][C^D]^{-1}, [n] = -[g][S^D]^{-1}$$

$$[\varepsilon^\sigma] = [\varepsilon^\varepsilon] + [e][C^E]^{-1}[e]^T$$

$$[d] = [e][C^E]^{-1}, [e] = [d][S^E]^{-1}$$

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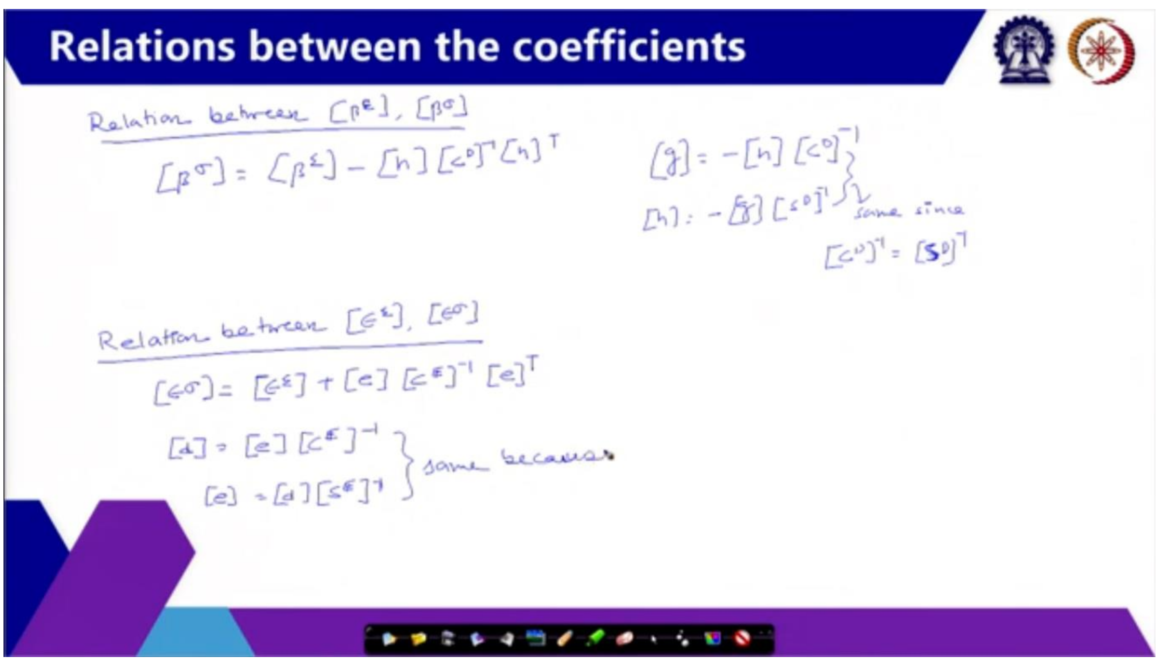

So, by this we related the different constants that arises in different constitutive relations. So, with that I would like to finish this lecture here I would see you in the next lecture.

$$[\varepsilon^\sigma] = [\beta^\sigma]^{-1}, [d] = [\beta^\sigma]^{-1}[g], [g] = [\varepsilon^\sigma]^{-1}[d]$$

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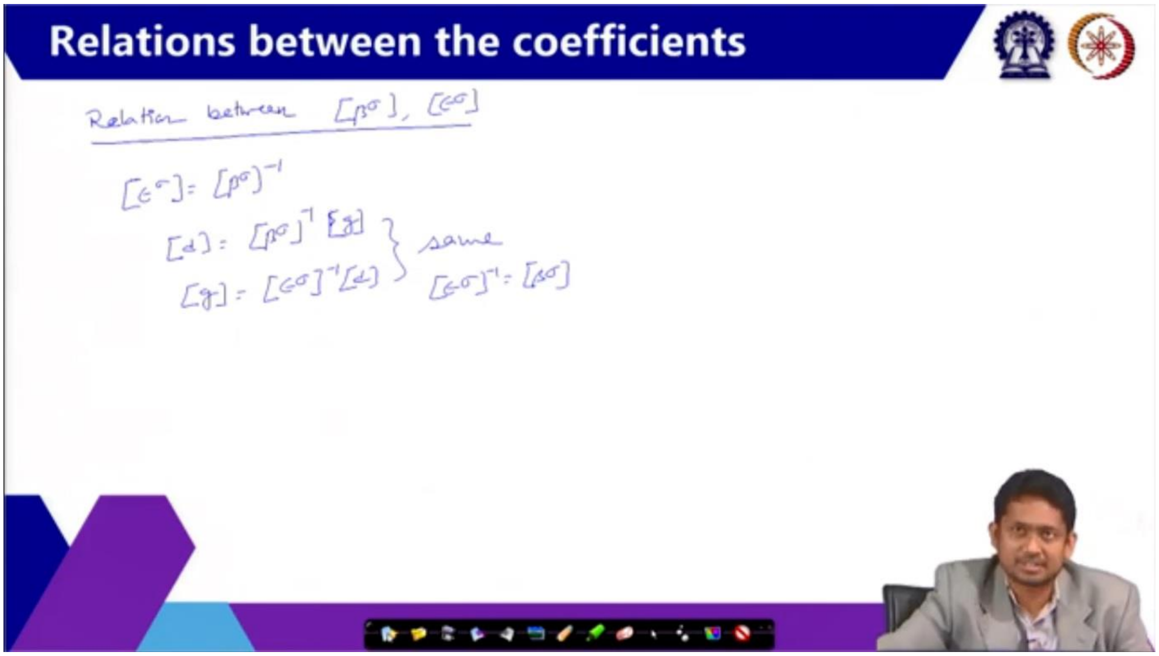
Relations between the coefficients
Relation between [βσ], [Cσ]
[Cσ] = [βσ]^-1
[d] = [βσ]^-1 [g]
[g] = [Cσ]^-1 [d]
same
[Cσ]^-1 = [βσ]

Thank you.