Introduction to CFD
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Lecture – 24
Numerical Solution of Unsteady Heat Conduction (Parabolic PDE) (continued)

We continue our discussion on parabolic partial differential equation. Last time, we learned how to do the Von Neumann stability analysis and we applied that technique to do the stability analysis of the FTCS scheme for discretizing parabolic partial differential equation.

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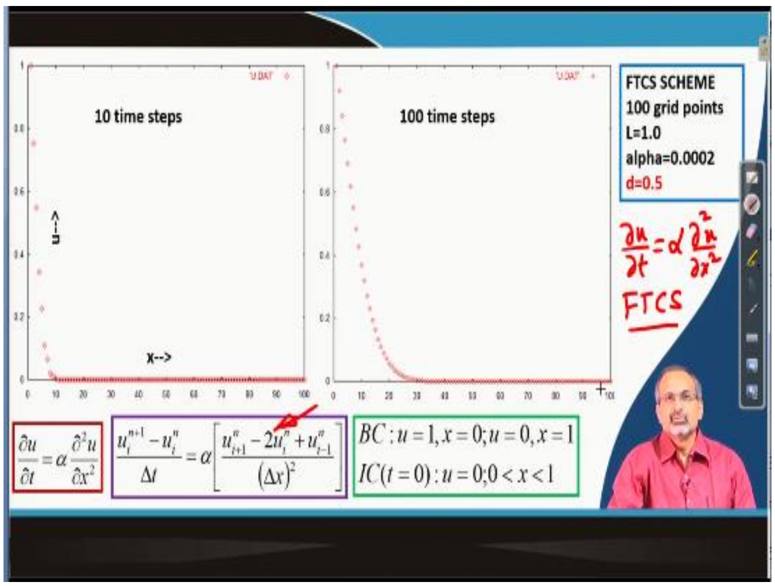

So, in this lecture, we will have a look at some of the simulation results, and you will also discuss that the outcome of the stability analysis taught us that the diffusion number should be restricted to a value of half at the most. So, if you maintain that stability limit, then how the simulation results look like and if you are not maintaining it, then what is the outcome. So, before we proceed further with these calculations, let us try to recapitulate the FTCS scheme.

So, we were discretizing the equation the del u del t is equal to alpha del u del x square. So, this equation when we discretize it with forward time central space discretization then we come up with a discrete form like this. And we need to take note that the scheme has first order accuracy in time and second order accuracy in space by virtue of the schemes that we have used to discretize the time and space derivative terms. So, if we were to write the order of accuracy, you could write it this way.

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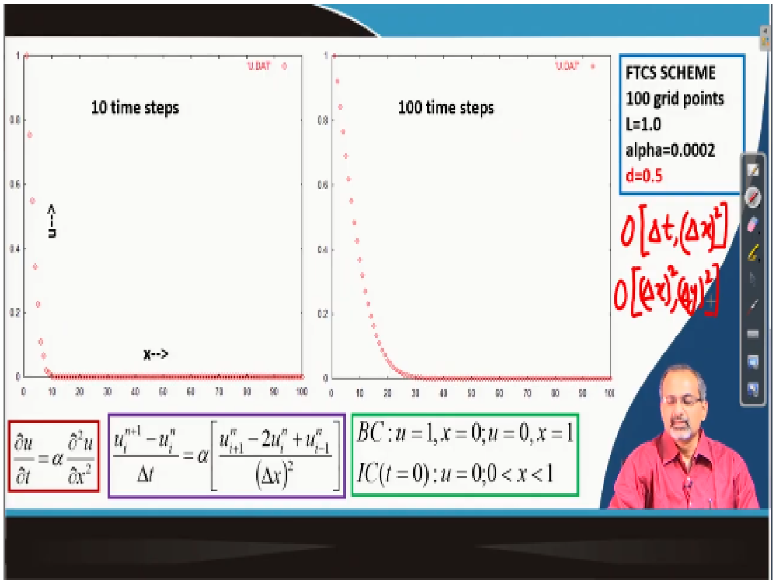

So, here we are looking at a time step as well, in addition to the space steps, which we did not see in elliptic equation, for example, in elliptic partial differential equation. When you were doing a multi-dimensional problem, which involves both x and y directions, and if we were using CD2 scheme to discretize, then the order of accuracy will be shown like this. So, there is no temporal dimension or time dimension in Laplace equation.

So, the accuracy will be expressed only in space. Why? For follow up for a parabolic partial differential equation, you will be seeing both time step as well as space steps indicated in the order of accuracy representation. Now, coming to the problem at hand which we are trying to solve numerically through a simple computer code. So, here we have a one dimensional domain in x.

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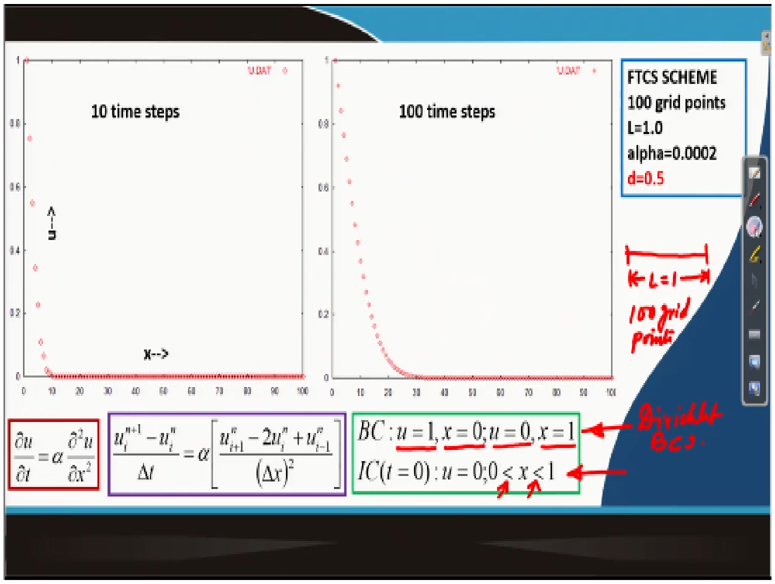

Let us say the length of the domain L is equal to unity and we have put in 100 grid points spanning this domain. So, you can find out the delta x which will be produced as a consequence and then we have imposed certain initial condition as well as boundary conditions at the end of the domain. So, you can see how the initial condition and the boundary conditions have been specified here in this green block.

So, initial condition says that obviously at t = 0 that is what we mean by initial condition u is equal to 0 everywhere apart from the boundaries that is why we use less than here, we do not put a less than or equal to because if we were to put equal to then we would reach the boundaries. So, any other point which lies a little away from the boundaries and covering all of the internal domain has a value of u = 0. That is the initial condition you are imposing. And what are you imposing at the boundaries, you are imposing as u = 1 at x = 0 and u = 0 at x = 1.

So, you are imposing Dirichlet boundary conditions at the ends of the domain. So, these things have to be kept in mind when we look at the simulation results. So, we are now starting to do the calculations over these 100 grid points apart from the boundary points of course, because that is where the values remain fixed and provided to you in the form of Dirichlet boundary conditions. So, you are not having a mixed Dirichlet-Von Neumann problem here.

So, if you were to have a Von Neumann condition at some end then the value at that boundary would have to be updated with time. So, here it does not arise because you have Dirichlet boundary condition. Now, having said that if you look at the solution here in the plots you are plotting the x coordinate along the x direction, x coordinate varying from in terms of the grid points. Let us take it in the form of grid points.

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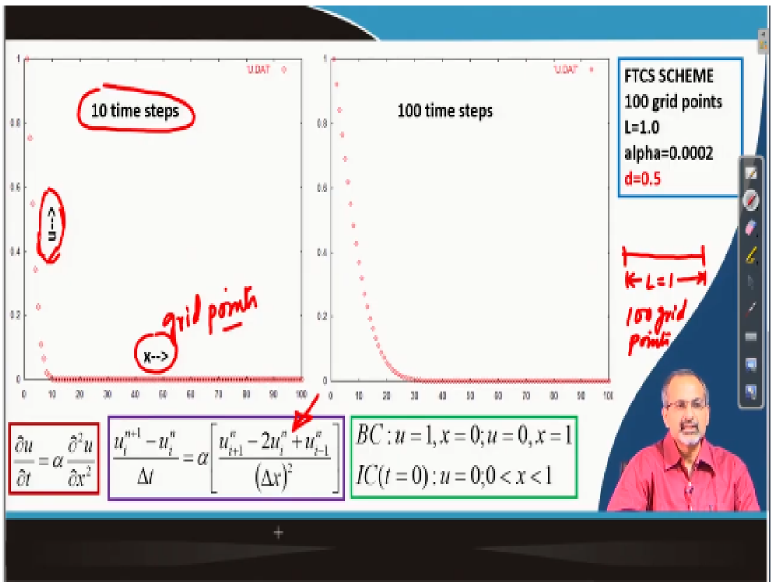

And then on the y axis you have the u, value of u represented and this is how the solution looks after 10 steps of calculation. So, each calculation step as we remembered would involve this discrete equation, this discrete equation will be solved at all the grid points i's between 2 to 99. So, leaving apart i = 1 and i = 100, which are the boundary points at all intermediate points, you would be applying this discretized form of the equation and updating the value of u.

So, as you keep updating from the first step to the second step and so on. Based on the initial and boundary conditions, you would be able to reach this distribution after 10 time steps. Of course, remember that this distribution is true for this discretized form of the equation. So, if you were to use different discretization schemes for approximating the time derivative and the space derivative.

Then the solution may look slightly different from this because the order of accuracy would then change. So, that is something that we have to keep in mind. So, ideally, one could try comparing the discretized equations solution quality by comparing with an analytical solution if available. So, in case that is available, then you have the convenience of comparing incidentally it is available for this particular partial differential equation that we are handling.

We will probably discuss it in a later lecture. So, having said that for now, we are concerned more about how the solution looks like as far as the FTCS scheme is concerned.

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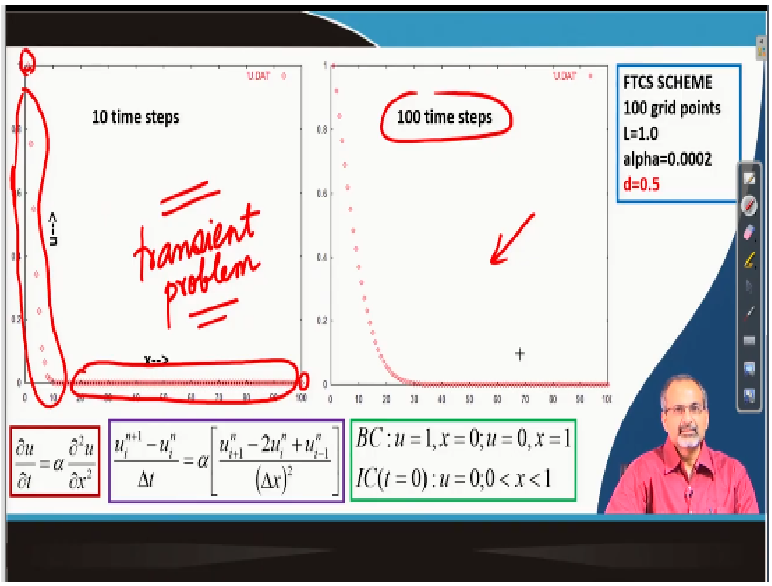

So, at the end of 10 steps, what you notice is that of course, the first grid point will always have the boundary value. Because it is set at that condition, also the last grid point will have the other boundary condition. And what is noticeable is that a number of grid points close to the left boundary have started gaining value because they are in close proximity to the left boundary value.

While many of the remaining points are lying silent as though they have not responded yet to the present presence of the left boundary value. So, it takes time for the solution to evolve. And that is why we say this is a transient problem. That means, the solution develops with time. Now, if you move to more number of time steps. Let us say one order higher at 100 time steps the solution looks what like, like what you see on the right hand graph.

Of course, again the grid points are indicated along x direction and the value of u along the y direction. And now, you see many more points I have started responding to the presence of the value of u = 1 on the left boundary as though the message is gradually diffusing into the rest of the domain. In fact, that is how it works because the second order term on the right hand side is indeed a diffusion term.

It is taking care of diffusion of the value of u from one boundary to the other. That is how it works. Now, if we look at the stability aspects of the solution till now, the 2 time steps solutions that we have looked at, they do not seem to show any kind of awkwardness. In the sense that some of the properties, which we expect from this kind of a calculation that the values are expected to be bounded between the maximum and minimum that you have in the boundaries.

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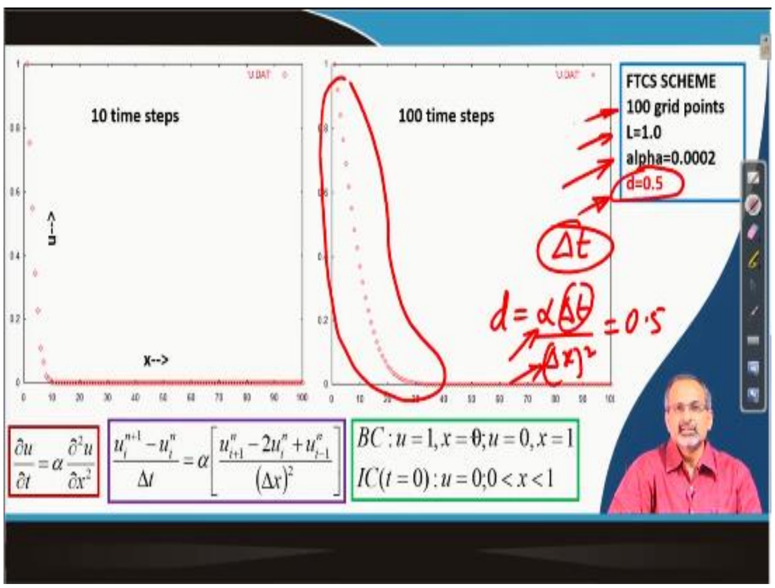

So, that is what you are seeing that some of the values are getting closer to the left end, while many of the other values are lying close to the right end boundary condition, but no value has overshot any one of them either on the positive side on the negative side. And that is how it is expected to be over here. And they are evolving with time because there is a time derivative in the governing partial differential equation.

So, that time derivative is actually taking care that the solution evolves with time and also the solution diffuses in space, because you have a second order derivative in u standing on the right hand side of the equation. So, the solution seems to be reflecting the physics. Now, we said earlier that we have hundred grid points and the length of the domain is unity and so on. For this simulation, we have chosen a value of alpha which is a diffusion constant as .0002.

You could have other values of alpha reflecting a particular physical situation, but having chosen the alpha and the delta x defined by the grid points et cetera. We finally get a d, value of d = 0.5 by the delta t that we have chosen. Which means, it is like saying that d which is defined as alpha times delta t by delta x square has been set at 0.5 with suitable choice of delta t after you have already said the values of alpha and delta x.

And you know that going by the Von Neumann stability analysis this was a limiting value of d for stable calculations. So, we are keeping ourselves bounded by that definition and since we are handling a linear partial differential equation. Von Neumann stability analysis should be guiding us well with the stability condition. Until now, the way the solution is proceeding seems to be a stable solution.

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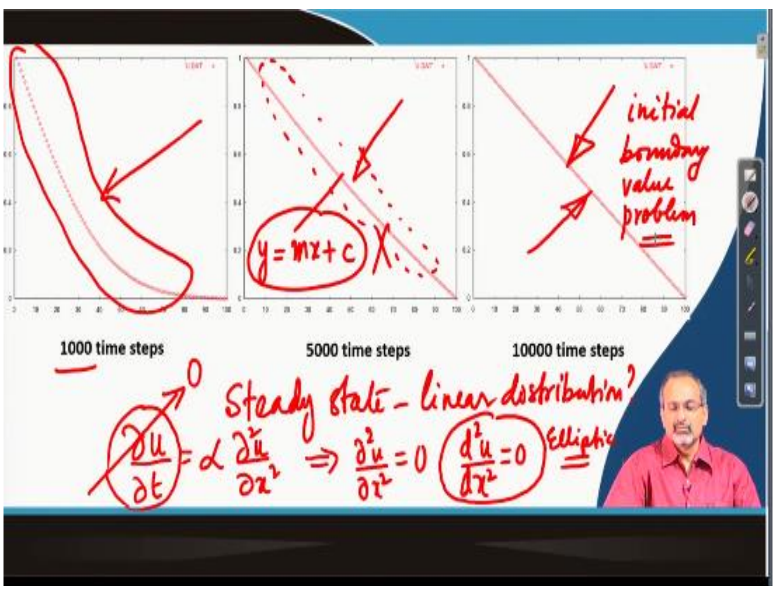

If we go further in time, we can look at solutions at much later time steps. So, this is 1000 time steps from the beginning. And now, you see that there is a wider diffusion of the value of u = 1 from the left end of the boundary towards the right end of the boundary. It has almost reached close to the right hand boundary. However, you can see a nonlinear nature in this curve, which means that it is yet to reach the steady state linear distribution.

Now, why do we say that at steady state, we expect a linear distribution? Now, if we look back at the governing partial differential equation, as long as things change with time, we would have a contribution coming from the left hand side of the equation, which influences the solution of u. Now, at very, very long times, if you are really approaching a steady state or an equilibrium state, you would not be changing with time anymore.

That means, at each and every grid point, the time derivative will approximate towards 0 as it does so, you are actually getting closer to the condition that the second derivative of u goes to 0 and this basically becomes an ODE now because you purely becomes a function of x. And for that we know that it has a linear form if you integrate this equation twice, you will have a linear form for u.

And then if you impose the boundary conditions on the left and right ends of the domain, then you will be able to get the distribution of u and at fairly large time steps. That is what happens you get a linear distribution. So from the nonlinear pattern that you are seeing 1000 time steps, if you move the solution further, a few thousand steps more you reach. The solution at 5000 time steps.

You can figure it out if you watch very carefully that there is still non linearity here, but the non linearity has weakened up. Here, by nonlinear I am meaning the non linearity of the curve, there is no non linearity in the governing partial differential equation, we have to understand this aspect carefully. So, what we mean by this is that this straight, this is not a straight line. So, this is not representable by a formula of this kind, this own whole good here.

So, non linearity is meant in that sense, but, there is absolutely no non linearity in the governing partial differential equation and what have we achieved here. Here, of course, we have achieved a linear distribution of u that is because asymptotically this condition has been reached. And now, this governing partial differential equation has actually reached a condition where it can be treated as an elliptic partial differential equation.

We have seen this earlier when we discussed about Laplace equation in one dimension, that there would be a linear distribution of the dependent variable in such a case. So, all through we saw that the solution remained stable, well behaved and bounded by the boundary values. So, we have seen a case where we have looked at the solution of initial boundary value problem that means, we started the problem from initial condition and right from t = 0 the boundary conditions were specified.

And we time step the solution over it a large number of time steps maintaining the boundary conditions all through till we reached a steady state solution and all through the solution remained well behaved and stable, because we had satisfied the constraints imposed by the Von Neumann stability analysis. And the constraint was that d would have to be kept less than equal to 0.5. And we chose a value of 0.5 for to d precisely and as expected, the solution remained stable.

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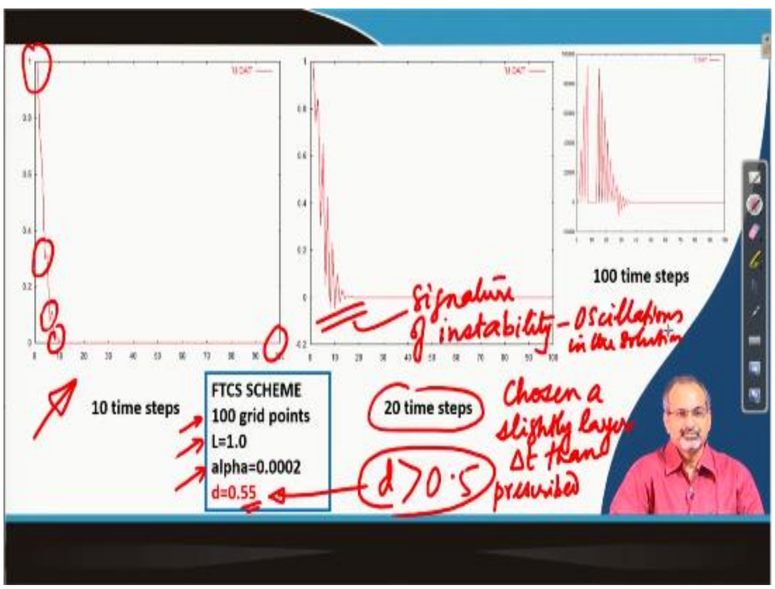

What if we did not go by the Von Neumann stability analysis outcome and we chose a d value which was other than 0.5 and on the wrong side. In the sense that we end up taking a value of d which is greater than 0.5 and that is precisely what is shown over here. So, we have to note that we have taken a value of d which is marginally larger than 0.5. It is not very significantly large, larger than 0.5, but it is certainly more. So, with a value of 0.55. Let us see how the solution emerges. So, other things are all remaining the same.

So, we have the same 100 grid points and unit length and alpha values as we did in the previous case. However, we have chosen a delta t in such a manner that finally d overshot the limit of 0.5. That means, we are trying to time step the solution more rapidly than what is prescribed by the Von Neumann stability analysis. So, what we have done essentially is chosen a slightly larger delta t then prescribed and that whether that choice finally influences.

The solution adversely is or not is what comes out from these plots. So, when we go to 10 time steps from the initial condition. We see that we are able to maintain the boundary condition on the left, on the right because they are anyway fixed at whatever values we have chosen. But, here, if we notice there are certain oscillations which are getting produced. However, the trend broadly remains the same as it was for the previous case of d = 0.5.

Now, if we take the solution a few time steps further. So, just 20 time steps and then we notice that many, many more oscillations have now started coming into the solution and the oscillations are becoming stronger. That means the jumps between consecutive grid point values are becoming larger and larger. So, this certainly is a signature for instability or signature of instability that means, you are seeing some oscillations in the solution.

If these oscillations become larger and larger and larger and so on, and grow unbounded then finally, they can make the solution. So, erroneous that the solution will lose any meaning or significance and also the numerical calculations will go out of bounds. And therefore, finally the computer code will crash producing not a number kind of situation. So, what would be the distribution of u over the domain, a few steps later in time, we have just reached 100 time steps and it looks very disturbing.

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So, it may still be satisfying the boundary conditions, but what is happening in the in-between grid points is very worrying and if you notice the kind of values that have been achieved on the left end of the domain which is indicating the values of u.

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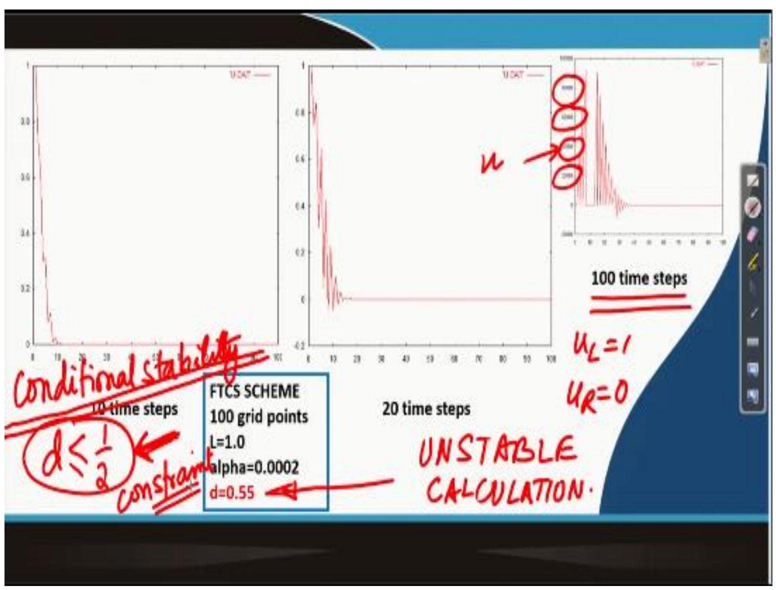

You have very, very significantly large values which have no connection with the boundary values. So, u on the left end was supposed to be 1, on the right end is 0 and you are producing values of the order of few tens of thousands in between. Which means certainly the solution is becoming increasingly erroneous and going out of bounds. If you were to run the solution for a few more time steps, it would finally crash.

So, what it proves is that Von Neumann stability analysis could predict well in advance that this would happen with the wrong choice of spatial steps delta x and delta t for a given value of diffusion constant. You may end up producing values of d which are well in excess of 0.5 for FTCS kind of discretization. And then if you are choosing to use FTCS for discretizing your governing equation, then this choice would lead to unstable calculations.

So, what you can see in the above plots is a typical indication of an unstable calculation. So, the condition that Von Neumann stability analysis derived for us bounding the value of d for FTCS scheme is called as conditional stability. That means, solution using FTCS algorithm will remain stable subject to the condition that you choose a value of d which is less than equal to 0.5. So, such conditions, when found through Von Neumann stability analysis would be termed as conditional stability.

So, if you are having unconditional stability for certain schemes, then such specific conditions will not come up at all. So, you will be able to show that the amplification factor in such cases would remain less than equal to 1 under any circumstances. If such is the situation, you will call such a situation as unconditional stability. That means, it will remain irrespective of your choice of delta x, it will remain stable irrespective of your choice of delta x and delta t and alpha.

So, conditional stability is stability of the scheme under the given choice such that you are able to satisfy the constraint. So, there is an issue of satisfying a certain constraint. If you satisfy the constraint, then the scheme will produce stable solution. So, we got to see through this numerical example, that it is indeed like that, that if you are satisfying the constraint, then it produces a stable solution.

The moment you violate the constraint, it will end up producing an unstable calculation. So, we saw an instance of conditional stability.

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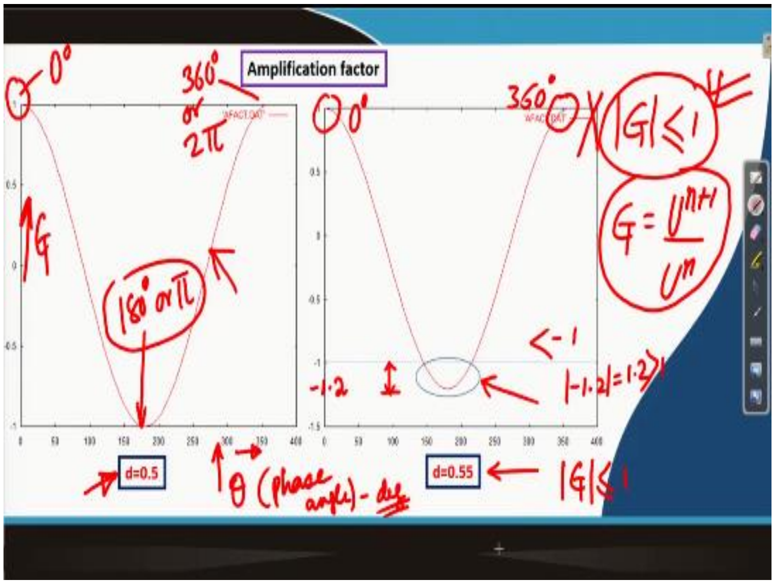

Now, if we were to plot the amplification factor for the two cases that we just looked at, it will show you some very interesting trends. Remember that you have to satisfy this condition for the amplification factor, where amplification factor is defined in this manner. So, when we chose d = 0.5 in the first example, how would the amplification factor vary. What the entire range of phase angles.

So, what we have plotted along the x axis is phase angle. We remember that this phase angle will vary between 0 to 2 pi. So, what we have plotted is in degrees. So, that is the x axis and what we have plotted along the y axis is G. So, that is for both the graphs that we are seeing in this slide. So, if we follow the plot for the d = 0.5 case, the maximum positive value of G that we see is attained for theta = 0 and for theta is equal to 2 pi. These are the two points.

0 degrees and this is 2 pi which is 360 degrees. And the maximum negative value is minus point, -1 which is reached at 180 degrees or pi. So, this is the behavior of G. The amplification factor for d = 0.5 which produced the stable calculations. So, this distribution certainly satisfies this constraint that mod G should be less than equal to 1 because mod of -1 is one which satisfies the constraint less than equal to one.

And of course, on the positive side, it automatically satisfies. How about the d = 0.55 case. So, we have satisfied mod G = 1 on the positive side because the maximum values again are at 0 and

360 degrees. But we have a problem area, here which is in circled in blue and that is where it has gone below -1 that means, it has become less than -1. There is a portion of the curve which has dipped below -1. So, if it has gone below -1 may be around -1.2.

So, mod of -1.2 will produce 1.2 which is greater than one. So, 1.2 is certainly greater than 1 and it does not satisfy the condition mod G less than equal to1 and that is where the problem is. So, when it comes to those phase angles, the solution produces erroneous results. And as we compute repeatedly over different time steps, these errors which are occurring from that phase angle range would corrupt the result more and more and more.

So, as solution proceeds, you will see more of these oscillations coming in the solution that you produce. So, we have to be very careful about this that we strictly follow the mod G less than equal to one condition. If we do not do so, that means if the finite difference form is failing to do so. For improper choice of parameters; whenever conditional stabilities coming into picture. Then we will fail to produce stable calculations.

In a later lecture, we will talk about how implicit schemes produce unconditional stability and from stability perspective, they are better choice then explicit algorithms because explicit algorithms most often would have conditional stability. Thank you.