Introduction to CFD
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Module No # 04
Lecture No # 17
Numerical Solution of Steady State Heat Conduction (Elliptic PDE)

Over the next few lectures we are going to talk about numerical solution of steady state heat conduction or elliptic partial differential equations in general.

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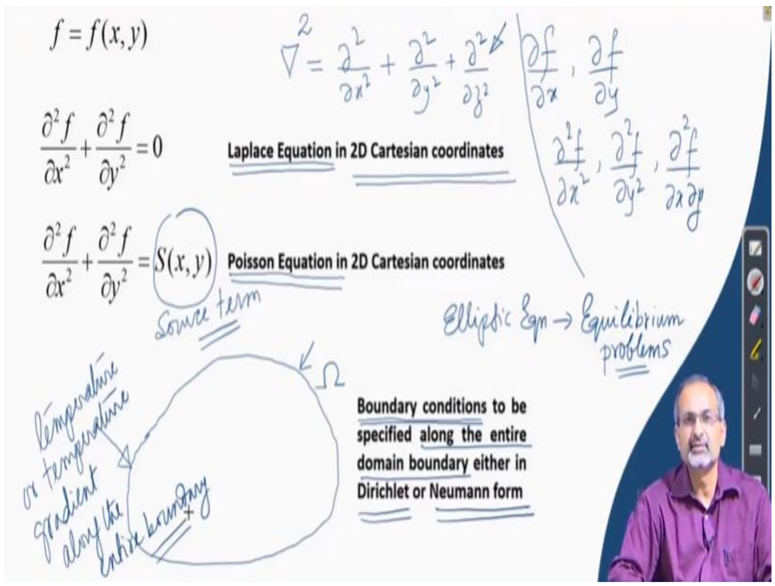

We have a function f of x and y so the function f is dependent on 2 independent variable x and y. In that case we could define partial derivatives of the function with respect to both x and y and there could also be mixed derivatives. For example we could have first derivatives of, f defined as del f del x, del f del y and we could have second derivatives of, f with respect to x with respect to y or also mixed derivatives with respect to both x and y.

We are going to talk about a very important equation Laplace equation by enlarge when we talk about the elliptic partial differential equations. And in this partial differential equation we find that second order derivatives of, f are involved. And they are either with respect to x or with respect to y but they are not mixed derivatives. Additionally we note that we are looking at the governing equation the Laplace equation in 2 dimensions and in Cartesian space.

Remember that we are importantly looking the Laplacian operator del square which in Cartesian 2D coordinates can be written like this. And in 3D coordinate can be written additionally with the third term included which corresponds to the second order partial derivative along z. And of course when we talk about approximate solution of Laplace equation we have to replace these second order derivatives by finite difference formula.

If you have a source term figuring on the right hand side of the equation with we have indicated by the function S which in general can depend on the spatial location. So it is a function of both x and y then we have Poisson equation again represented in 2D Cartesian coordinates. We recall from our previous discussions that when we are talking about solution of elliptic partial differential equations then elliptic partial differential equations fall in the category of equilibrium problems.

Incidentally steady state heat conduction equation happens to be an equilibrium problem. Just to sight an example if you have a domain which is given by this curve let us say the domain boundary is defined by omega. Then when you try to define Laplace equation or Poisson equation in general, an elliptic partial differential equation in this domain it is essential that you define the boundary conditions all along the domain boundary either in Dirichlet or Neumann form.

So if it is a heat conduction problem it balls down to defining temperature or temperature gradient along the entire boundary. And having given this boundary condition what Laplace equation would do is it will provide us the steady state temperature distribution within that domain defined by or bounded by omega. Now in principle if you impose certain temperatures or temperature gradients along the boundary and the inner part of the domain is not in equilibrium with those conditions that we have imposed at the boundary.

Then it will take a certain finite extent of time for the inner part of the domain to come to equilibrium with the boundary condition. So this will involve transient behavior that means a time dependent behavior by means of which the temperature will change at each and every point within the domain and then finally come to an equilibrium. Laplace equation does not give you an answer to this transient problem that means it will never tell you that what goes on in between to reach the equilibrium temperature distribution.

But it addresses the problem has a direct equilibrium problem itself which means that given a boundary condition what will be the final temperature distribution at equilibrium state is what is going to be answered by Laplace equation. So this aspect as to be kept in mind when we try to solve Laplace equation or try to find out interpretations from such solutions.

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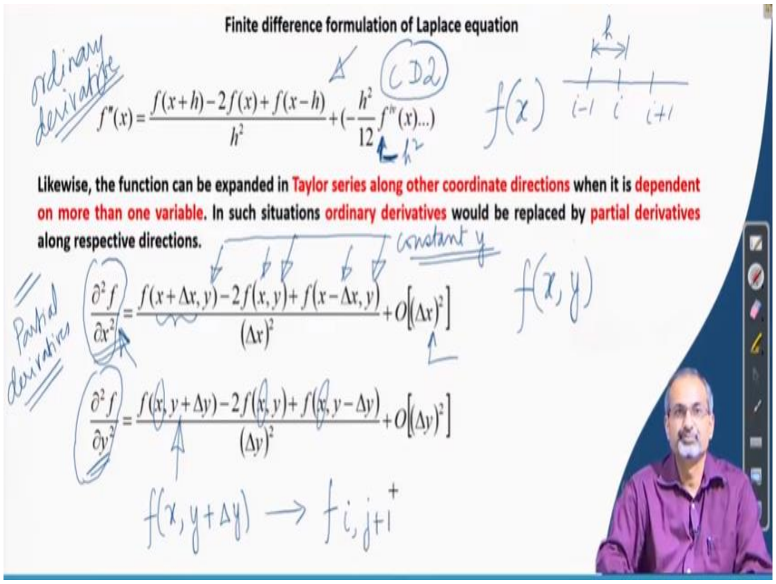

Since we want to handle second order partial derivatives in Laplace equation we would look back at the Taylor series, or Taylor table based finite different approximations that we have developed in due course for second order derivatives. You may recall at that time we were talking about functions which were only dependent on x. So they were essentially ordinary derivatives that we were taking about.

And this is a formula that we have derived which can be stated as a CD2 formula for second derivative. Because it as a stencil which looks like this centered, around the point i and it has the leading error term h square which means it gives you second order accuracy. Now when you are talking about functions which are dependent on more than 1 independent variable like the one we are talking about now.

So you now have a function f of x y we talk about the derivatives in partial sense however the Taylor series approach applies equally in this case 2. And therefore you can end up using the same kind of formula that you have derived earlier for approximating the ordinary derivative using finite difference formula over here as well. But only thing that you have to keep in mind is that when you are taking the second derivative and you are representing a finite differential formula for it and you are taking a second derivative with respect to x.

You are using functional values for constant y so only x changes here from x + delta x to x to x – delta x. But as you can see the y remains the same so this derivative applies for a constant y. This shows the order of accuracy it is second order accurate in terms of the grids spacing along x direction. In a similar manner when you are taking the finite difference approximation for the second derivative along y you can see that x remains constant while you move from one grid point to the other along the y direction.

You can very easily understand that these values which are represented at specific values of x and y could also be represented in terms of the grid index. Like for example this point may have a grid index which looks like this let us do a little more detailing on this.

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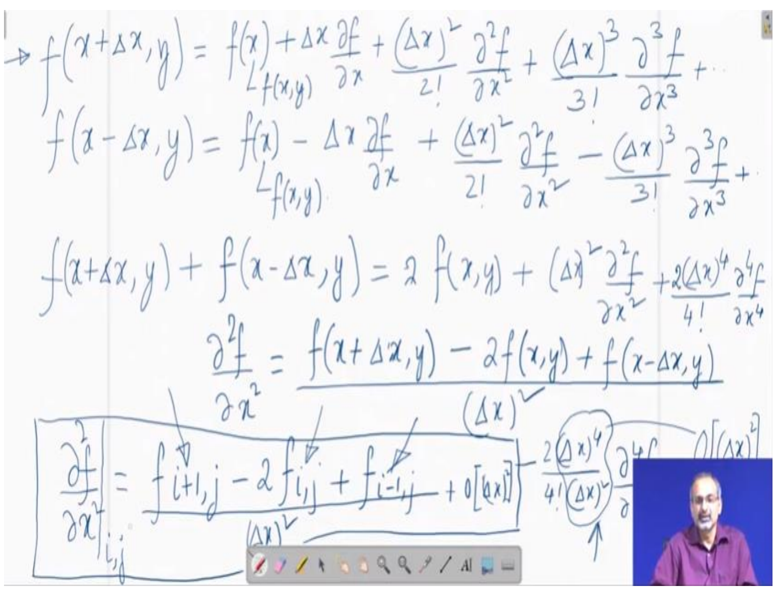

So we will write down the Taylor series along the x direction to express the value of the function f of x + delta x y. So let us have a look at how the Taylors series look like you can notice that in the series we are now having terms involving partial derivatives of different orders with respect to x. And you can see that we are defining all this for a given y so strictly speaking this is f of x, y. So we now have the Taylors series for f of x + delta xy and f of x – delta xy involving partial derivatives of, f with respect to x.

Now if we sum the 2 equations what do we get? So the left hand side when summed up gives you this and the right hand side let us write it as f of x, y. So right hand side gives us so this would be 2 times delta x to the power of 4 by factorial 4 and then multiplied by the fourth derivative and so on. Now if you just transport transpose the terms you will get an expression for the second derivatives again let us make it more specific here.

So this is the leading error term in your truncation error and you can figure out that this will reduce to delta x squared. So we have a series 2 scheme for the partial derivatives with respect to x. So this if you want to express it in the grid index form how would you write it? First of all you can simplify this to f of i + 1, j so we would no longer retain the geometric location which is given by the xy coordinates but shift to the grid nomenclature.

And therefore we replace the x + delta x / i +1 x – delta x / i – 1 and the y remains constant for all the terms and therefore there is no change in the j index. And the point x essentially refers to i and of course you are calculating this derivative at the, i, j grid point. So this is how it works so if you follow a similar procedure and expand in a Taylor series with f of x, y + delta y and f of x, y – delta y, you should be able to get a similar looking formula for second derivative with respect to y at the grid point i, j.

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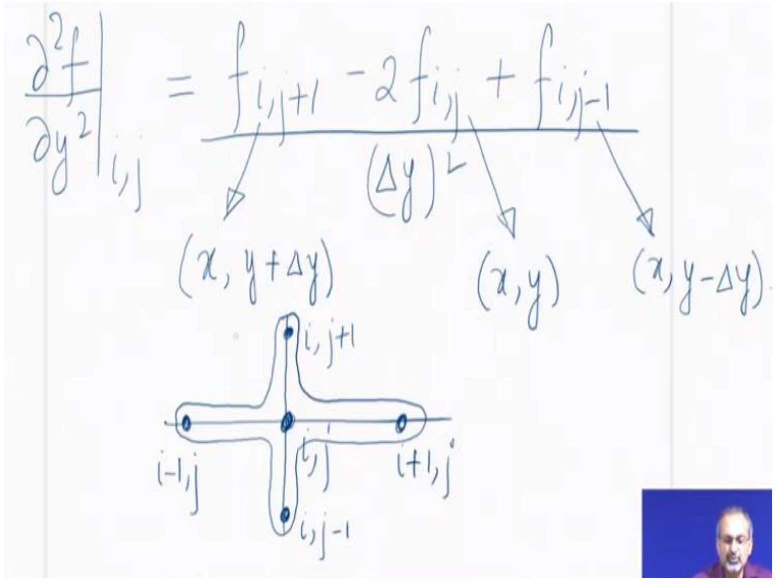

We can in fact try writing it intuitively so we can write it as so you can recognize that if you want to look at the geometric location of this point this would correspond to x, y + delta y. This would correspond to x, y and this would correspond to x, y – delta y so these are the geometric locations of those grid points. So now we have to be comfortably navigating between the geometric locations and that grid point locations even in 2 dimensions space.

So till before this most often we have been talking about a 1 dimensional grid and there we had often reduced partial differential equations to ordinary differential equation. But now onwards we will more often talk about problems where the function depends on more than 1 independent variable. Here we have dependence on 2 spatial independent variables the function could also be dependent on spatial variable as well as time.

Even, in that case the function as dependence on more than 1 independent variable and therefore all such problem will involve partial derivatives. So just looking back at what we are talking about in the grid sense we have used up a stencil like this in expressing the 2 partial derivatives which are of importance to us in as far as Laplace equation is concerned. Even if you include the source terms and try to solve Poisson equation there would not be any fundamental change.

The additional issued be that you have to accommodate source term on the right hand side of the equation, but the discretization on the 2 partial derivatives on the left hand side of the equation remains the same as it is for the Laplace equation. So now if you; plug in the 2 finite difference expressions that you formulate it for the 2 partial derivatives.

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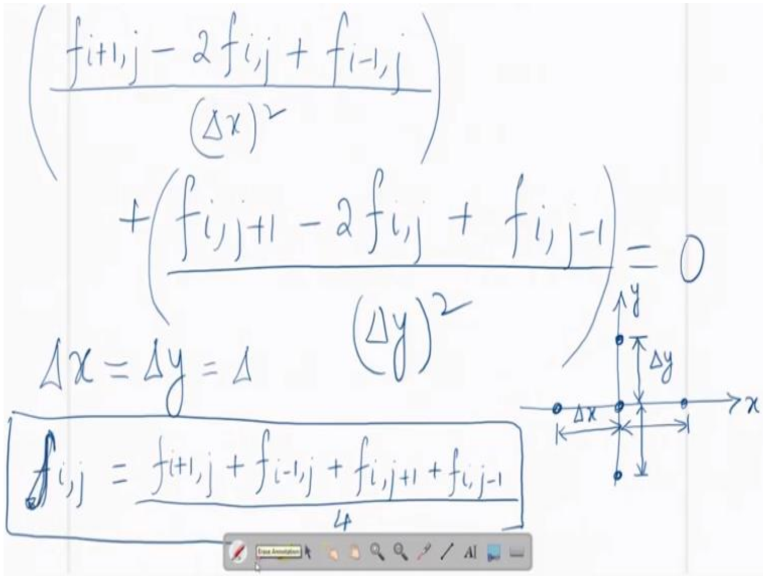

You have the discrete form of Laplace equation available so this is how the discrete form would look if you have different grid spacing's along the x and y directions. Of course you have a uniform grid along both the directions but you may be having different spacing along the 2 directions and therefore delta x and delta y may not be the same. However if delta x and delta y are same then this equation reduces to a still simpler form and we can show that this equation can then be written as.

So let us say delta x = delta y = delta and then we can understand that both the collection of terms that you have due to the 2 derivatives will have a common denominator. And therefore you will have the equation coming up as 4i+1j sorry fi+1j + fi-1j + f ij+1 + fij-1 -4f ij. So if you transpose it to the other side you have a 4f ij equals this if you divide all through by 4 then what you essentially have is an expression looking like this.

So this is rather a simplified form of Laplace equation when you have equal grid spacing's along the 2 orthogonal directions. We have been talking about the CD2 discretization or the second order central difference based discretization of Laplace equation till now. You could have more, higher order accurate schemes to discretize the derivatives. For example if you go in for a fourth order discretization that is by using the CD4 scheme that we have talked about earlier. You know that first of all you will lead a wider stencil to do that

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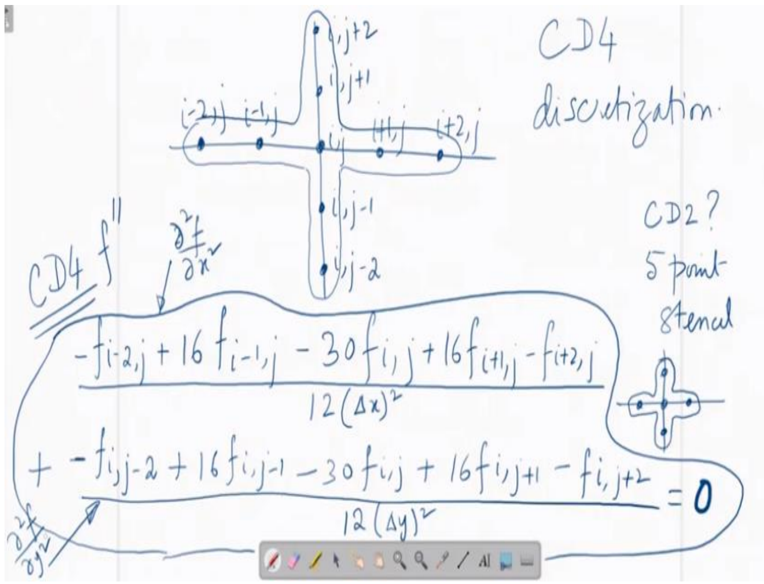

So let us say if we are talking about a CD4 discretization we would end up needing more number of points in the stencil. So when we say more we need to just think back and remember that what was the stencil size for CD2? It was a 5 point stencil. So if you recall we just drew this points and we could see that those actually forming a 5 point stencil for us. So the question is that what; would the stencil size look like for CD4 discretization?

So we can intuitively understand that it would look like this because we have already familiar with how CD4 works for ordinary derivatives for the second order finite differencing. So you have ij as the reference grid point and then all these neighboring points showing up in the stencil. Now we can obtain a CD4 discretization for the second derivative using a Taylor table approach more conveniently

If we do that we will find that it will looks like this so this is the representation for the partial derivative along x direction. So this is the approximation of del 2 x del f square using the fourth order central differencing. And then we add on the second derivative along y direction of course the formula looks more complicated because you have a wider stencil and it could be a good idea to try obtaining this discretization using a Taylor table approach.

One thing that we can understand is that it will certainly enhance our formal accuracy but it would be more difficult for us to handle a scheme of this kind when we go closer to the boundaries. Usually the kind of strategy that, we have followed in order to tackle boundary conditions using the (()) (28:16) scheme which we showed in the previous lecture. We also use similar strategies when we handle this explicit finite difference approximations by switching over to lower order accurate schemes in general as we approach the boundaries.

Because that would involve smaller stencils and therefore would be more convenient to represent thank you.