Aircraft Structures - 1
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Lecture No -20
Castigliano's Theorems

Welcome back to aircraft structures one course this is Professor Anup Ghosh from Aerospace Engineering IIT Kharagpur. We are in continuation of the week 4 lectures in that series in the total sequence the lecture number is 20 we will learn Castigliano's theorem today that is very, very popular and this is sometimes it is said it is a variation of the different energy methods only. But let us we will try to establish the equation first and then we will solve a few example.

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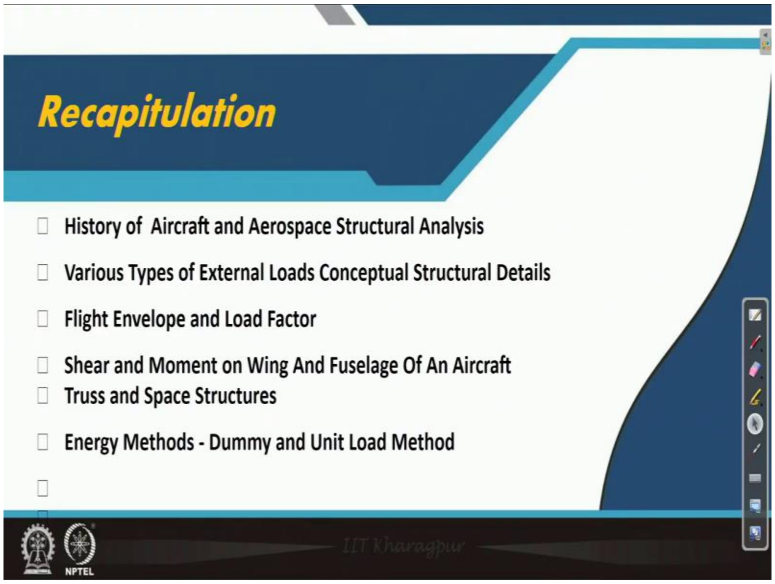

Before we start anything as usual we are supposed to recapitulate things. In this recapitulation we have done history of aircraft and aerospace structural analysis, various types of external loads conceptual structural details flight envelope and load factor. We have done shear and moment on wing and fuselage of an aircraft has also been covered. We have also done truss plane truss and also space truss or the space structures we have solved.

And in last three lectures we have considered with many examples dummy load method and unit load method. And in this week class will or this lecture we will try to understand or solve the series sorry Castigliano's theorem and let us start that.

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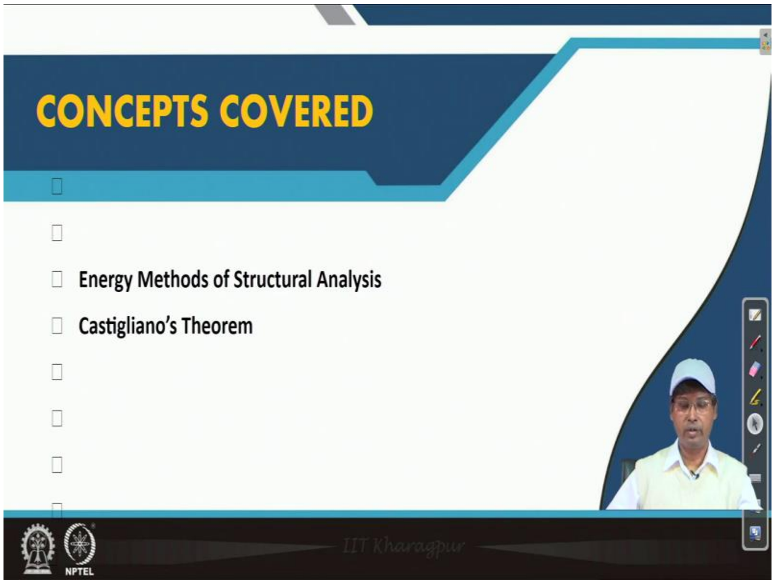

So, Castigliano's theorem is the main topic let us try to cover.

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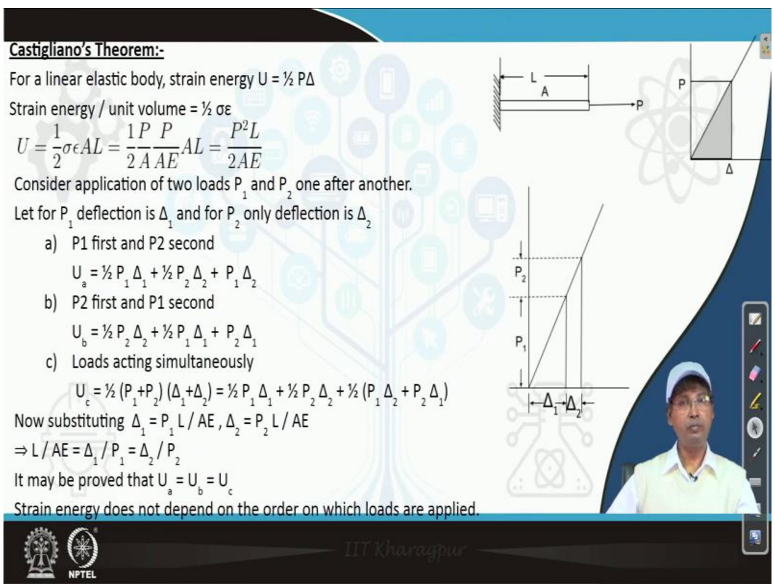

So, Castigliano's theorem let us try to establish first, we are considering again one bar axially loaded by P of length A sorry length L and cross section A, just load deflection curve is shown here. Now the strain energy for a linear elastic body strain energy is equal to half P Delta strain energy per unit volume we did not really define as half of Sigma into Epsilon and if we put those values of half Sigma and epsilon that gives us the value as it goes to P Square L by A twice A considering application of two loads P 1 and P 2 one after another.

We may have three different scenario in one first case that P 1 is applied first P 2 next P 2 is applied first P 1 next and in the last case both are applied simultaneously. So, let us see how the energy changes or how what is the net amount of energy and what is the relation between those energies. So, in the first case let us say it is a, P 1 first P 2 second. So, P 1 first means half of P 1 Delta 1 P 2 comes next because of that the displacement is Delta 2 that is half P 2 Delta 2 but P 1 remains in the system that is the reason work done is done by that P 1 is equals to P 1 Delta.

Similarly if we go for the second one the P 2 first and P 1 second half P 2 Delta 2 for the first half P 1 Delta 1 for the second one and then P 2 remains that is why P 2 and the second displacement is Delta 1. Similarly if we go for the second necessary loads acting simultaneously that means P 1 plus P 2 is creating a displacement Delta 1 plus Delta 2 this scenario is depicted here in this with Delta 1 P 1 corresponding to Delta 1 P 2 corresponding to Delta 2 the sequence is not shown there may be more many figures for that but I think these are easy and usually you can guess it.

So to do that you multiply those quantities and we see that behalf of P 1 Delta 1 plus half of P 2 Delta 2 plus half of P 1 Delta 1 plus P 2 Delta 1 we get the total load. Now since we have a relation between the displacement and the applied load Delta 1 is equals to P L by AE Delta 1 is equals to P 1 L by AE Delta 2 is equals to P 2 L by AE from this relation we can easily find that that Delta 1 by P 1 is equals to Delta 2 by P 2.

Now if we use this relation if we do simple there were a little bit with this 3E relation you can easily prove that you may consider that as a home assignment is it you can easily prove that this is the U a U b and U c are the same that means the sequence is not important while we apply the load. Strain energy does not depend on the order on which loads are applied.

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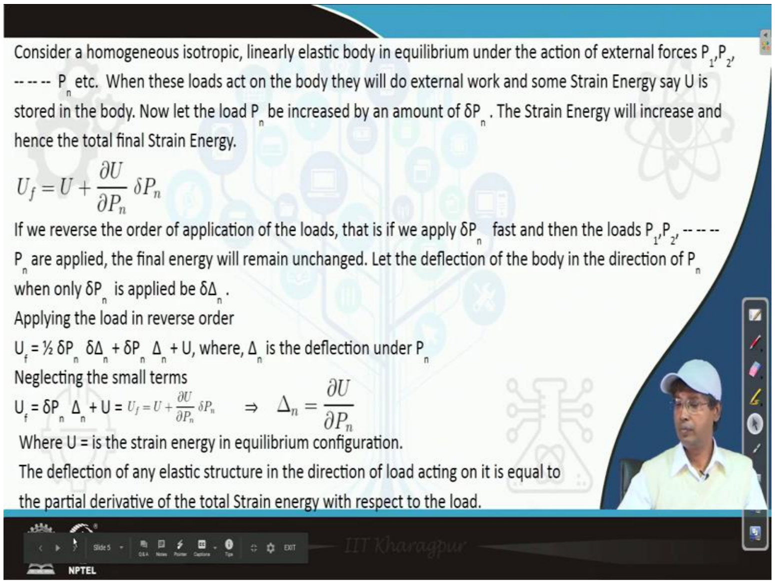

Consider a homogeneous isotropic linear elastic body in equilibrium under the action of external forces P 1 P 2 and so on as P n. When these loads act on the body they will do external work and some strain energy say U is stored in the body. Now let the load P n be increased by an amount Delta P n this is important. The strain energy will increase and hence the total final strain energy will become U because of those what is stated here and plus because of Delta P n.

And Delta P n is multiplied by the change rate of change due to Delta P n, P n that is the reason the del U Delta P n is the gradient multiplied by the Delta P n. So, if we reverse the order as we have done in the previous case of application of the load that is if we apply Delta P and first and then the loads P 1 P 2 to P n, the final energy will remain unchanged that we have already proved. Let the deflection of the body in the direction of P n when Delta P n is applied is equals to Delta n small Delta or variation small variation of Delta of capital Delta.

Applying the load in reverse order what do we have following the previous concept what we have done that this is half of Delta P n Delta n and this is because the Delta n is created by the inset of loads and that increases this and they energy you what we are not explicitly writing it because this was existing this is coming extra where Delta n is the deflection on that P n. Now neglecting the small terms that means this particular term this is very, very small both are very, very small amount and if we neglect that so you F is equals to Delta P n capital Delta n plus U is equals to this one because in the previous case energy is this.

And that leads to that capital Delta n is equals to del U by P n where U is the strain energy in equilibrium configuration. The deflection of any elastic this is known as the Castigliano's second theorem we will see the theorems in the next slide. The deflection of any elastic structure in the direction of load acting on it is equal to the partial derivative of the total strain energy with respect to the load.

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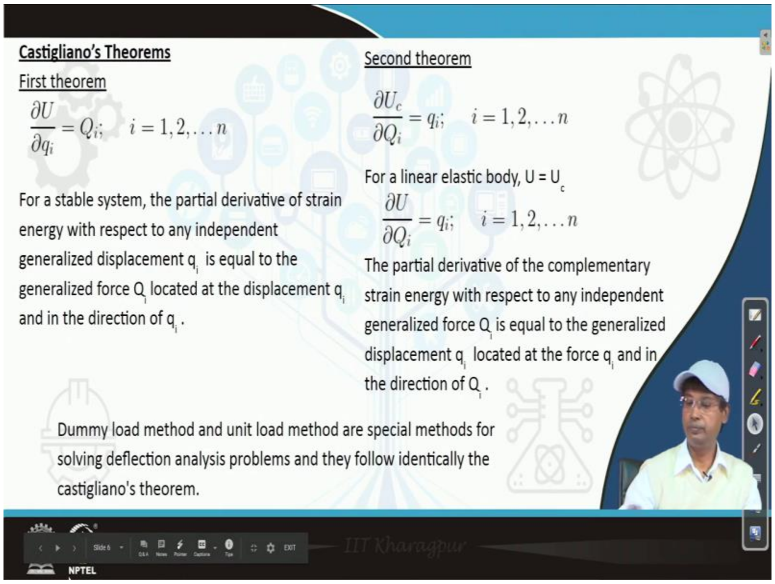

So, the second theorem as let us read first then we will read the first one del U by del qi the partial derivative of the complementary strain energy with respect to any independent generalized force Qi is equal to the generalized displacement small Qi located at the force qi and in the direction Qi. So, these two things are very, very important located at the force Qi and in the direction Qi.

So, with that it is better to see and this because we consider why do we say the statement with respect to complementary energy first because we have seen in the first lecture of this week at this model that if the relationship is is not linear it is not linear elastic the derivation partial derivation of complementary energy gives the displacement not the strain energy. So, that is the reason we say that first and then for a linear elastic body as it is mentioned here we can consider that del U del Q capital Q i is equals to small qi displacement in the ith position in the direction of applied load Q i is small qi.

So following that things we can also prove in a different way that is not in the scope of the study so we may refer advanced books for this but let us learn that first theorem del U del Qi is equals to small qi or the derivation with respect to the displacement gives as the force. For a stable system the partial derivative of strain energy with respect to any independent generalized displacement qi is equal to the generalized force Qi located at the displacement qi and in the direction of qi.

So with this it is just opposite to that so it is easy to remember the first theorem. Now it is an important point to note here that the dummy load method and unit load methods are special method for solving deflection analysis problems and they follow identically the Castigliano's theorem. So, we this note if you look at it if you solve the problems carefully if you look at it in many times you will understand you will have the idea that these methods are not much different but it is some variation of one is the some variation of the other.

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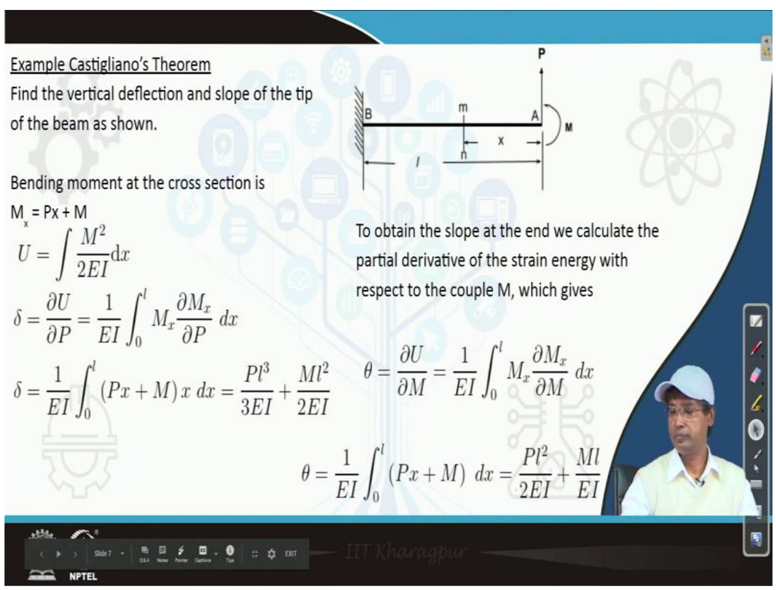

So, let us solve a problem but the problem is again a cantilever beam. In the cantilever beam there are two loads applied at the tip. One is vertically upward P and a moment n. So, it is asked that that what is the vertical deflection of Point A as well as what is the slope at that point. Since M is there and since P is present there we need not to apply any dummy load or unit load we if we consider derivative with respect to P or the energy we will get the vertical deflection.

If we consider a partial derivative with respect to the M we will get that slope. So, that is in this example is not very tough it is the easy to understand and it explains the method very well but what I would suggest that in the last three lectures whatever problems we have solved using dummy load method and unit load method you better try to solve those methods using Castigliano's theorem.

Let us solve this one again this as I have mentioned at any section as it is given here a section which is x apart, x is measured from this portion the moment is this P x and M and it is simply that U is M square by 2 EI for deflection what we have done we have considered as I mentioned for deflection it is vertical upward deflection and in that direction P is applied so there is no problem we are simply directly con considering derivative of partial derivative of that energy and we get that 1 by E I equals to 0 to L Mx del Mx del P dx.

So while we while we substitute this values M x P x is the derivative value this x and this is Mx and then if we integrate it we get that PL cube by tri EI and Ml square by twice EI is the deflection. So, if we solve without the M we generally get this value and this is the contribution of M that is quite clear. So, for the slope before we go for the slope better we try to draw it. So, this is the slope we are trying to find out and this is the deflection what we are trying to find out and then we see what is done here.

Then to obtain the slope at the end we calculate the partial derivative of the strain energy with respect to the couple M which keeps theta equals to del U del M and M x del M del M x del M is there and that gives us since this is there is only one will come so no point nothing is mentioned yet only a Mx is present here and that is integrated and it gives that L square by 2 PL square by twice EI Ml by EI.

So with that example we conclude today's lecture of Castigliano's theorem and as I have suggested already you please better try to solve the other problems what are covered in the last examples in last lectures to solve these equations.

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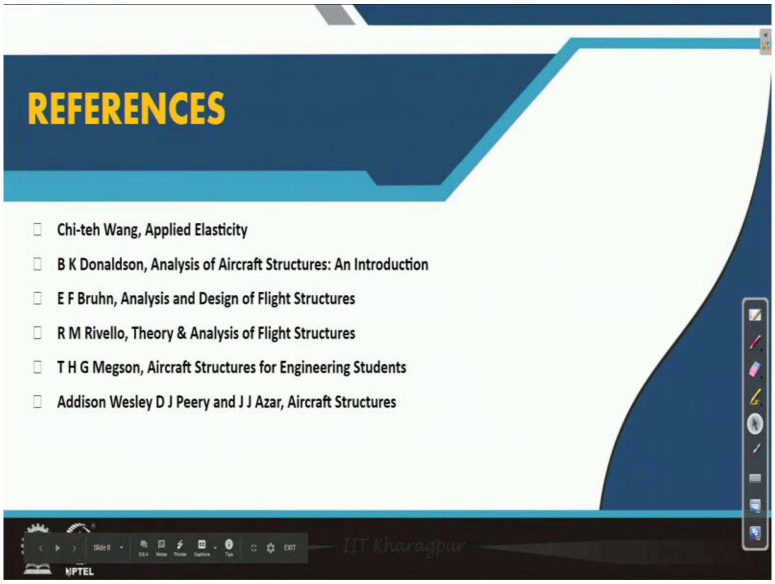

So, next slide is a simple repetition of the references.

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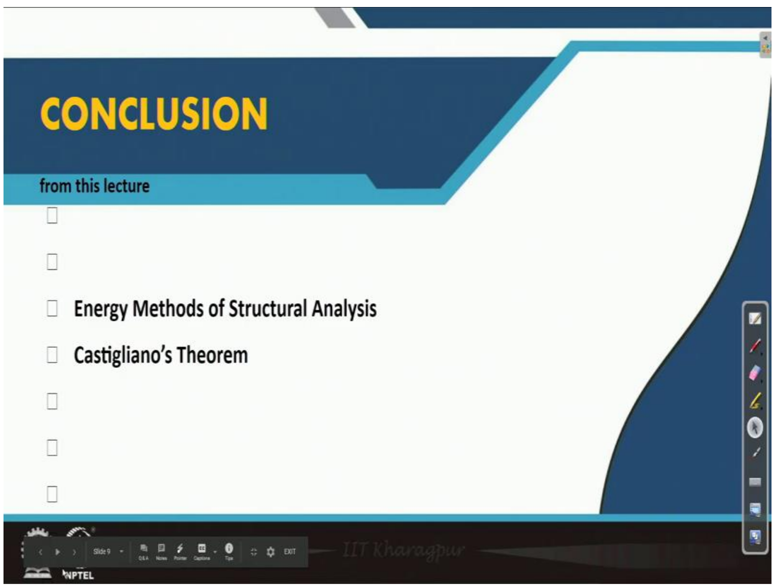

And after that we have one slide what we have learnt today that is Castigliano's theorem that also comes in the energy method and we will follow in next class the Rayleigh Ritz method and I thank you for attending this class the next class we will learn the Rayleigh Ritz method, thank you.