Aircraft Structures - 1
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Lecture No -17
Introduction to Energy Methods

Welcome back to aircraft structures one course. This is Professor Anup Ghosh from department of aerospace engineering IIT, Kharagpur we are at the beginning of fourth week of the course that is known as module 4, this is lecture number 17 we will get introduced to the energy methods, principles of energy methods we will discuss in a very, very brief way. The concepts presented here is difficult to present in this few words.

So for further query or inquisitiveness to satisfy the inquisitiveness please refer to books advanced books available on variational calculus or energy methods related books. So, with that let us proceed.

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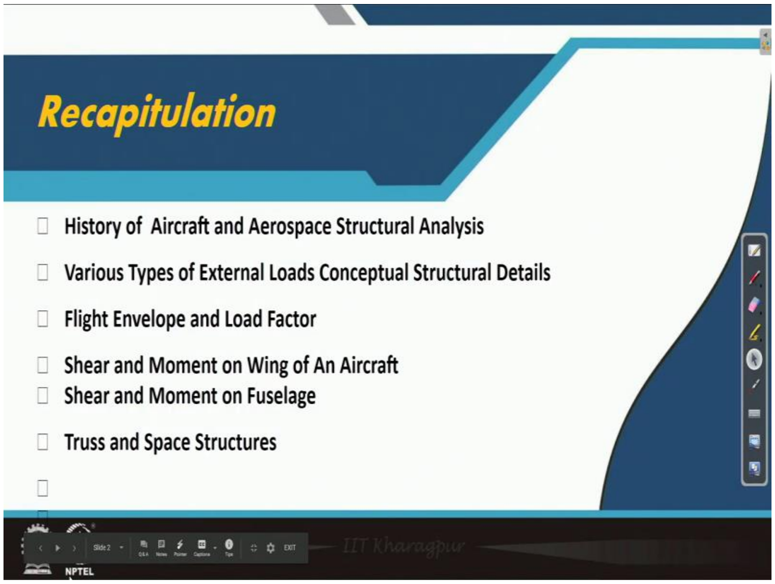

As usual we need to recapitulate what we have done. We have done we have done history of solid mechanics or structural analysis and then brief history of development of aircraft then flight envelope and loads, load factor how load comes into details of fabrication and internal fabrication details of structures. Then we have come across loads coming to the wing and fuselage of aircraft how the bending moment shear force has come in to the wing and fuselage.

And then in the last week we got introduced with the truss system. In the truss system the advanced way of analyzing three-dimensional structures or three-dimensional trusses we have seen we have solved a few problems related to aerospace engineering. And then will this week will proceed further with the energy methods.

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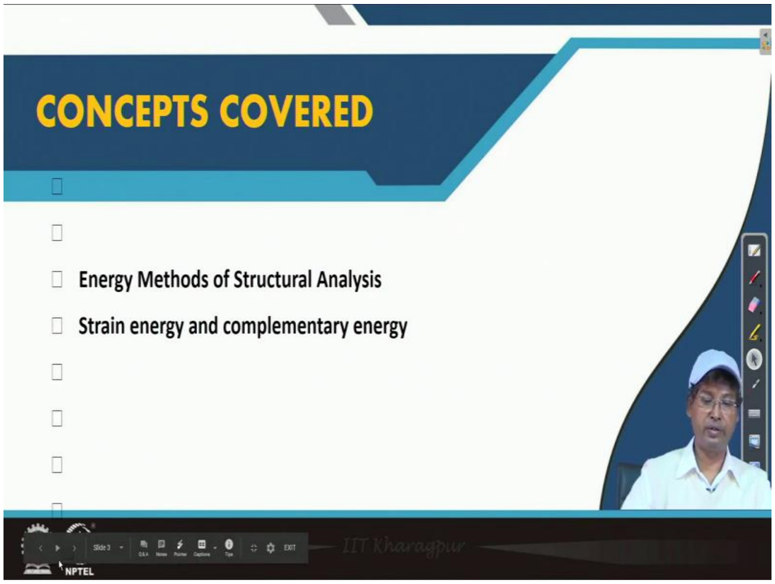

Energy method of structural analysis that is what is our aim to learn various methods we learn starting from the stationary value of potential energy to Castingliano's theorem to Raleigh's method many, many methods will come slowly and we learn dummy load method, unit load method all those methods will come slowly and we will learn those things let us cross it.

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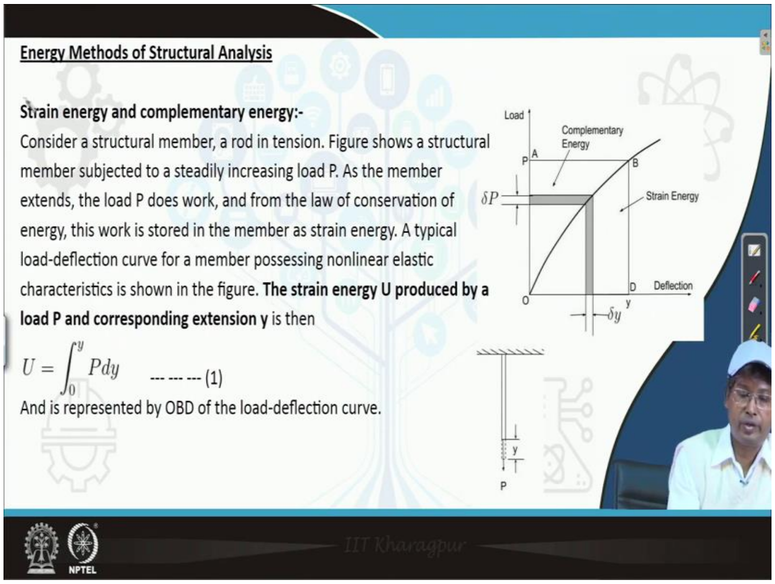

So, energy method of structural analysis we are starting, strain energy and complementary energy. The concept of strain and complementary energy is the first topic we are getting into. Consider a structural member a rod in tension this is the figure you should refer for that figure shows a structural member subjected to a steadily increasing load P. As the member extends the load does work and from the law of conservation of energy this work is stored in the member as strain energy.

A typical load deflection curve for a member possessing nonlinear elastic characteristic is shown in the figure, please mind it this curve represents nonlinear elastic material that is the reason we see a curve it is not a straight line. The strain energy U produced by the load P and corresponding extension Y is then U equals to integration from 0 to y P dy and is expressed sorry and is represented by OBD of the load deflection curve OBD this portion represents that energy U.

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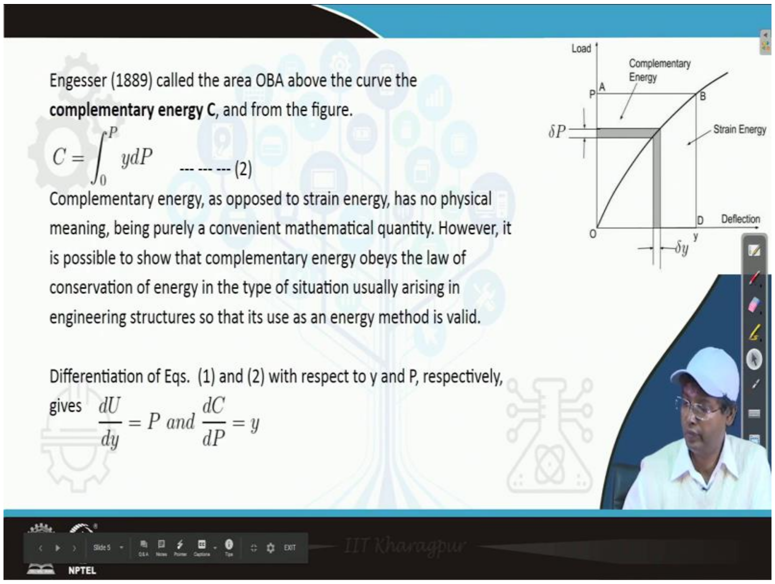

Engesser in 1889 called the area OBA above the curve as the complementary energy C and from the figure we see that C is equals to integration from 0 to P y dP. Complementary energy as opposed to the strain energy has no physical meaning. Physical meaning of strain energy is described in the previous slide being purely a convenient mathematical one quantity so it is a purely mathematical quantity.

However it is possible to show that complementary energy obeys the law of conservation of energy in the type of situation usually arising in engineering structures so that its use as an energy method is valid. So, we will be using that one and that whatever is shown said here that we can use it for structural analysis that will slowly establish. Differentiating equation 1 and 2 to is here the 1 is with U, so with respect to y and P respectively gives that dU dy is equals to P and dC dP is equals to y.

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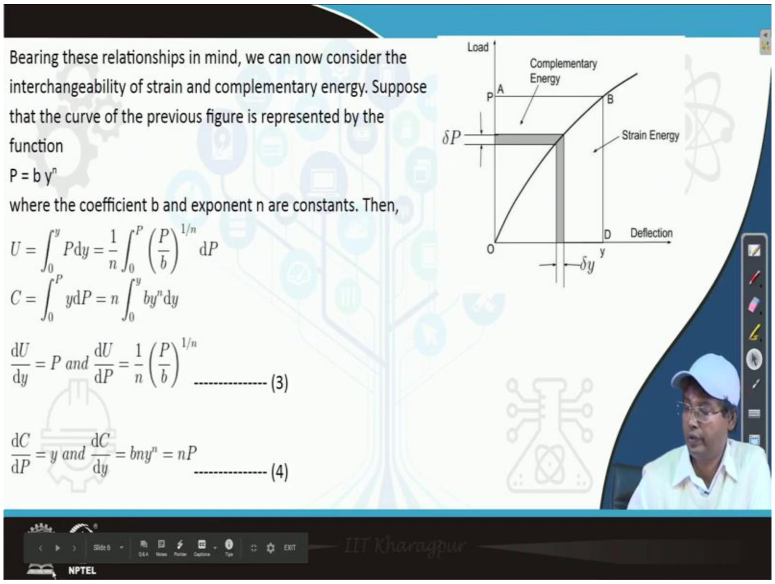

Bearing these relationships in mind we can now consider the interchangeability of strain and complementary energy. Suppose that the curve of the previous figure is represented by the function P equals to b Y to the power n where the coefficient b and exponent n are constants. Then if we do a simple calculus we can find that you may be expressed as we have said earlier or using this function we can express it as 1 by n integration 0 to P P by b to the power 1 by n dP.

Or C may be expressed as this also n integration 0 to y b y to the power n dy and if we take the derivative as it is given in the last slide what we get that dU dy is equals to P and dU dP is not having a straightforward equation it is having the effect of non-linearity is quite clear 1 by n P by b to the power 1 by n. Similarly dC dy is becomes b ny to the power n or nP.

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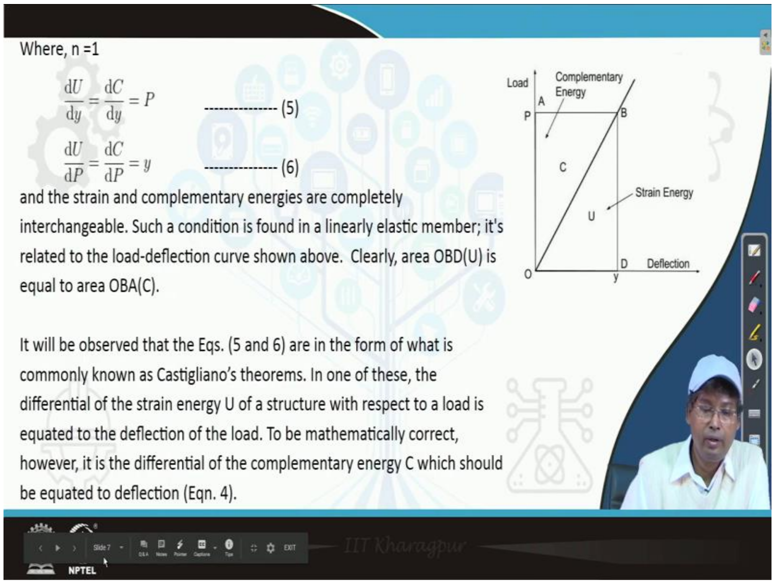

Now it is the most common case is the linear elastic one that curve is shown here. This is the curve for linear elastic one and for n equals to 1 it becomes a linear elastic material and in that case dU dy becomes equals to P as well as dC dy becomes equals to P whereas the other way dU dP and dC dP becomes equals to y. And the strain and complementary energies are completely interchangeable. Such a condition is found in a linear elastic member it is related to the load deflection curve shown on the right hand side.

Clearly the area OBD is equal to the area OBA strain energy and complementary energy. It will be observed that the equations 5 and 6 are in the form of what is commonly known as Castigliano's theorem. This is more popularly known as Castingliano's theorem. In one of these the differential of the strain energy U of the structure with respect to a load is equated to the deflection of the load.

To be mathematically correct however it is the differentiation of the complementary energy C which should be equated to the deflection. So this is more appropriate it says instead of this that we have if you look at the equations in the previous page that you can easily understand.

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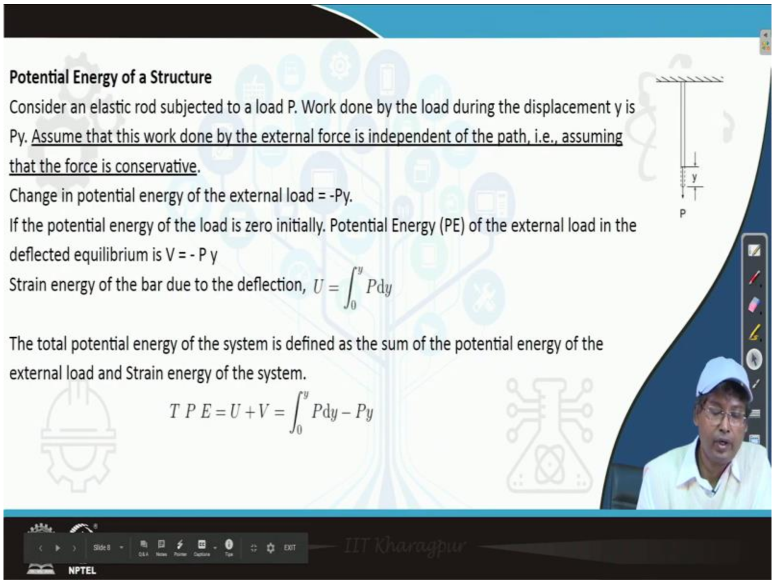

Potential energy of a structure: Consider and elastic rod subjected to a load P work done by the load during the displacement y is Py. Assuming that this work done by the external force is independent of the path that is assuming that the force is conservative there is a big proof for that in advanced books let us assume this to continue. Change in potential energy of the external load is equals to minus Py. If the potential energy of the load is zero initially potential energy of the external load in the deflected equilibrium is V equals to minus Py strain energy of the bar due to the deflection is U equals to what we have already seen integration 0 to y P dy.

The total potential energy of the system is defined as the sum of the potential energy of the external load and strain energy of the system that is what U+ V and that we makes it that total potential energy is equals to integration 0 to y P dy - P into y.

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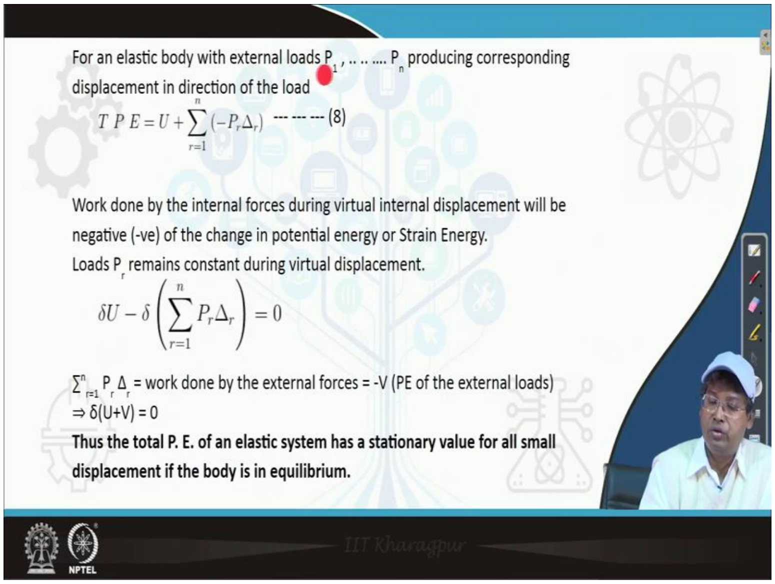

For an elastic body with external load P 1 P 2 P 3 P n producing corresponding displacement like Delta 1 Delta 2 Delta capital Delta l in direction of the load, the total potential energy becomes U plus summation of R equals to 1 to n minus P r Delta r. Work done by the internal forces during virtual internal displacement will be negative, if the internal forces are virtual it is negative of that change in the potential energy or strain energy.

Load P r remains constant during the virtual displacement so we can write that it is virtual change of energy Delta U minus Delta summation of P r Delta r from 1 to n. Again if we look at the summation of P r Delta r is the work done by the external forces which may be said as the minus of V potential energy of the external loads. So, summing up this with this concept we can write that the change of any small change or variation of U + V is equals to 0.

And in language if we write that thus the total potential energy of an elastic system has a stationary value for all small displacement if the body is in equilibrium. So, with that concept let us proceed further we will see we will need to use this concept to solve problem.

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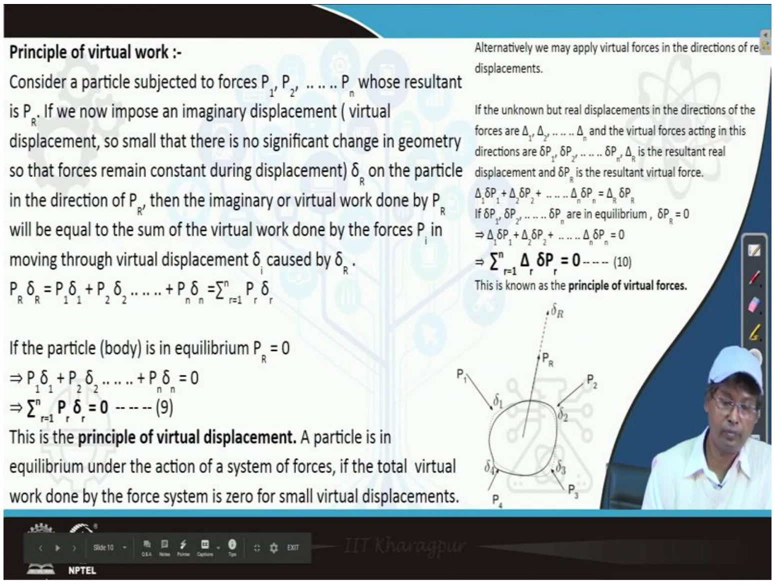

Principle of virtual work principle of virtual work: Consider a practical solid particle here it is shown subjected to forces P 1 P 2 2 P n whose resultant is P r resultant is shown here as P r, if we now impose an imaginary displacement Delta R on the particle in the direction of P r then the imaginary r virtual work done by P r will be equal to the sum of the virtual work done by the forces P i in moving through the virtual displacement Delta i caused by Delta r.

So it says that if there are P 1 P 2 P 3 P 4 and many more up to the n and the corresponding virtual displacements are Delta 1 Delta 2 Delta 3 Delta 4 then and those are having resultant as P r and Delta r what we can write that P r Delta r is equals to P 1 Delta 1 plus P 2 Delta 2 and summation like that and in a summation form it is like that. But before we go further here in this bracket I skipped this it is introduced that the virtual displacement is so small that there is no significant change in geometry so that the forces remain constant during displacement with this concept we introduced the virtual displacement.

Now if this is what we have P r Delta r equals to this if the particle or the body is in equilibrium so resultant P r is definitely is equals to 0. So, this side becomes 0 and or in any portion of the this also is equals to 0 or this is equals to 0 and as a summation form we write that P r Delta r is equals to 0 and we say that this is the principle of virtual displacement. A particle is in equilibrium under the action of a system of forces if the total virtual work done by the force system is zero for small virtual displacements.

Similarly as we have introduced here the virtual displacement we can introduce here as a virtual force that is there is in this portion I have kept in small font because it is almost repetition of the same thing. Only instead of virtual displacement the same principle and concept may work in the same way and we may get one more equation where it is virtual forces acting and we say that is as the principle of virtual forces.

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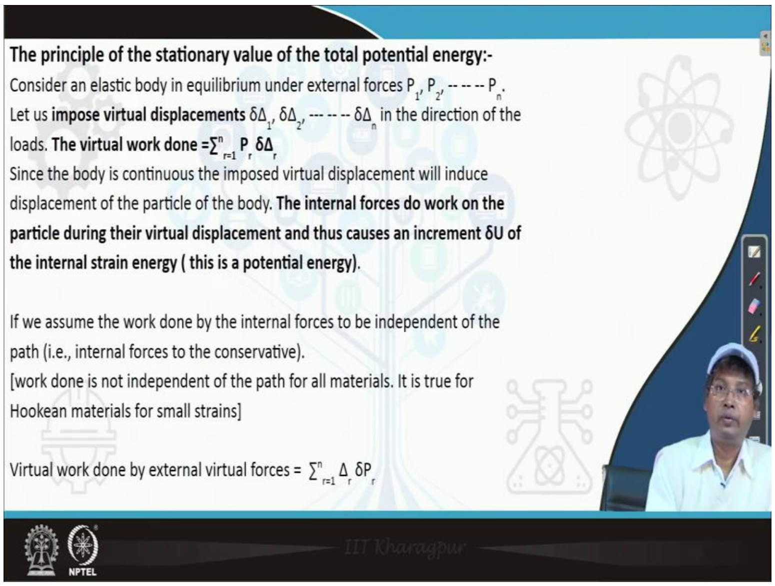

The principle of stationary value of total potential energy will be discussing now before that we let us define again the virtual work done in 2 form one in the form of the virtual displacement and other in the form of virtual forces. Consider and lasting body in equilibrium under external forces P 1 2 P 2 2 P n let us impose virtual displacement Delta Delta 1 to Delta Delta n in direction of the loads and then already you have learned that the virtual work done is equals to P r Delta Delta r summation over r to n.

Since the body is continuous the imposed virtual displacement will induce displacement in the particle of the body the internal force do work on the particle during the virtual displacement and thus causes an increment of the strain energy that is Delta U of the internal strain energy this is a potential energy. And then similar way with similar concept if we follow for the virtual forces if we assume the work done by the internal forces to be independent of the path.

Internal forces to be conservative we can straightforward say that the virtual work done by external virtual forces is equals to Delta r Delta small Delta P r summation over r equals to n but in this point the assumption what we say that is not always true that is only true for Hookean material but considering Hookain material will proceed further.

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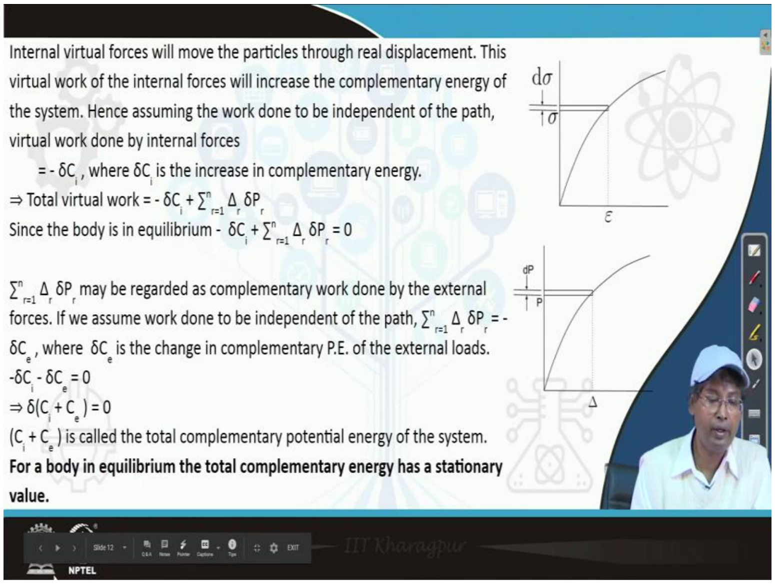

Internal virtual forces will remain sorry internal virtual forces will move the particles through the real displacement. This virtual work of the internal forces will increase the complementary energy of the system hence assuming the work done to be independent of the path virtual work done by internal forces is equals to minus of Delta C i where Delta C i is the increase in complementary energy.

So, the total virtual work becomes minus Delta C i + Delta P r capital Delta r multiplied by small Delta variation of P r and virtual P r summation over r equals to 1 to n since the body is in equilibrium this total system or the total virtual work becomes equals to 0 and summation of Delta r Delta P r may be regarded as complementary work done by the external forces. If we assume work done to be independent of the path summation of capital Delta r small Delta P r summation over r equals to 1 to n is equals to minus of Delta C e where Delta C e is the change in complementary potential energy for the external load e represents the external loads.

And then from this equation we can directly have the equation as minus Delta C i - Delta C e is equals to 0 and then we can say that variation C i + C e is equals to zero, so we say that this C i + C e is called the total complementary potential energy of the system for a body in equilibrium the total complementary energy has a stationary value like the total potential energy what we have already done.

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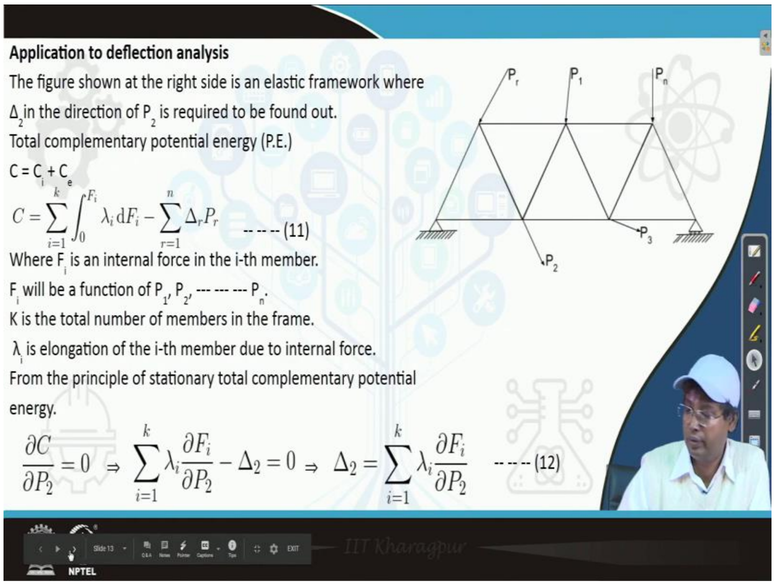

So, let us apply the concept of stationary value of the energy into problem-solving and we think with an example we will see how that we can use and solve a problem. This figure what you see here is a truss in this figure there are forces starting from 1 2 3 4 to r 1 2 3 2 r to n, the figure shown at the right side is an elastic framework where Delta 2 in the direction of P 2 is required to be found out.

Total complimentary potential energy is equals to C i + C e as we have got in the previous example we can write that 1 as summation of i equals to 1 to k for 0 to F i for individual member lambda I delta F I, so if i is here the individual member forces. So, if we name the member 1 2 3 4 5 6 like that it will come as lambda i F i where F i is the internal forces in the ith member F i will be a function of P 1 P 2 to P n, k is the total number of members in the frame and lambda i is elongation of the ith member due to the internal force.

So once we find out this value we can find out the total potential complimentary potential energy. Now as it is said it is having a stationary value what we can see that from the principle of stationary total complimentary potential energy we can say that the partial derivative of total potential complimentary potential energy with respect to P 2 since we want the Delta 2 becomes equals to 0 and then we go for partial derivation with respect to this, this lambda i remains same Delta Fi Delta P 2.

Since Fi is a function of P 1 P 2 and P n we get this portion from this and whereas this portion except all other Delta except Delta to all other things vanishes because those are not function of P 2. So, this in a straightforward way gives us that Delta 2 is equals to summation of i equals to 1 to k lambda i which is elongation of a member at member and then del Fi del P 2 as the partial derivative of each member with respect to the force P 2.

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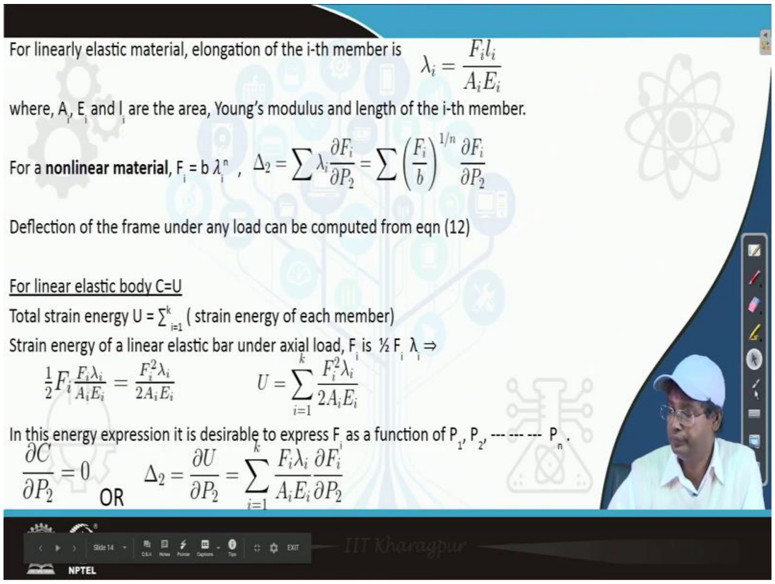

For linear elastic material elongation of ith member is lambda i F i l I E i divided by A i E i this is a very well-known formula we have already come across this many times where A I E i and l i are the area Young's modulus and length of the ith member respectively. For a nonlinear material this portion is just introduced to you to keep in mind that the case is not always true. In case of nonlinear material what we can observe that if it is F i equal to b lambda I to the power n that same way we can find out the solution.

But it the equation changes a little bit we need to substitute that value and to carry out that operation anyway the equation 12.1 whatever is there in the previous space that will give us the Delta 2. For linear in elastic material where we have already seen that C is equals to U the total energy U is equals to i equals to 1 to k strain energy of summation of strain energy of each and every member. Strain energy of linear elastic bar under axial load F i is half of a F i lambda.

So that half i half of a F i lambda this is value of lambda is written here and total U becomes summation of a F i square lambda divided by 2 A i E i and that is what since we have seen this and C is equals to U in this energy expression it is desirable to express F i as a function of P 1 P 2 to P n and we get following the discussion what we have done in the; with respect to last slide that Delta 2 is equal to del U del P 2 and this equation, this is the equation summation from i to k F i lambda i divided by A i E i and partial derivative of Delta del F i del P 2 partial derivative of each and individual member forces with respect to P 2.

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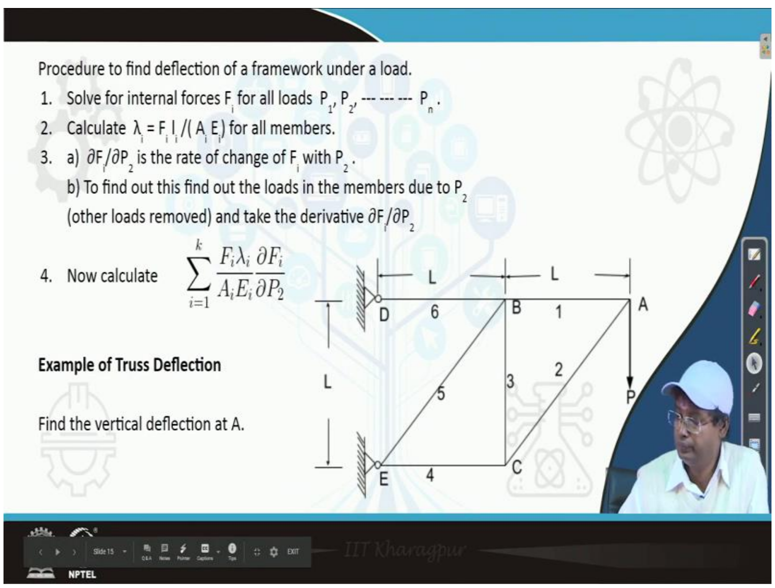

Procedure to find deflection of a framework under a load: This is the standard procedure following similar to this procedure we will solve this problem. Solve the internal forces F i for all loads P 1 P 2 P n calculate lambda i equals to F i l i by A i E i for all members find out the partial derivative of del of Fi with respect to P 2 that is del F i del P 2 is the rate of change of F i with respect to P 2 to find out this find out the loads in the members d F i dP 2 other loads removed and take the derivative of del F i del P 2.

Now calculate the expression what we have said in the last phase same expression and find out the deflection in the desired direction. Here P 2 is symbolic with respect to the previous discussion previous figure but this is not always P 2 definitely it is the direction of the desired force where we want to find out. For this case in this problem it will be with respect to P because there is no other force in this member.

If there are other forces in the member then we need to modify it with respect to that force. So, this is a truss where we need to find out the vertical deflection at A, P load is acting here. So, if it is a vertical length deflection is acting in this direction following this formula if we make it del F i del P and if we complete this we will get the solution.

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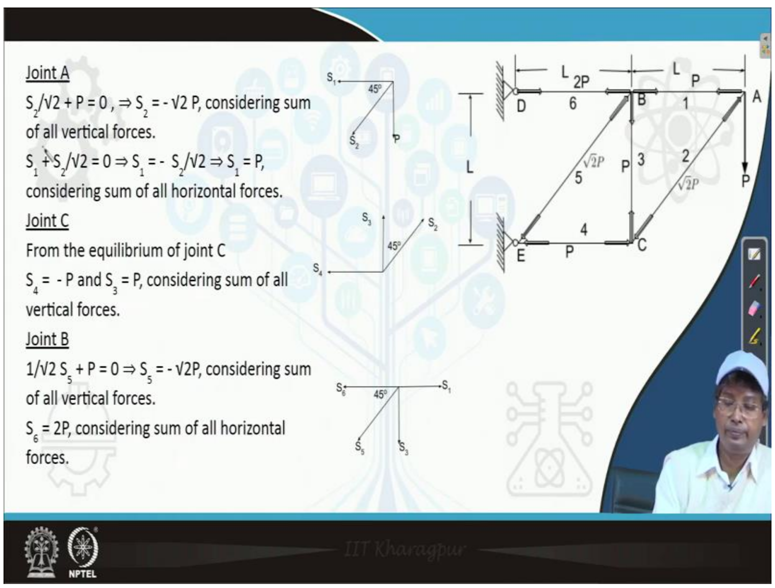

So, to go further let us first proceed for the solution of this truss joint a this is joint A, this is joint A S 1 S 2 and P is acting, this is 45-degree simple equations are there. so in the vertical direction both equals to zero that gives us S 2 equals to minus of root 2P. So, this is a compression member as it is, so and similarly we get that ish one is equals to P considering the horizontal equilibrium solution of FX equals to zero with respect to this point and that gives that this is a tension member then again we come to join C 2 and C is this S 2 S 3 S 4 S 2 is already found out is 4 is equals to minus P and since s 2 is equals to root 2 P minus root 2 P it is minus P and S 3 is equals to P, similar with following the horizontal Direction equilibrium.

So, joint b if we come join b is a similar way we can find out S 1 is known now, S 3 is known now the only two unknowns that is S 6 and S 5, so S 5 if we want to find out we need to consider equilibrium in this direction and similar way we get that S 5 is equals to minus or root 2 P and S 6 is equals to 2P. So, all the member forces are now known with respect to this we will use a table to carry out the further calculations.

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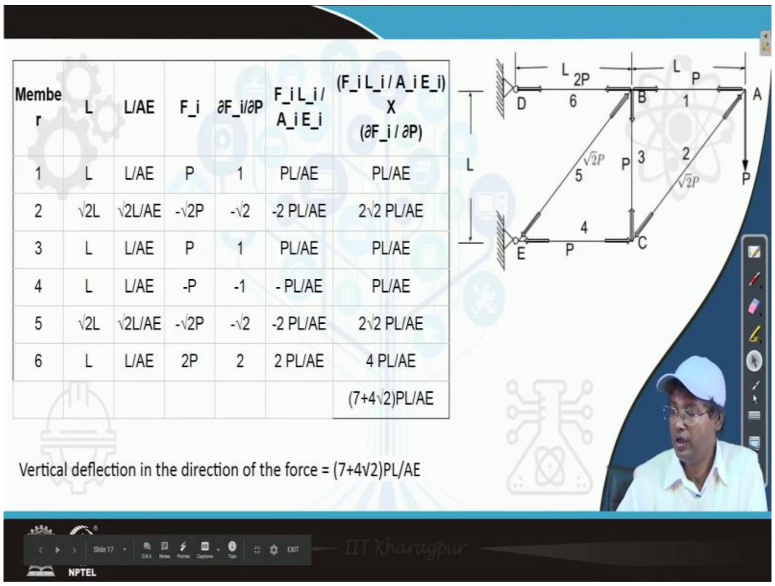

And we put the values in this table we have put the length of each and every member this is the main member 1 2 3 4 5 length is given as l root 2l like that whatever is that these two are more that is why this two are more 2 and 5 L by AE calculated from here is constant it is assumed that all the member having same cross-section and it is made from same material. Member forces in the previous slide we have found out that same forces are put here in all the forces those are also put in this figure.

So you can easily match those figures and then we are considering that del F i del P so this is the important step or is it is better to follow carefully what I am doing I am taking derivative of this, this is 1 this is root 2 this is 1 this is minus 1 minus root 2 and this is minus root 2 this is 2 and then we calculate that F i l i A i by A i E i and we get these values it is nothing but multiplication of these two column we get and finally we get the summation here.

So the vertical deflection in the direction of the force is equal to 7 plus 4 root 2 multiplied by PL by AE that is the final answer. So, with a little concept of energy method we can easily find out the deflection of a truss at a certain point and it works very well to find out the member forces.

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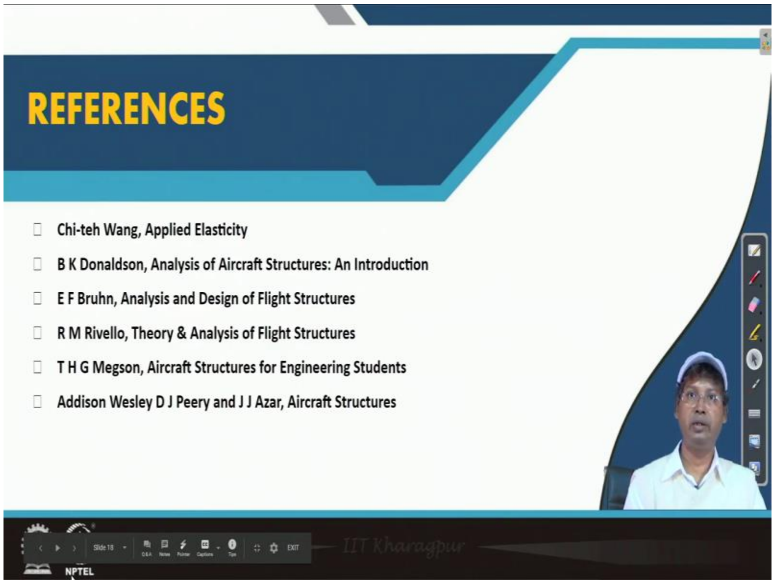

With this let us try to come to the end of the of today's lecture references our standard references.

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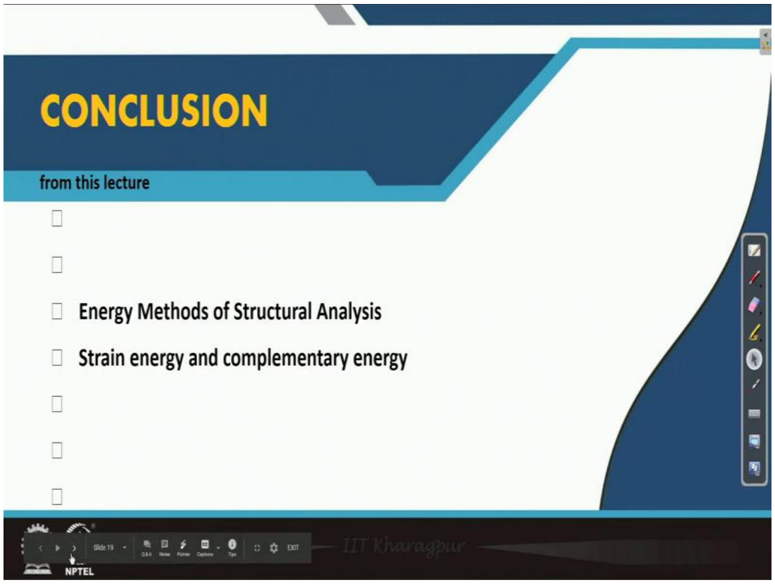

We every week bring their slide and in this slide we see that the strain energy and complementary energy is introduced and we have said that it is having a stationary value. Total complementary energy or total potential energy and using that property we can easily find out deflection of a point of a truss it is not only truss it may be applicable for any other structure where we can find out similar way the energy expression.

So those problems we will see with those problems will come beam problems another slowly but before that we will get introduced to other methods with respect to trust and maybe with some tricky way of solving problems. So, with that introduction to future lecture let us end today's lecture thank you for attending it will meet again in the second lecture of module 4 next time, thank you.