Space Flight Mechanics
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Lecture No - 77
Orbit Determination (Contd.,)

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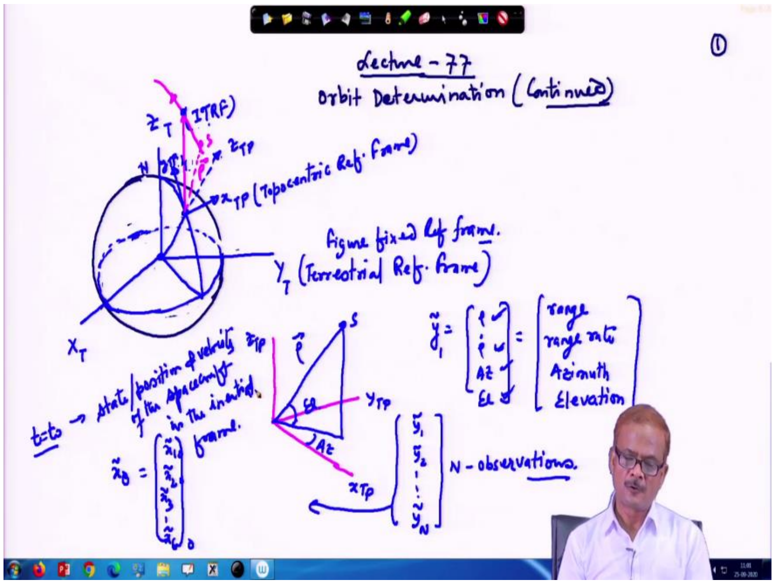

Welcome to lecture 77 we were working with orbit determination in that context in the last lecture we have to determine the Jacobian matrix. Now we progress further so in the orbit determination as I have stated earlier that the objective is to determine. This is the inertial reference frame XYZ and Earth is here and with respect to this inertial frame Earth is rotating and we can show the this equatorial plane of Earth which is not coinciding with the XY plane.

But right now for simplicity just assume that they are the same. I want to show you something what we are trying to do. Or either let us forget about inertial frame and we assume that this is our Terrestrial from X_T, Y_T, Z_T and it is the Terrestrial reference frame which is fixed to the earth at its centre and also called figure fixed reference frame that I have describe you ITRS International Terrestrial Reference Frame.

In this we have somewhere ground station location Greenwich Meridian maybe anywhere we are not worried about right now about this and from there and this is the outward normal z direction east direction we are taking along this and the north direction we are taking along this. This is your north pole here say. And then the observation is being made from this place. Satellite is here another satellite moves in the orbit.

This is your ρ and next time your ρ will change the way you will look here in this frame this we call as topocentric frame z_{TP} and y_{TP} topocentric reference frame x_{TP}, y_{TP} and z_{TP} this is a $\vec{\rho}$ satellite is being observed. This we will take azimuth, elevation so you are making your observation from this place. So what you have available we write this is $\tilde{y}$ equal to , $\dot{\rho}$, Azimuth (Az) and Elevation (εl).

$$\tilde{y} = \begin{bmatrix} \rho \\ \dot{\rho} \\ \text{Az} \\ \varepsilon\text{l} \end{bmatrix}$$

Called the range, range rate, azimuth, elevation so these are the four observations were making at any particular instant of time you have automatic machines. We presume that we have all the facilities available. So that ρ, $\dot{\rho}$, Az and εl can be directly measured. So once we do the measurement; let us say this is the first measurement we have done. This way you have $\tilde{y}_1$, $\tilde{y}_2$ up to $\tilde{y}_n$.

You are made N number of observations and you need to determine what the value of $\tilde{x}_0$ will give, $\tilde{x}_0$ which consists of its $\tilde{x}_1$, $\tilde{x}_2$ in this 0 we can keep it outside $\tilde{x}_3$ up to $\tilde{x}_n$.

$$\tilde{x}_0 = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_6 \end{bmatrix}_0$$

That means at the initial state is equal t_0. What is the state slash position velocity, position and velocity of the spacecraft inertial frame. So how do we start?

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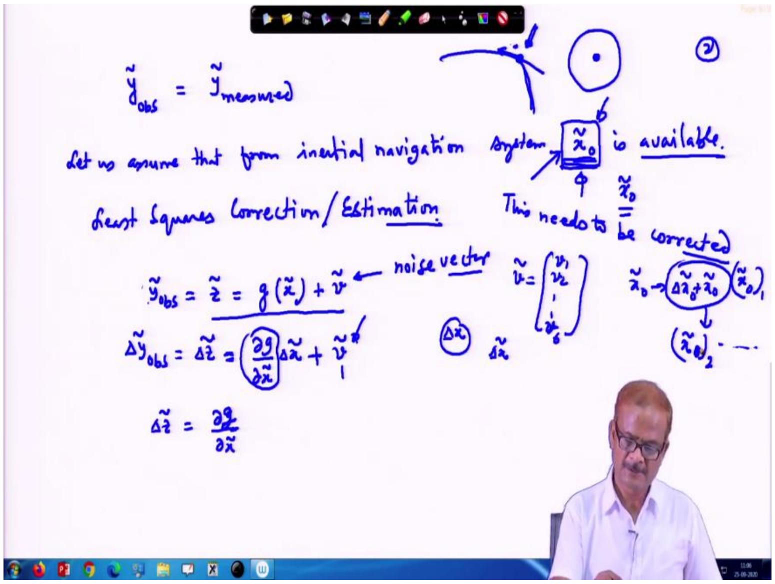

For this what is required is that we are making observations from the ground station and this write as $\tilde{y}$ which you are doing here. So this becomes your observed value and here we can write let us say this first one this is the first number of observation what we write here the ($\tilde{y}_{\text{observed}}$) observed to indicate this is being measured y observed is ($\tilde{y}_{\text{measured}}$) measured using instruments.

So let us assume that from inertial navigation system $\tilde{x}_0$ is available but it is not correctly known. This implies that your rocket has over performed underperformed. So wherever you wanted to inject the satellite in the orbits, which is orbit you are looking for and you are going and injecting your satellite here in this place. And this position is mathematically fixed. But while you are going here so your rocket is over performing underperforming. So with velocity which will inject that will differ it may go beyond this point.

If the velocity and position will not remain the same moreover your inertial navigation system it may not be exactly correct value at the time of injection. So, if there at the time of injection some error there in the state which we are writing as $\tilde{x}_0$ then your orbit will change considerably and if this problem happens you do not know in which orbit is going you are pointing your tracking system in some direction and your satellite is somewhere else it cannot be tracked in that condition.

Therefore this value needs to be corrected and for doing this what we do, we apply the least square correction, which we are doing also called the least square estimation. So we write like this. Let us say $\tilde{y}_{obs}$ this is your $\tilde{z}$

$$\tilde{y}_{obs} = \tilde{z} = g(\tilde{x}) + \tilde{\vartheta}$$

where $\tilde{\vartheta}$ is your noise vector. $\tilde{\vartheta}$ Implies ϑ_1, ϑ_2 up to ϑ_6 all the measurements are contaminated by noise.

And in this format it is least square estimation. We apply in the form of the matrix conversion and other things it is not possible. So what we do that we write it like this

$$\Delta y_{obs} = \Delta\tilde{z} = \frac{\partial g}{\partial \tilde{x}} \Delta\tilde{x} + \tilde{\vartheta}$$

$\tilde{\vartheta}$ is noise vector it is kept like this. While you are linearizing the system but this noise does not get altered. So here this part this quantity we can rewrite in the form $\frac{\partial g}{\partial \tilde{x}}$; this whole thing we will write.

What we are trying to do is to solve for Δx. The correction required in x_0. So we need if we can correct for the initial value $\tilde{x}_0$ is known if we correct it by certain quantity Δx_0 this is your updated value. The next again you correct this and this is your $\tilde{x}_0$ but after the first correction and after the second correction, this will again become 2 and so on. So after few iteration you will see that this converges the; your error in the observed value.

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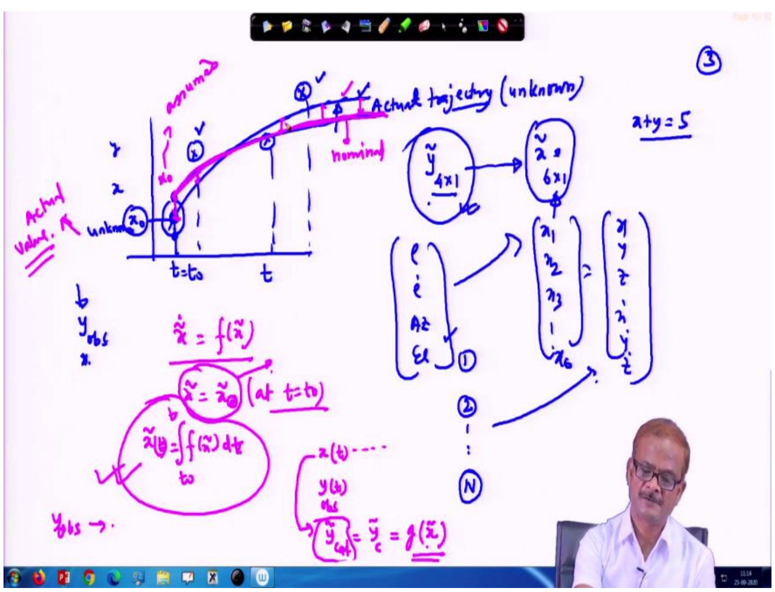

Actually, this is a little difficult topic for the beginners and I will make you understand what I am doing exactly in a different way. This is the actual trajectory in one dimension I am showing. Actual trajectory this is t and here x and y both are plotted. Some relationship is there so this is your exact actual trajectory which is unknown and these are the observed values around this. You have done at different instant of time.

So, using this observed values you have to determine this trajectory. This is the objective here. Instead of doing this in just in one dimension we are doing this in 6 dimensions. In our case, case is little complicated in the way that $\tilde{y}$ this is 4 x 1 Matrix or vector and $\tilde{x}$ this 6 x 1 Matrix or vector. So this is the 4 dimensional vector and this is a 6 dimensions vector and from 4 dimension to using 4 expression as I was explaining in the last class. That is we try to solve for x + y = 5 will get infinite number of solution using just one set we cannot determine this set.

Just using ρ, $\dot{\rho}$, Az and εl we cannot determine this x_1, x_2, x_3, up to x_6 which is nothing but your xyz and it $\dot{x}$, $\dot{y}$, $\dot{z}$ in is not possible and therefore we choose this multiple states of this. At the first instance of observation and at the instant of the observation so total N number of observations we make. And from there then we try to determine this part. And in that course this least square estimation it appears.

I will come to this point. What we have done that if you see that if the; between the y observed is known to you. This is itself contaminated with noise. So this is not correct. This is not the exact value but it is because of the presence of this, this also gets contaminated. This is not your exact measurement. So what we do that we assume that. Say here this is your x_0 at this point this is your x_0 and corresponding value t is equal to t_0.

But we are aware of this values these are unknown. Let us say that assumed nearby value here in this region. It is coming either here or here because it is in one dimension. And then I propagate the state. Once we propagate the state you see that some error is with the real trajectory actual trajectory there is difference it is not the same here. You have the; this much of error, here this much of error, so these are the errors.

This is not the actual one and this is actual and this is nominal. If you are assuming initial value and propagate the state $\dot{\tilde{x}}$ is equal to f($\tilde{x}$) where $\tilde{x}$ = $\tilde{x}_0$ at t = t_0. And if we assume this that this is available from the inertial navigation system, then we can propagate it. This can be done using $\tilde{x}$= f($\tilde{x}$)dt integrate from t_0 to t this gives you x(t).

And very simple in MATLAB you have Runge-Kutta scheme so you can use that or if you are looking for very accurate then there are many other methods which can be used for doing the precise integration method. The method is not of your which Numerical method I should used to integrate I am presuming that those things are left for afterwards once you are advanced then only you can think of to bring all those things.

So here once it is propagated then you have the x(t) available at different instant of time and y(t) observe is also available to you. So this x(t) this can be converted to y calculated which we write as y c tilde this is y calculated $\tilde{y}_c$ and how we are calculated using this equation. So once we calculate you can see that there is no noise. Only thing the y calculated will differ from the y observed because the initial value of the state we are assuming is equal to x_0 this is not correct.

This is coming from the inertial navigation system. This is not correct and therefore once you integrate it this will follow this rule it will go like this. So this is not your the right thing then you need to; using this observation y observation the question is using this y observation, can we correct this x_0 and this is x_0 assumed and this is x_0 unknown actual value, the initial state which is unknown. We have to find out this state and we are assuming the state to be here and accordingly the trajectory if we propagate using this scheme, so it is going along this direction.

And then these are the error here. So this one time dimension its appearing very simple but in multiple dimension it becomes complex especially here in this case the kind of relation we have having.

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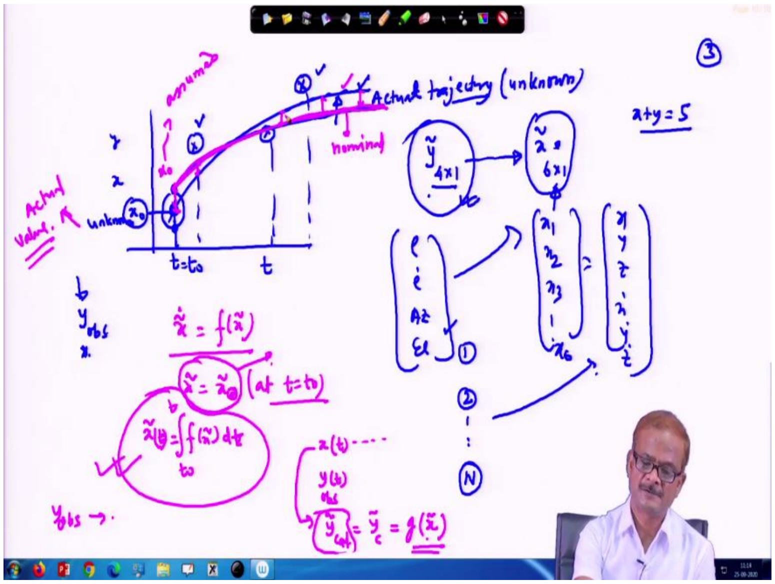

Therefore the question is that y observed is available and from there if we subtract this y calculated this is Δy. So using this can we determine the $\Delta\tilde{x}_0$ which is the correction required at the initial time. This is the question and this we have to address here. So this Δy because y has a relationship given like this y equal to $\tilde{x}$. So therefore $\Delta\tilde{y}$ we can write using the chain rule.

$$\Delta\tilde{y} = \frac{\partial g}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial \tilde{x}0} \Delta \tilde{x}_0$$

And if we write this as the H matrix this quantity we need to determine. Remember this is a $\frac{\partial g}{\partial \tilde{x}}$ this is at the same time. Here, x is at time t while this is x_0 is time t_0. So if we know all these quantities then we shall be able to and let us this we write as, say if you remember that $\Delta\tilde{x}$ this we have written as φ after in the transition Matrix format. So basically this quantity here it appears as your transition Matrix.

$$\Delta\tilde{x} = \varphi(t, t_0)\, \Delta \tilde{x}_0$$

And this transition Matrix we can write as $\Delta\tilde{y} = H\varphi\Delta \tilde{x}_0$. So now, this is casted in the form of matrix equation and therefore $\Delta \tilde{x}_0$ this can be written as Hφ and let us say this is we write as the pseudo inverse those who are some basic requirement is there in the linear algebra so that you can understand it. So then this Δx_0 is decided and this Δx_0 then we applied to the; and write this as the faster first titration we get this equal to $\tilde{x}_0 + \Delta\tilde{x}_0$.

$$(\tilde{x}_0)_1 = \tilde{x}_0 + \Delta\tilde{x}_0$$

Every time we have to determine after first propagating from this point to this point the state this is $\tilde{x}_0$, $\tilde{x}_t$ we are doing all this calculations and then we are applying to; this we are applying to this after finding out this Δx_0, then this is the updated value. Now we will replace with this replace this with $(\Delta\tilde{x}_0)_1$. Once this is replaced again the new value which may be after correction it will lie somewhere here and then we propagate.

And your actual trajectory may be somewhere here. This is your actual trajectory which is unknown and this is updated trajectory and this is updated trajectory if this way if you keep titrating the next again after this gets updated. So this is $(x_0)_2$ that means after update latest say this comes to somewhere here in this point. And once we propagate this comes closer to this and after doing all this corrections applying this correction. So next time you have $(\tilde{x}_0)_3$ after few alteration this will go to the actual trajectory.

How do we know that this converts to the actual trajectory this will be known from the fact that this Δy will it has to get minimised in the sense of x^2. If we can minimise Δy tilde transfer $\Delta\tilde{y}$ and here writing this as the error function 1 by 2 times like this. And these quantities can be minimized it should reduce over the iteration once we were doing like this going from one trajectory to another trajectory after applying all this correction.

So this will minimise initially say it might be some numerical value let us it is 100 as you progress through this iteration so it will come down come down and it will go to internet 1 and then actually come in fraction. After some tolerance is reached you can stop the iteration and the process that we do here is to call the batch processing. We take all the data points here on the trajectory and then do the processing.

So for the beginners it is little difficult to understand and moreover because we have very little classes already. I have crossed 8 lectures. Only 5 lectures we have meant and with few lecture we will trying to wind it up and I will supply of material on this the hand handwritten material. So that it will become a ready reference to you to understand later on once you are not in touch with me.

So till this extent we have done this. The next thing will be this comes at the end. So in the process for a let me summarize.

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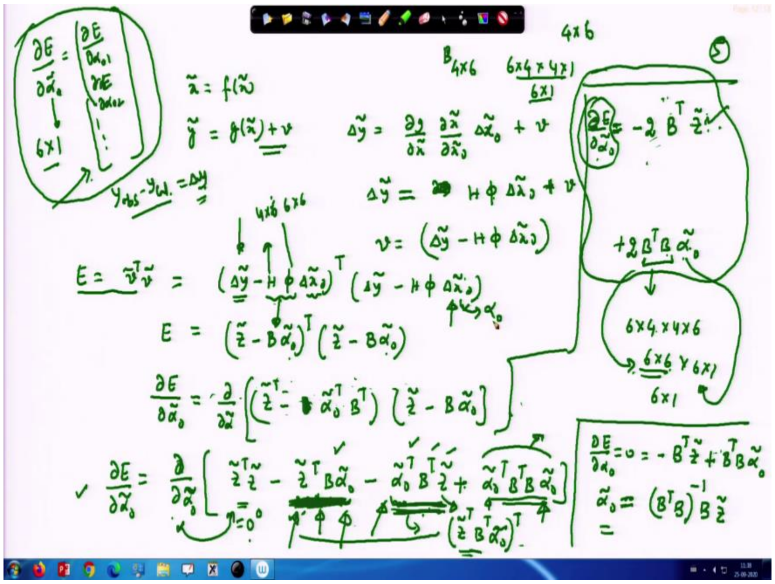

So, what is required that $\tilde{x} = f(\tilde{x})$ this equation is given to us. And also $\tilde{y} = g(\tilde{x}) + \vartheta$ is given to us. Here this is the observation. It comes with noise. So we linearized this part

$$\Delta\tilde{y} = \frac{\partial g}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial \tilde{x}_0} \Delta\tilde{x}_0 + \vartheta$$

We write like this and then this is H times phi and delta x tilde 0 + nu. So, minimize this error. This is our objective.

$$\Delta\tilde{y} = H\varphi\Delta\,\tilde{x}_0 + \vartheta$$

At the next time so in this linearized form, it appears as $\vartheta = \Delta\tilde{y} - H\varphi\Delta\tilde{x}_0$. We are writing in this format and this quantity you would like to minimize.

$$E = \vec{\vartheta}^T\vec{\vartheta} = (\Delta\tilde{y} - H\varphi\Delta\tilde{x}_0)^T\,(\Delta\tilde{y} - H\varphi\Delta\tilde{x}_0)$$

What is the benefit? It will be visible to you. Now here for convenience let us assume that this Δy we write as $\tilde{z}$ and this part we write as B matrix. $\Delta\tilde{x}_0$, let us write this as $\tilde{\alpha}_0$.

So, if we want to minimize, with what we want to minimize? So how much correction is to be given here in this place. This quantity is fixed by y of z - y calculated and this is your Δy. So how much correction I give here so that this quantity gets minimized and for that then because Δ is 0 we are replaced by α_0. So therefore this we differentiate with respect to α_0 and set it 0. There after you just solve it. So this is your $\tilde{z}$ you can expand it also and then work it out.

I will take a shortcut physically here this can be written as two times the quantity we need to; let us write it first expand it and then we will do the other part this is B^T becomes because this is vector so $\tilde{\alpha}_0^T$ B^T and then $\tilde{z}$– $B\tilde{\alpha}_0$. So here then it will get reduced to once the difference this quantity will be equal to 0. Here, α_0 is here. So we get here the first quantity here so we have $\frac{\partial f}{\partial \tilde{\alpha}_0}$ $= -\tilde{z}^T.B$

Then the next one here we have to little we have to be little careful in writing this. So, if you look here at this quantity; so this is a vector and you are taking the inner product of this vector here and therefore this turns out to be a scalar. But then we differentiate this is scalar with respect to a vector. So this will again become a vector and this is also a scalar and then why you differentiate with respect to a vector so this becomes also a vector.

These two can be combined together and let us go on the next page or we can continue here. So then we differentiate this part again this is not differentiated. This is not differentiated we have to differentiate with respect to this. So the quantity then if you look into this part, so this can be again written as $(\tilde{z}\alpha_0)^T$. So it is the same quantity only thing is listed in this format.

We need to differentiate it one by one. Let us complete this part, first this part. This part is 0. The ones we differentiate this becomes 0 this two we have to decide and this part is the transpose of this, this we can writer as $B^TB\tilde{\alpha}_0$. This is the way we can write. So this is your third parts, so this comes with plus sign and because this quantity and this quantity and this quantity both are same.

Rather than writing in this format this is z^TB tilde we have written whether this is consistent with this notation or not. I would like to write here

$$\frac{\partial E}{\partial \tilde{\alpha}_0} = \begin{bmatrix} \frac{\partial E}{\partial \alpha_{01}} \\ \frac{\partial E}{\partial \alpha_{02}} \\ \vdots \end{bmatrix}$$

$\frac{\partial E}{\partial \tilde{\alpha}_0}$, this is because of 6 dimensional vector 6 into 1. I would like to write this is $\frac{\partial E}{\partial \alpha_{01}}, \frac{\partial E}{\partial \alpha_{02}}$ alpha and so on. I would like to present this in vector format. If I do so then at that time you can see that if I am writing here in this format, it will not be consistent with that.

Because here z let us say z here in this case is 4 into 1 vector. And B is 4 into B we have written somewhere on the previous page Hφ we have written as B. So Hφ is written as B, 4 into 6 this let us say this is 4x6 and this part is 6x6 this is equal to 4x6 your B is 4x6 and if we operate on this by; this part was; first will take this part. So in this part B transpose is 6x4 and this part B then become 4x6 so then this become 6x6 Matrix.

So this particular part is 6x6 matrix. And here in this place we have to decide how much this will come and in what format it should come so that we finally get it here in this format. Rather than writing here in this format if we write it this way that because we know that this and term they are the same. These are the scalar terms and they will give the same value because both are exactly the same only thing they differ by transpose here.

Because it is a the inside a scalar field does not matter and therefore we write this quantity as two times we can write this as B^T therefore written as B^T so that it becomes B is 4 times 6 B is of dimension 4 x 6 B^T become 6 x 4 so B^T and then $\tilde{z}$ this becomes 4 x1 so this becomes 6 x 1 size and here also this is 6 x 6 x 6 x 1 size and this α_0 is 6 x 1 size so this becomes 6 x 1 so this way then the consistency exist.

So, this part we have to particularly take care of that once you differentiate it and otherwise you just write expansion of this differentiate and you will see that it gets reduced to this format. And because α_0 is present here and α_0 is present here, so for that reason 2 will also appear in this place. And finally what we are doing we are taking this quantity is equal to 0 and here we can summarize the thing.

$$\frac{\partial E}{\partial \alpha_0} = - B^T\tilde{z} + B^T B\tilde{\alpha}_0$$

$$\tilde{\alpha}_0 = (B^T B)^{-1} B\tilde{z}$$

So $\frac{\partial E}{\partial \alpha_0}$ this equal to 0 this becomes - B^T 2 can be eliminated $\tilde{z}$ this equal to what sorry this is plus this plus $B^T B\tilde{\alpha}_0$ and we try to solve it for $\tilde{\alpha}_0$ this will give us $(B^T B)^{-1} B\tilde{z}$. This part we should verify by taking this Matrix rather than you can differentiate it. You can also differentiate it. But finally put here in this format both of them can be combined and it can be written in this way.

And once you have written in this way. I can expand and write all the mathematics, but we do not have time. While time to be very brief I also get the problem that how to present you so that we can finish this fast within few lectures. So what α_0 we are written α_0 is nothing but your Δx_0. So this way you have estimated Δx_0.

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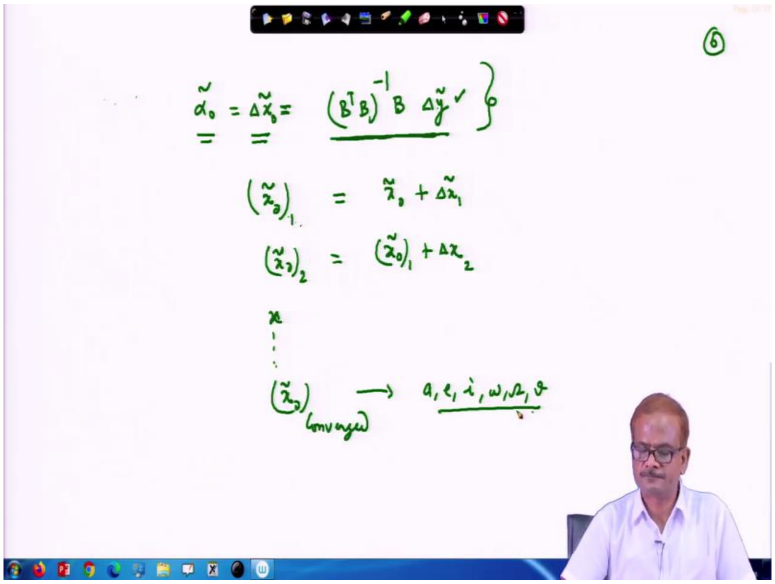

So your α_0 is nothing but Δx_0 which is the quantity $(B^T B)^{-1} Bx\Delta$ and z is your y, Δy we have written Δy we are replaced by z this is your $\tilde{z}$. This particular part so then this comes with $\tilde{z}$ is equal to; z is $\tilde{y}$, $\Delta\tilde{y}$ and here this is Δx_0. So at a particular instant of time so instead of working like this, this is just for one data set let us say. But if you have multiple data set so you will see that at first instant you are available.

Then other instant you will have this available for Δy_1 similarly $\Delta\tilde{y}_2$ and you have up to $\Delta\tilde{y}_1$ and each of them can be represented in terms of Δx_0 so that means the representation if we are doing here this gets complicated a in that case. Once you combine but ultimately the result will be the same it will come here in this format and from there then this Δx_0 can be estimated once you have estimated this x_0 so $(\tilde{x}_0)_1$, this is after the first iteration.

After doing this operation you get this $\tilde{x}_0$ plus we will name this as 1 not as the new so that we knew that after first iteration, now this is follow this process. So next time the $(\Delta\tilde{x}_0)_2$ will be is $(\tilde{x}_0)_1 + \Delta x$ and here we can put as one. And this we can put as 2 corrections given at the second iteration. Similarly if you keep on doing this you get finally the converse result Δx_0 conversion and once you have got this then a, e, I, ω, **Ω**, θ all this can be determined using the standard equation. So we stop here and will continue in the next lecture.