Space Flight Mechanics
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Lecture No - 63
General Perturbation Theory (Contd.)

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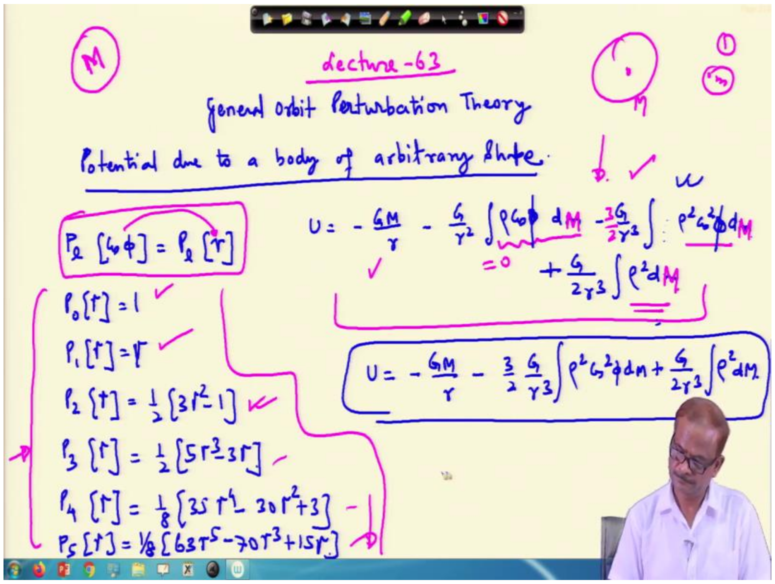

Welcome to lecture 63 we are discussing about potential due to a body of arbitrary shape will continue with that.

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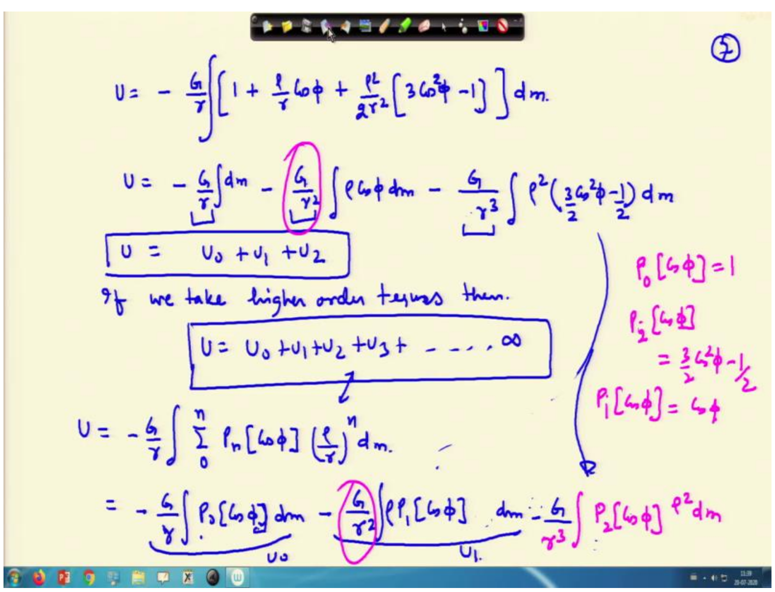

If you remember that in the last lecture we were deriving this expression for the potential in that case we have got this expression. This expression we have got for the potential up to the second degree. If the; same equation I have written here in this place minus GM minus G if this is the second term and the third term I have expanded and it has been written in this format. So, this 3 by 2 also we can take it outside here in this place rather than placing inside.

$$U = -\frac{G}{r}\int dm - \frac{G}{r^2}\int \rho cos\phi dm - \frac{G}{2r^3}\int rho^2(3\cos^2\phi - 1)dm$$

And this we have to integrate to get in a simple format moreover the function the Legendre polynomial that we have written here in this place. So this Legendre polynomial few of them I have listed here. Pl cos ϕ this is a Legend polynomial so first ϕ I have replaced in terms of Γ and then this can be written like this $P_0\gamma$ is equal to 1 3 1 γ. So this way if you know this series, so directly you can insert and you can get this expansion.

So you can do it for suppose, you know for 100 terms, you know this expansion this kind of series, this series is available till 100 terms. So you can just, you need to insert here in this place and you will get 100 terms and there after you need to integrate of course. Our main objective is to work with this not with this. This already we have used it and using this we have got this term rather directly by expanding also we have got this term.

Now, we need to estimate each of them. This quantity directly you can see that this is nothing but – Gr inside this is only m and we replace it with this is dm replaced by m and here we get and moreover because we are carrying out the expansion finding the potential due to a body which is use mass m representing by him. So it will be prudent that we show it by dm that is shown by small m so that the confusion does not arise.

This is not for the small mass. This is for the bigger one, M is located here. This is of mass M. Now estimate all these quantities. The first quantity is ok already it is done. The second quantity this will be the integration. This quantity will be equal to 0 this we have to show and this two quantities we need to work out.

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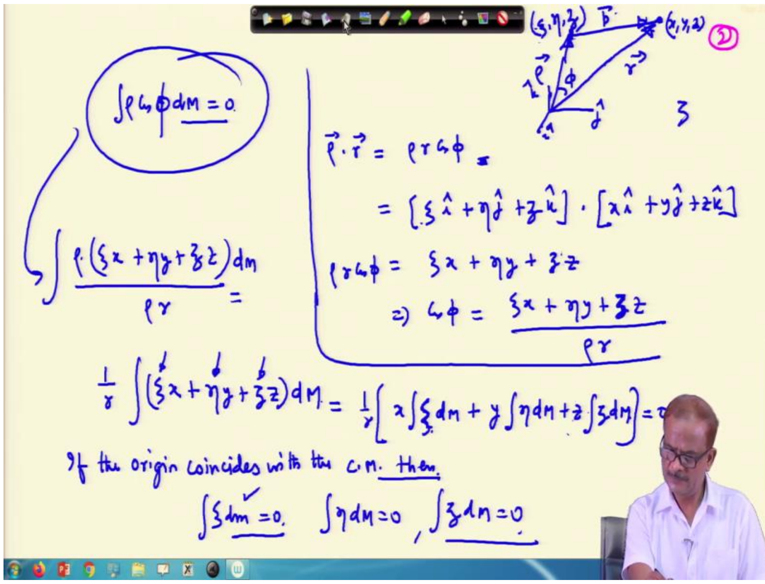

So we take the next quantity ρ cosθ dm this quantity 0 and what the cos θ perhaps it was ϕ we have used the notation ϕ not θ so will keep it ϕ. This is ϕ, so this angle is ϕ and hear this is ρ and in this place this is r and this victory have written as p and in the co-ordinate system we have chosen. This coordinate we write has xyz and this co-ordinate we write has ξ, η and ζ.

$$\vec{\rho}.\vec{r} = \rho \, r \cos\phi$$

$$= [\xi\,\hat{\imath} + \eta\,\hat{\jmath} + \zeta\,\hat{k}].[x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}]$$

So, then we have ρ. r equal to ρ r cos ϕ and this quantity will be then given by and of course you are the unit vector $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ or any other notation you can choose for unit vector. So this is ρ times r, insert this here this becomes ρ cos ϕ x + ξ x +η y + ζz divided by ρr, ρl by r we can take outside ρ cancels out. This quantity this becomes equal to ξ x + η y+ ζz dm.

$$\rho \, r \cos\phi = \xi\,x + \eta\,y + \zeta z$$

And then this we can expand as; so r is not dependent on the integration is over the body and for the bodies of coordinates we are using this. This will only remain inside the integration sign. This is what we are getting here. Now if the origin coincides with the centre of mass then we ξ dm this quantity will be equal to 0. This is the basic definition for the centre of mass location of the x location, y location, z location of the centre of mass.

$$\frac{1}{r}\int(\xi x + \eta y + \zeta z)dM = \frac{1}{r}\left[x\int \xi dM + y\int \eta dM + z\int \zeta dM\right] = 0$$

$$\int \xi dM = 0\,;\int \eta dM = 0\,;\zeta dM = 0$$

Because the origin is located there itself; so that is bound to be 0 and therefore this quantity will be 0. Earlier you have η dM this quantity will be equal to 0 and ζdM this quantity will also be equal to 0. So therefore this quantity vanishes which I have already written here in this place. So for the second term it vanishes then remains these two terms to the worked out. So we have to work out now U then we are getting as - GM divided by r - 3 by 2 G by r ^{3} ρ ^{2} cos ^{2} ϕ dM + G by 2 r ^{3} ρ ^{2} dM.

$$U = -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^3}\int \rho^2 \cos^2\phi \, dM + \frac{G}{2r^3}\int \rho^2 dM$$

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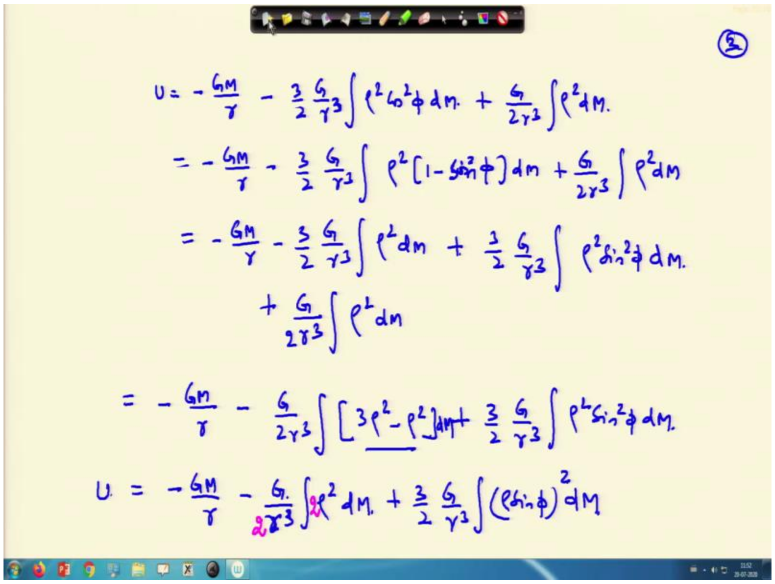

$$U = -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^3}\int \rho^2 [1 - \sin^2\phi] \, dM + \frac{G}{2r^3}\int \rho^2 dM$$

$$U = -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^3}\int \rho^2 dM + \frac{3}{2}\frac{G}{r^3}\int \rho^2 \sin^2\phi \, dM + \frac{G}{2r^3}\int \rho^2 dM$$

$$U = -\frac{GM}{r} - \frac{G}{2r^3}\int 2\rho^2 dM + \frac{3}{2}\frac{G}{r^3}\int \rho^2 \sin^2\phi \, dM$$

For convenience, we copy town to the next page. U equal to - GM by r -3 by 2 G r ^{3} -3 by 2 G r ^{3} and you have ρ ^{2} cos ^{2} ϕ dM and then + G by 2 r ^{3} ρ ^{2} dM. So the next step we carry out by writing cos ^{2} ϕ is 1 - sin ^{2} ϕ dM + G by 2 r ^{3} ρ ^{2} d M + 3 by 2 we do little rearrangement 2, 2 will cancel out once we subtract it so we get here G by r ^{3} and ρ ^{2} dM + 3 by 2 so this is U. Now we look into what these quantities are; let us maintain this quantity 2 here and write 2 here in this place.

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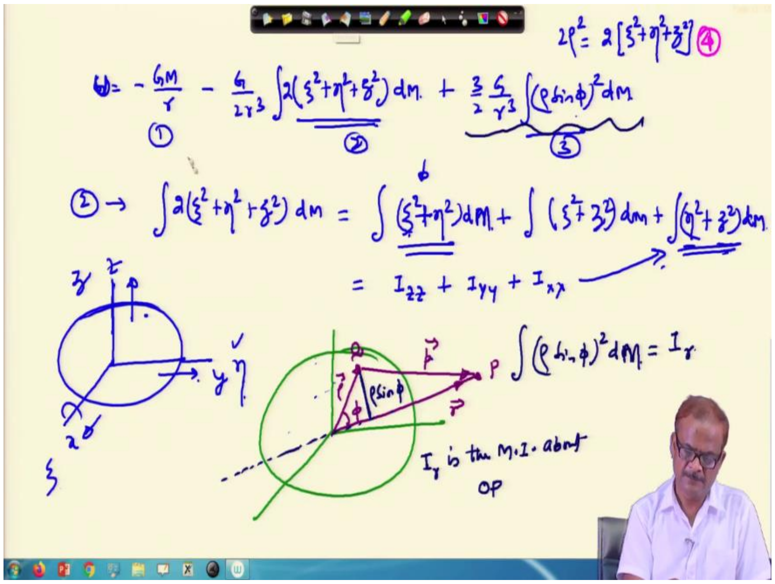

We have done U equal to -GM by r and then the quantity which is written here the second term – G by 2 r ^{3} and 2 ρ r ^{2} and what this ρ is? 2 times $\xi^2 + \eta^2 + \zeta^2$ so 2 times $\xi^2 + \eta^2 + \zeta^2$ dm and then the + 3 by 2 G r ^{3} and then the quantity which is written here ρ sine ϕ^2 dM. Now what this quantity is? Let us say this is the first term we have written and this is the second term and this is a third term we are having.

$$U = -\frac{GM}{r} - \frac{G}{2r^3}\int 2(\xi^2+\eta^2+\zeta^2)dM + \frac{3}{2}\frac{G}{r^3}\int(\rho sin\phi)^2 dM$$

$$\int 2(\xi^2+\eta^2+\zeta^2)dM = \int(\xi^2+\eta^2)dM + \int(\xi^2+\zeta^2)dM + \int(\eta^2+\zeta^2)dM$$

$$= I_{zz} + I_{yy} + I_{xx}$$

So second term we are working with 2 $\xi^2 + \zeta^2$ this quantity we can write as; now we are the body here, this is the x axis y axis z axis and the coordinates for the body we are just use any point in this the Xi and η and ζ we have indicated. So ξ is along this direction η is along this direction and zis along this direction. So $\xi^2 + \eta^2$ here we are taking ξ in this direction. In this direction we are taking it η; we are taking ζ along this direction.

So $\xi^2 + \eta^2$ times dm just gives you moment of inertia about the z-axis. Either you can refer to the book by Bear Johnston Mechanics or Irwin James or either look into the lecture on satellite attitude dynamics and controls. Given by me so in that lecture you will find all these details. So, these quantities $I_{zz} + I_{yy} + I_{xx}$ same way it is η and ζ, so η and ζ is involved like this; so this is the moment of inertia about the x axis.

And what this quantity is? Now I did not draw this figure we took point Q to be here and we took point P to be in this place. I will make it better point P is here and Q is here. This is your P vector. Here this is the r vector. This is the ρ vector, this angle we have taken as ϕ. So these perpendiculars distance from here to here, nothing but ρ sine ϕ. So the quantity this term we are looking into and this term then gets reduced to $(\rho \sin \phi)^2$ dM.

$$U = -\frac{GM}{r} - \frac{G}{2r^3}\left[I_{zz} + I_{yy} + I_{xx}\right] + \frac{3}{2}\frac{G}{r^3} I_r$$

This is nothing but this perpendicular distance ^{2} times dm and this we have to integrate. So this gives you moment of inertia about this axis which is the axis of this is the; this term is moment of inertia about the OP which we write as I_r. So I_r is the moment of inertia about OP.

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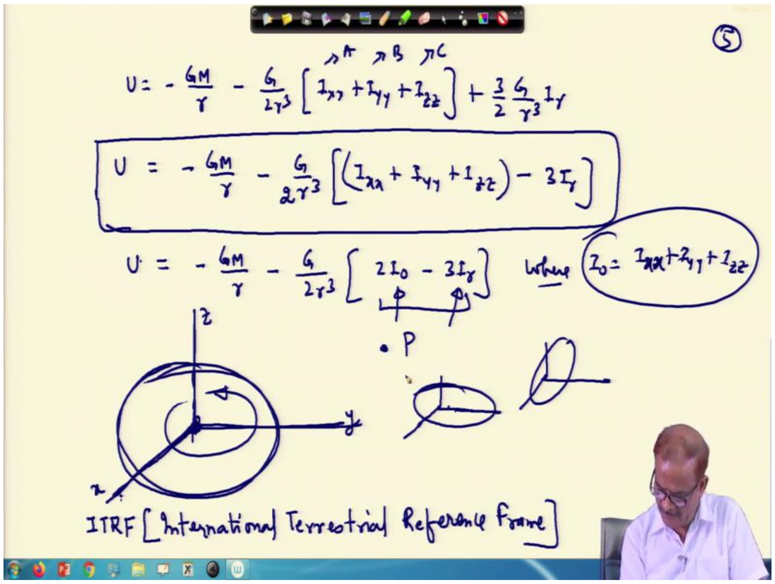

Therefore the expression U that get, this will get reduced to equal to - GM by r – G by 2 r ^{3} 2 r ^{3} and then times $I_{xx} + I_{yy} + I_{zz}$. In many places this is written as A, B and C instead of writing $I_{xx} + I_{yy} + I_{zz}$ just this is just written as ABC. And the next term is 3 by 2 G by r ^{3} + 3 by 2 G by r ^{3}. So this is the form we are getting here and sometimes this is also expressed writing is as $2I_0 - 3I_r$ where I_0 is equal to $I_{xx} + I_{yy} + I_{zz}$.

$$= -\frac{GM}{r} - \frac{G}{2r^3}\left[I_{zz} + I_{yy} + I_{xx} - 3I_r\right]$$

$$= -\frac{GM}{r} - \frac{G}{2r^3}\left[2I_0 - 3I_r\right]$$

An important point I would like to point out here that if you are finding potential at any point so at any instant of time your reference frame has to be attached to the body. Ok. So in the case of the Earth this frame that we use this is called the ITRF according to the recent notations is called the International Terrestrial Reference Frame. So, here xyz this is referring to your ITRF and this rotates along with the earth.

So it is fixed inside the earth it rotates along with the Earth. This is not inertial frame. Here while we have been working, if we write the equation of motion ok that we do in the inertial frame, but the potential ones we are describing we can do it using the frame which is attached to the body and moving along with that. Because in that case describing the moment of inertia that; moment of inertia term we have derived here it becomes easy.

If the frame is fixed, ok in orientation as in the case of the inertial frame and the body is rotating so sometimes body is like this, other time body has gone here in this place. So you can see that all these terms then will be variable. There will be a function of time and it will be difficult to treat in general way this particular expression and therefore this same is attached to this and then this is brought out. So, to particular instant of time the potential we have got.

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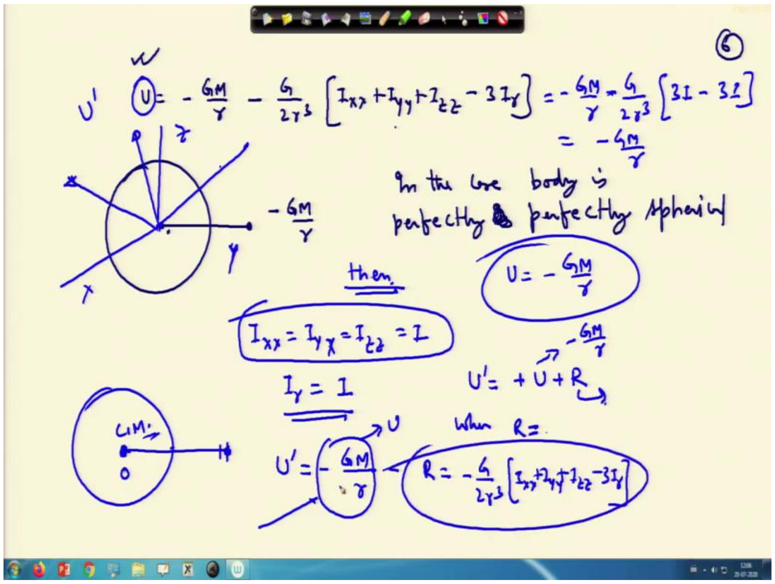

So, thus we have got this equation U equal to we will copy it on the next page U equal to - GM by r – G by 2 r^3 Q $I_{xx} + I_{yy} + I_{zz} - 3 I_r$. Now this will serve for working out our problem. So, here we are using the notation U but if we remember that if the body is perfectly spherical so in that case potential at a point; this is given by –GM by r and how we can see we can directly see from this place, in that case body is perfectly spherical which is not the case in the earth in the case of earth.

$$I_{xx} + I_{yy} + I_{zz} = I$$

In the case body is perfectly spherical then I_{xx} equal to I_{yy} equal to I_{zz} is equal to I and I_r also in that case it becomes equal to I. Because moment of inertia along any axis you take it will be the same. Therefore I_r will be equal to I and of course the I_{xx} here x y and z about the x y and z axis also this will be the same therefore this gets reduced to - GM divided by r – G by 2 r^3 and 3I – 3I so the other term vanishes and we get U equal to - GM by r which is also due to a point mass.

So in the case of a perfectly spherical body its acting the potential due to this at a point it acting just like this is the whole mass is concentrated about the origin at a point O which we are in this case we are taken as the centre of mass. So, that implies that the U we are writing here this consists of 2 terms - GM by r and the other term instead of writing this has U we write suppose instead of you writing this as U we write this U' and this quantity we write as U.

$$U' = U + R$$

You can immediately see that U' this will be equal to -U - R where R equal to simply or simply write plus because that notation we are using here U is equal to - GM by this quantity is -GM by r and this quantity then we write as R equal to – G by 2 r^3 $I_{xx} + I_{yy} + I_{zz} – 3I_r$. So, this acts, this particular part this is acting like a perturbation potential if body is not perfectly spherical.

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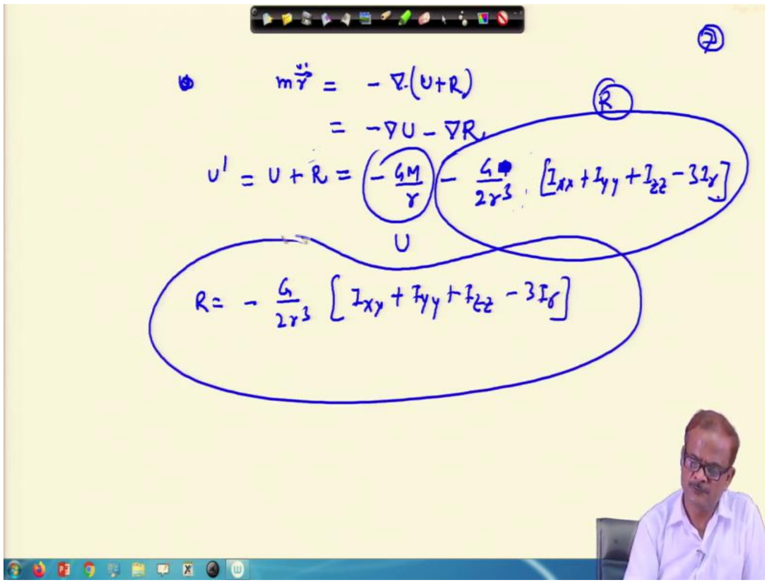

$$m\ddot{\vec{r}} = -\nabla(U+R)$$

$$U' = U + R = -\frac{GM}{r} - \frac{G}{2r^3}\int\left[I_{xx} + I_{yy} + I_{zz} - 3I_r\right]$$

So therefore we will use the common notation instead of writing this what we have written and $m\ddot{\vec{r}} = -\nabla(U+R)$ and this quantity is and this quantity appears as GM by r ^{3} G by 2 r ^{3} m is not there and then $I_{xx} + I_{yy} + I_{zz} - 3I_r$ not this side; this is with ΔU operator. So we should remove this, this is only part we write simply as U + R. So, this quantity is here U and this quantity is; we change the notation.

So this we are writing as this whole thing we are writing as U' instead of writing U we are writing as U' therefore it becomes easier for us to use the earlier notation. And this perturbation potential R we are; this is written as integration sign already we have integrated so this is not present check integration sign already removed, so this – G by 2 r ^{3} $I_{xx} + I_{yy} + I_{zz} - 3I_r$ and this perturbation potential will be used in the equation for the Lagrange Planatery equation and it can we solve to held result we are looking for ok, so will stop this lecture here and continue in the next lecture. Thank you very much.