Space Flight Mechanics

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Lecture – 38
Restricted 3 - Body Problem (Contd.,)

Welcome to lecture number 38, we have been discussing about the 3-body problem and in that context then we moved to the restricted 3-body problem, so we will continue with that today, we are going to discuss about the Lagrange points.

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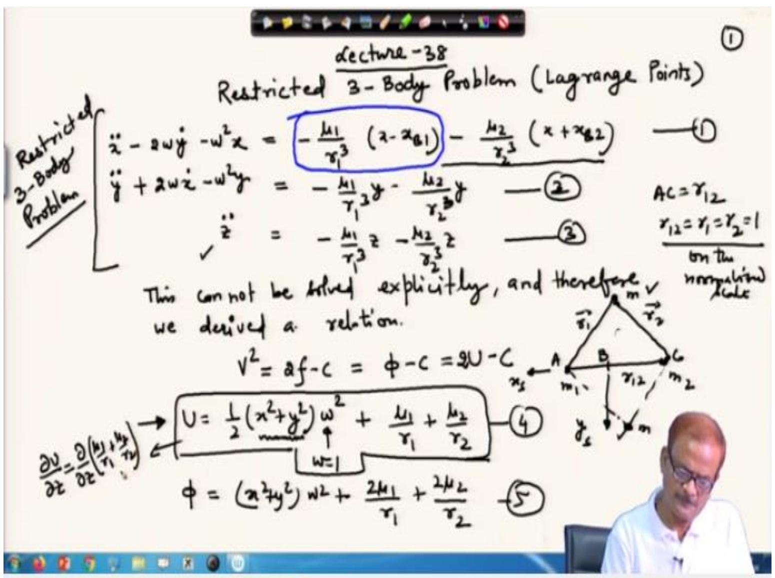

So, already we have observed that the restricted 3-body equations is governed by

$$\ddot{x} - 2\omega\,\dot{y} - \omega^2 x = -\frac{\mu_1}{r_1^3}(x - x_{B1}) - \frac{\mu_2}{r_2^3}(x + x_{B2})$$

and similarly, along the y direction in the synodic frame, we have

$$y + 2\,\omega\dot{x} - \omega^2 y = -\frac{\mu_1}{r_1^3}\,y - \frac{\mu_2}{r_2^3}\,y$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3}\,z - \frac{\mu_2}{r_2^3}\,z$$

so these 3 equations we got for the restricted 3-body problem and as we have stated this cannot be solved explicitly and therefore, we derived a general relation.

And that relations in the short form we have written as

$$V^2 = 2f - C$$

or the same thing we have also written y – c or either we have written it in terms of u – c; 2u – c, where u we defined as 1/2 $(x^2 + y^2)$ and if you remember, ω^2 terms also appears but ω^2 if you assume

$$\omega = 1$$

, so this term does not get involved in the u term otherwise this ω^2 term this remains.

And therefore, from here we have ϕ equal to

$$\phi = (x^2 + y^2)\omega^2 + 2\frac{\mu_1}{r_1} + 2\frac{\mu_2}{r_2}$$

so this we have already worked, now the question arise is, the last time as we have been discussing that Lagrange points they exist and out of that we have worked for on the normaliser scale already we have looked into that equates the; what I mean here that if you have mass m_1 and this is mass m_2, then we have looked at r_1 will be equal to r_2.

So, this is your r_1 and this is r_2, so magnitude wise mass m is here, either it is a possible here in this place or either here in this place, so m can be here or either m can be here and in this direction, we have taken the each direction of the synodic frame, this is B, Barycentre and this direction indicates the y direction of the synodic frame, so this till this extent we have done it okay and thereafter also we observed that if we write this as the r_{12}, the distance between m_1 and m_2, let us say this is A and this is C. So, AC equal to r_{12}, so also we have observed that this will be equal to r_2 equal to 1 on the normalised scale, so we are going to look into this matter again.

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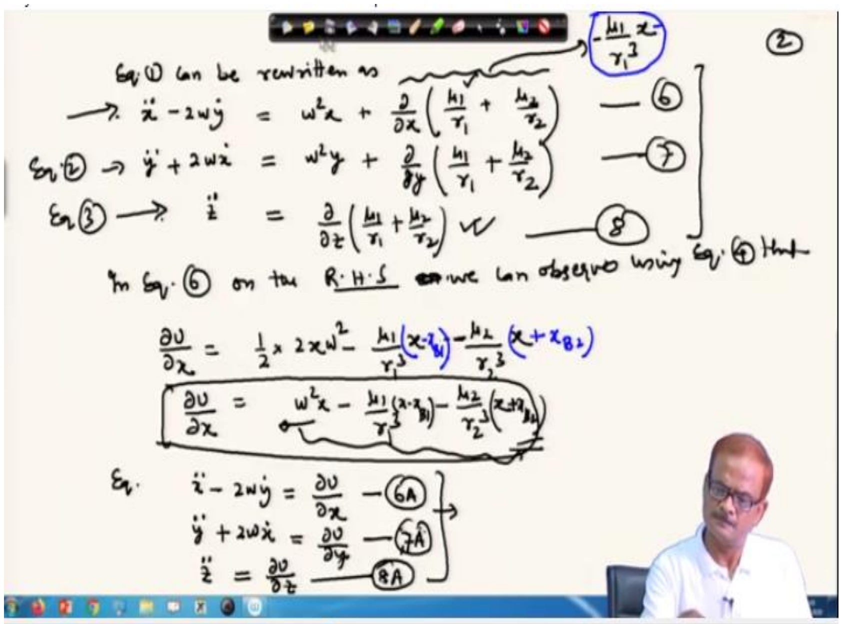

So, the equation we have written there, the first equation; equation 1 can be rewritten as

$$\ddot{x} - 2\omega\,\dot{y} = \omega^2 x + \frac{\partial}{\partial x}\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right)$$

y double dot can be written as, this can be written as

$$\ddot{y} + 2\,\omega\dot{x} = \omega^2 y + \frac{\partial}{\partial y}\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right)$$

and $\ddot{z}$ this will can be written as in the same way,

$$\ddot{z} = \frac{\partial}{\partial z}\left(\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right)$$

and this is our equation number (8). Now, if we look into the in equation (6) on the right hand side, if you look that quantity is nothing but $\frac{\partial u}{\partial x}$.

So, here u already we have defined, so if I differentiate it, so 1/2 times, if I take the differential of in equation (6) and on the right hand side we can observe using equation (4) that ∂u/∂x equal to 1/2 times 2x, partial differential with respect to x we are taking and thereafter the rest of the quantity they will appear as r^3x, this already we have done, so you can look back, refer to the earlier lectures.

And of course here ω^2 term was there, so this is

$$\frac{\partial u}{\partial x} = \omega^2 x - \frac{\mu_1}{r_1^3}(x - x_{B1}) - \frac{\mu_2}{r_2^3}(x + x_{B2})$$

so the equation (1) was rewritten as like this, equation 2 is rewritten like this and equation 3 is rewritten like this, so in the equation (2) on the right hand side, if we observe; okay, we have plus sign here in this place, so that we need to take care of.

Here, if you see, actually what is happened, this is fine, so this is also okay, r_1^3 actually, equation 1 if we are writing it like this, so here already we have taken into account, if you see, if the first term, this term is differentiated, so this will appear as μ_1/r_1^3 with minus sign and times x and this quantity is nothing but what we get here in this place, x minus; here one more term we have to add, so we will add that particular term also, this will be r_2 once we differentiate, so this is x (x+x_{B2}) and here this we have this (x – x_{B1}) and with minus sign here.

So, this way here also we do the correction, so this is (x – x_{B1}) –(x – x_{B2}), now if you look here on this term, this particular term and look here in this place, it should have same thing, - μ_1/r_1^3 (x – x_{B1}) similarly the other terms and here (x + x_{B2}), so and this term obviously we have brought it from the left hand side. So, what it implies that our the equation 6 is nothing but

$$\ddot{x} - 2\,\omega\dot{y} = \frac{\partial u}{\partial x}$$

this is the equation (6).

So, let us write this as (6A), okay, along the same line you can write

$$\ddot{y} + 2\,\omega\,\dot{x} = \partial u/\partial y$$

this is 6B as a 7, we write this as 7A because this written as 7, this is 7A and

$\ddot{z}$, this will be simply

$$\ddot{z} = \frac{\partial u}{\partial z}$$

going back here, if you differentiate this, so ∂u/∂z, this quantity there is no z appearing in this term, okay therefore, this term will be 0, only thing you get here.

So, this will be equal to ∂u/∂z equal to μ_1/r_1, + μ_2/r_2, so this is what we get and we have written here in this place. So, the above equations, so this we write as (8A), so what we see that the equation (6), (7), (8) can be reduced here in this format.

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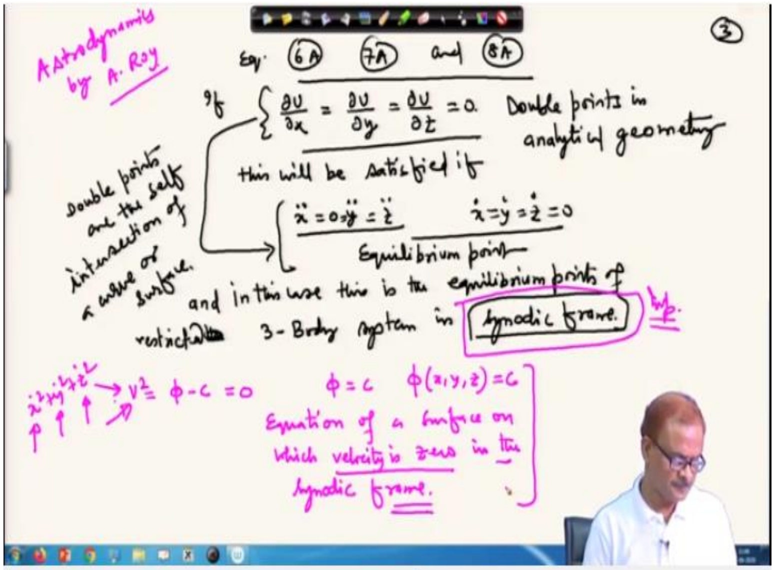

Now, equation (6A), (7A) and (8A), if

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

, so this refers to your double point in analytical geometry and when this is going to be true once all this quantities on the right hand side are 0, so this is satisfied by; so this will be satisfied if

$$\ddot{x} = \ddot{y} = \ddot{z} = 0$$

means they are all 0. Generally,

$$\dot{x} = \dot{y} = \dot{z} = 0$$

So, this defines your equilibrium point, so this is satisfied, these double points are nothing but equilibrium point, double points are the self-intersection of a curve or surface and in this case, this is the equilibrium points of the 3-body, restricted 3-body system in synodic frame, this is very important, this is synodic frame not in inertial frame, this part is important to note, okay. So, we got the Jacobian integral derived already as you know and that was derived in the synodic frame, you can also do the same thing in the inertial frame.

So, I am not giving that derivation here but you can look into the book by on Astrodynamics, it is a given in the list of books by Archie E. Roy, so that part I am skipping here. Now, v square equal to ϕ – C, if I set it to 0, so

$$\phi = c$$

or ϕ which is a function of

$$\phi(x, y, z) = C$$

this gives us equation of a surface on which velocity is 0 in the synodic frame, again this is in synodic frame because this v you have defined as $\dot{x}^2$, $\dot{y}^2 + \dot{z}^2$, okay.

And $\dot{x}$; x, y and z in our context it has been described in the synodic frame, okay and therefore, the velocity is 0 in the synodic frame and on this we will discuss further but toward the end of this topic.

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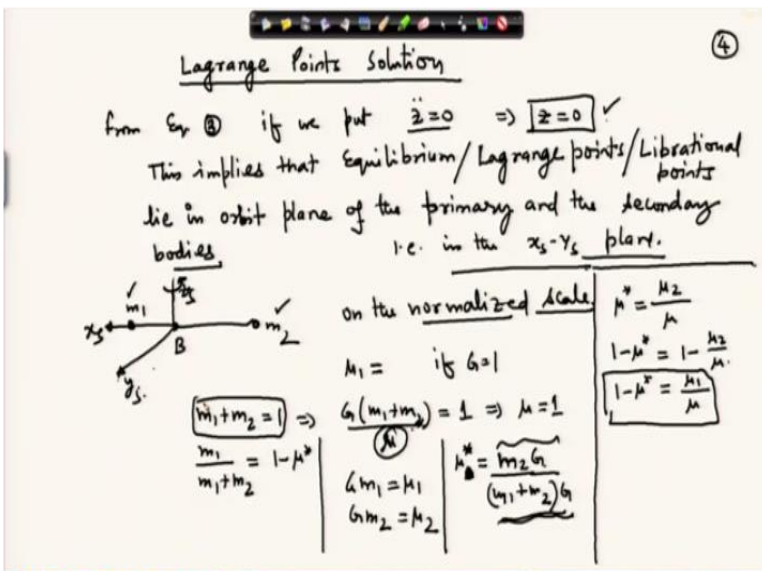

So, now we look into the Lagrange points solution, from equation 1 or let us say equation 3, if we put

$$\ddot{z} = 0$$

, so in that case what do we get, on the right hand side we have this equation here, $\ddot{z}$ equal to 0, so for acceleration is 0. So, this part at the equilibrium point if you are putting at 0, so the solution to this will be

$$z = 0$$

and this already we have discuss also.

So, this implies z equal to 0 and we have written earlier, this implies that the equilibrium /Lagrange points/Librational points, these are also called librational points lie in the orbital plane of the primary and the secondary bodies. Why this is so, z equal to 0 because in this our case, z was up and here we have taken x and this we have taken as y_s and this was the point B, m_1 is located here and m_2 is located here and this is rotating.

So, we assume that we chose the synodic frame such that x_s and y_s lies in the plane; orbital plane of the masses m_1 and m_2 and because of that we are getting this result that z equal to 0, so that means all the Lagrange points will lie in the plane xs, y_s plane that is in the x_s, y_s plane, okay thereafter we normalised the scale, so on the normalised scale what we observe that μ_1 can be written as; so what is mean by normalisation that we have

$$m_1 + m_2 = 1$$

Total mass we assume it to be unit therefore, m_1 by m_1 + m_2, this we will write as 1 - μ^*, okay let us multiply this also by G times m_1 + m_2 and if G equal to 1, then we consider this remains 1 and this quantity is nothing but mu, so this is mu equal to 1, this quantity is mu, G times m_1 we write it as μ_1, G times m_2 we have written as this as the μ_2, so μ_2 star or μ^* we will define as; μ^* will be; see μ^* is the mass of m_2, okay.

So, this is m_2 divided by m_1 + m_2 times G, this is in the normalised form, so μ^* is m_2 times G is μ_2 and below this quantity is mu, similarly 1 – μ^*, this will be equal to, so this is the normalised form because they are having the same dimension, okay and we are dividing it and normalising it, so reducing it to the scale 1 and for this we have already discuss also.

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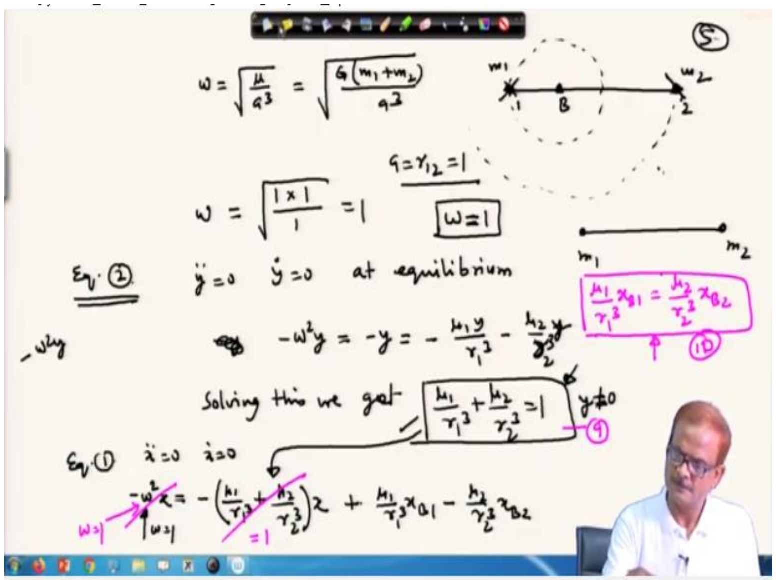

And we know that ω equal to

$$\omega = \sqrt{\frac{\mu}{a^3}}$$

and mu is nothing but G times m_1 + m_2 divided by a^3, so here in this case the angular velocity of each masses, m_1, m_2 and this is a barycentre, so angular velocity this will be moving about B like this and this will also be moving about B like this but simultaneously and what will be the period; period will be the same
as the motion of m_2 about m_1 or either m_1 about m_2, whatever the period the same period will be there, okay.

So, according to this scheme, then your if we write this as 1 and this is the 2, so a refers to r_{12} the distance between 1 and 2 and on the normalised scale, we take this as 1, G already we are taking this as 1, so this m_1 + m_2 as 1, so this gets reduced to ω equal to; so ω also on the normalised scale is 1 and benefit of using this is the ease of representing the system and less time taking and better understanding.

And it can be which once you have got the results, so you can solve, you can get back to the actual scale without any problem okay, so from the first equation we got here z equal to 0, so this is one of the conclusion. So, the Lagrange points what we are deducting that they are lying in the xy plane. Now, next we take equation 2, take equation 2 is related to the y axis and there what we see that $\ddot{x}$ is; y double dot we have taken to be 0.

And $\dot{y}$ equal to 0 at equilibrium and therefore, your equation that gets reduced to minus y ω^2 y with minus sign, okay, so here already or maybe we can write like this; - ω^2 y = - y, okay. On the right hand side what we have; so we have to write the whole equation, so this we have written as – μ_1 by y r_1^3 –μ_2 by divided by r_2^3.

And solving this we get

$$\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} = 1$$

$$\dot{y} = 0$$

, so under that condition this is valid, so that means if I have 2 masses m_1 and m_2, so using this you will get the solution where the y is lying of the axis, so this also we have already done, this I have taken a revision because in the next I am going to do on the actual scale some of the things that will be pretty complicated, so we can recall in this lecture.

Then, we can okay, once we have got this, then we can take the third equation, first equation, so from equation 1,

$$\ddot{x} = 0$$

$$\dot{x} = 0$$

so from there we get – ω^2 x equal to minus; rest of the terms I will arrange here and write in this place rather than again and again writing and rewriting and if we utilise this here in this place and we know that ω this equal to 1, so what do we get?

These 2 terms will cancel out, then this and this will cancel out because this quantity equal to 1 and ω is also equal to 1, so minus x, minus x from both sides will cancel out, so this yields

$$\frac{\mu_1}{r_1^3} x_{B1} = \frac{\mu_2}{r_2^3} x_{B2}$$

Now, what the result we have got here in this place, let us name it, 8A, this result with write as (9), and this result as (10).

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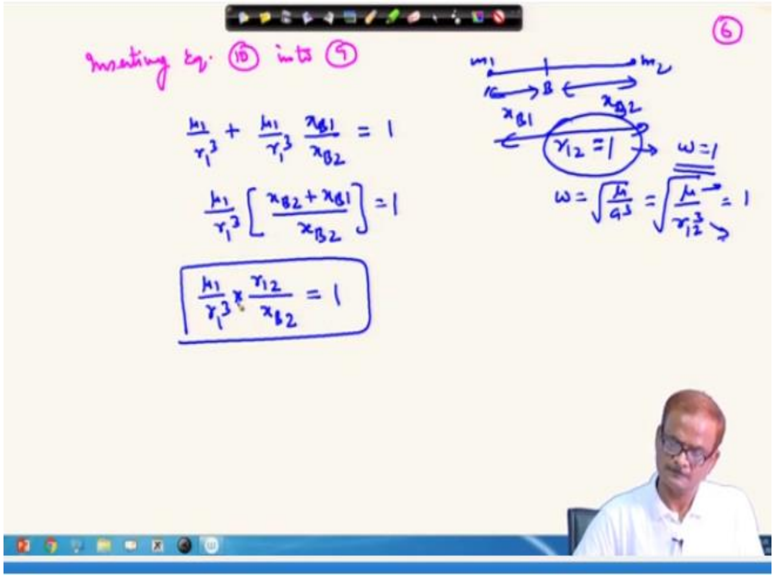

Now, insert (10) into (9); equation (10) into (9), so what do we get here, $\frac{\mu_1}{r_1^3}$ $+\frac{\mu_2}{r_2^3}$ you can see this quantity, this will be μ_1/r_1^3 (x_{B1} divided by x_{B2}) and this is μ_1/r_1^3 x_{B1} divided by x_{B2} and on the right hand side we get this is as 1, so μ_1/r_1^3, (x_{B2} + x_{B1})divided by; and (x_{B1} + x_{B2})is nothing but your r_{12}, this is barycentre, so this distance you have taken as $x_{B1}x_{B1}$, this distance you have taken as x_{B2}.

$$\frac{\mu_1}{r_1^3} \times \frac{r_{12}}{x_{B2}} = 1$$

In this whole distance we are writing as r_{12}, so on the normalised scale this quantity already we are writing as r_{12}, okay and based on this we have got ω equal to 1, so ω is nothing but mu by a^3 what we have written earlier equal to

$$\omega = \sqrt{\frac{\mu}{r_{12}^3}} = 1$$

and as we choose this one and this quantity is 1 by our assumption, so in that case we have got ω equal to 1. So, from here we get r_{12}, okay, so we will stop here and we will continue in the next lecture.