Space Flight Mechanics
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Lecture – 12
Classical Orbital Elements / Parameters

So, welcome to lecture number 12, so we have been discussing about the 2 body problem, in that context we are working with orbital parameters, so which we also call as the classical orbital elements.

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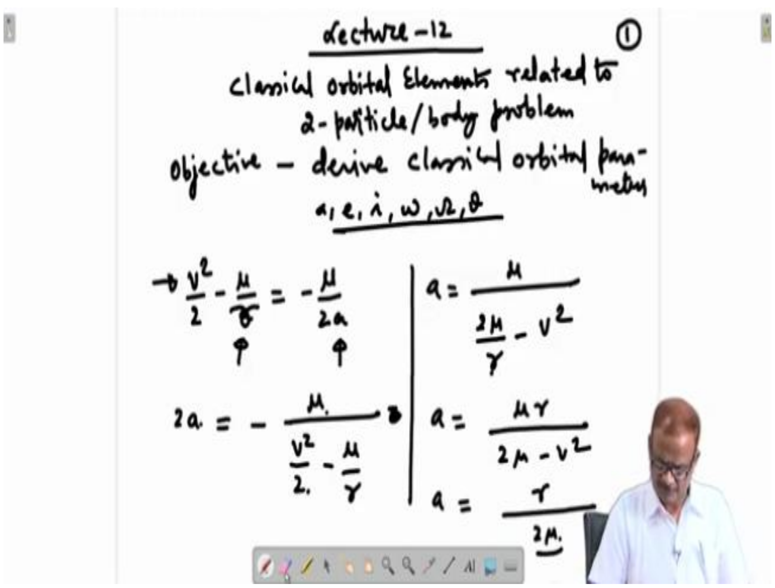

So, we start again working out those parameters, so if you remember in the last lecture we have derived, so here objective is to derive classical orbital parameters which are a, e, i, ω, Ω and θ, these are the parameters which we are going to work. So, first of all let us go into the vis-viva integral we have derived, this is

$$\frac{v^2}{2}-\frac{\mu}{r}=-\frac{\mu}{2a}$$

okay. So, if v is given and r is given, so a can be determined, we reorganise it to get that value.

So, we write this as 2a equal to - μ/($v^2$2/- μ/r) or a = μ by 2 μ by r – v^2; $v^2$by 2 – μ/r = - μ/2a, this is okay, v^2/2 – μ/r, we observe it inside, so this term will come here and v^2 will go on this side and if we little bit do the reshuffling here in this place, so we can write it as μ r divided by $2\mu - v^2$or a = r by $2\mu/$ r, we will go on the next page.

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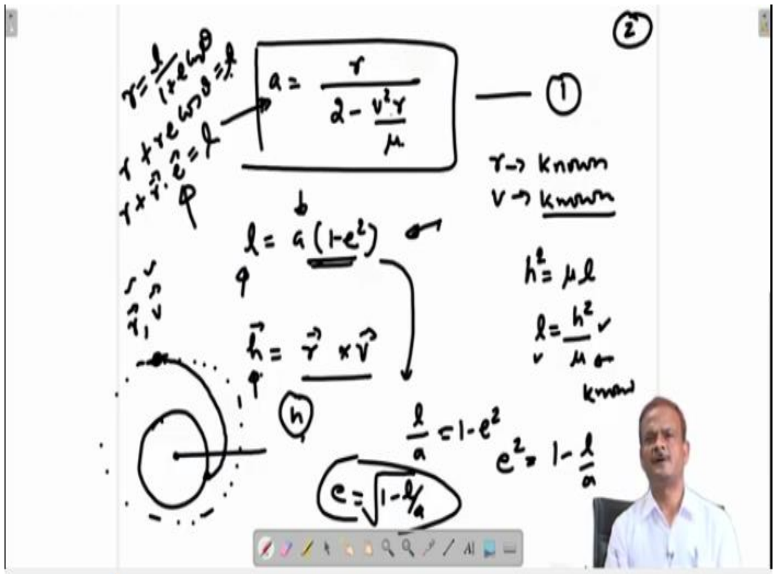

So, a equal to μ r by; r we have taken here in this place, so this is v^2r, this is 2μ and r goes in the numerator okay this is fine, so a equal to and then we divided by r again, we will divide by μ, okay, so on the next page, then will have

$$a = \frac{r}{2 - \frac{v^2 r}{\mu}}$$

so μ, μ cancels out, so we will not have μ here, we are doing here; so this is 2 minus; we are dividing the numerator, denominator by μ.

So, mu here cancels out and this we get as 2 here in this place and this is v r^2 divided by μ, so this is v r^2; v^2r divided by μ, so v^2r divided by mu, so this is the your first parameter determined here which is here. We have other parameters, rest 5 are remaining, so one by one we will be working out but this is the way of working for a parameter, now before we do anything we will try to work out the eccentricity vector.

And from here also you can get the e value, other ways of doing also this problem is there but this is one of the approach, so here what we are assuming that r is known and v is known, so if these 2 quantities are known, so then immediately you can determine. See, there are numerous ways of working problems in spaceflight mechanics and all the ways of working out it may not be possible to do it here in this place.

But as far as possible I will try to work out them like say

$$l = a\,(1 - e^2)$$

so if we know a from this place, so a is known and l how we can know; l we can know from; we know the relationship h^2 equal to μ times l, so l, h^2 will be h^2 by μ, μ is known because this is a planetary gravitational constant and h will be known from h equal to $r \times v$ because your r and v are known as the vector.

So, here in this case, what happens once the satellite is launched around the earth, somewhere here from the ground station suppose, satellite is launched, it is going in the orbit, now here it is injected in the orbit and thereafter it will follow certain orbit, okay it will follow certain orbit, so while this satellite is injected here, so at that time the inertial navigation system that sends you these values of r and v.

So, you know r and v from there, so if the h is from there the h will be known, h magnitude becomes known, h magnitude becomes known you can insert here, the l will be known and therefore from here, then e can be computed as

$$\frac{l}{a} = 1 - e^2$$

or $e^2 = 1 - l/a$ or

$$e = \sqrt{1 - \frac{l}{a}}$$

so this way we can also computed but I will show it through other way which will be used more frequently rather than using this.

Sometimes it may be very useful to work out using this expression rather than going into what I am going to derive but one of my objective as I told you earlier that while we have written

$$r = \frac{l}{1 + e\cos\theta}$$

and this we have deduced it in the form $r\ e\ cos\theta = \ l$ or this we have written as $r\ .\ e\ = \ l$, okay. So, here e is appearing as an eccentricity vector, so I am going to show that e indeed this is an eccentricity vector.

And it appears as an integral; integration constant in the expression of the motion, equation of motion expression, if we process it in a proper way, okay and from there also then we can derive a, so there are numerous ways of doing make into this space flight mechanics problem and I will try to give you as much as possible keeping in view the constant on-time, okay.

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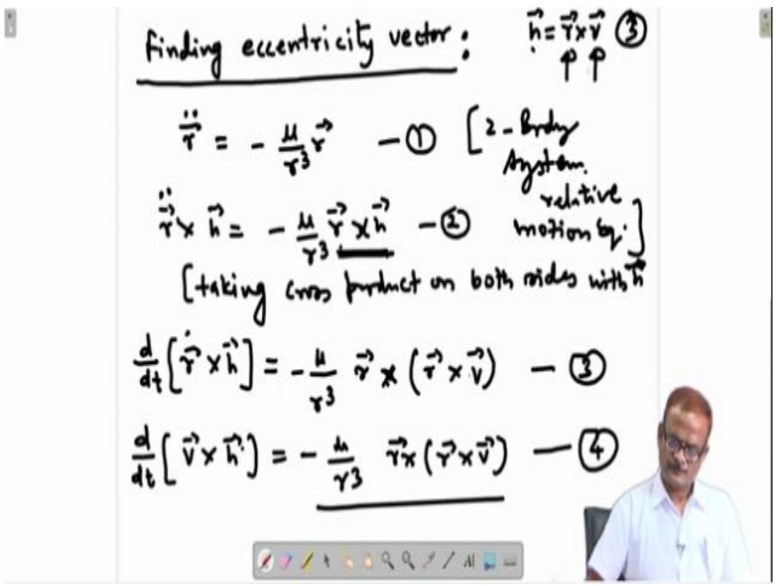

So, let us start working with the finding out the eccentricity vector, so finding; so later on I will tell you that what is the benefit of getting the eccentricity vector, even if you do not note an eccentricity vector it is okay but there are certain benefits which you will realise once I further progress. So,

$$\ddot{r} = -\frac{\mu}{r^3}\vec{r}$$

this is our basic equation we have been using for 2-body problem system relative motion.

Take cross product on both side with h which is the angular momentum vector, this is by taking with h,

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = -\frac{\mu}{r^3} \quad r^{\rightarrow} \times (\vec{r} \times \vec{v})$$

because h is a constant vector from our earlier work this is r^3, this is r^3; minus μ by r^3 r cross, this is v, so on the left hand side we have this is v×h, r × v or r ×h, it is in this format, so this part we are going to expand, the right hand side of equation (4) we are going to expand.

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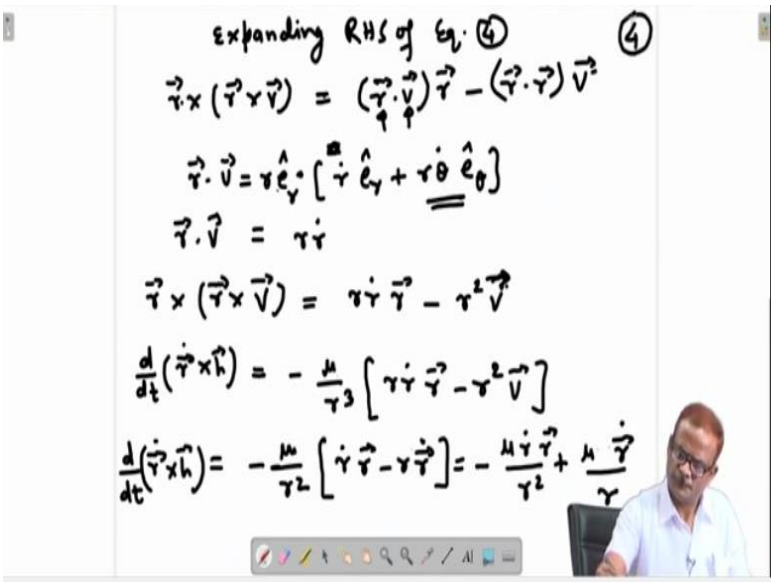

So, we have there $r \times v$, r.v r minus; remember r cross; here, r × h in this one, r and h they are perpendicular to each other because h is a vector which we have obtained as r ×v, okay so therefore, h is perpendicular to both h and v. So, r.v, what this quantity is; you can consider this as the component of vector v along the r direction, okay so I will write it in a more proper way.

So, here before taking up this whole thing let us write this r .v, this equal to r dot; it is a dot product with e θ, your r dot this quantity, here also this I can remove and write it in proper way, r equal to r times $\hat{e_r}$ dot. Now, this is dot product okay, so this becomes r$\dot{r}$, $e_r$dot, er dot that is unit vectors, so this you get 1 and this part get vanishes, so

$$\vec{r}.\vec{v} = r\dot{r}$$

so your $r \times v$, this gets reduced to r $\dot{r}$ $\vec{r}$-r^2v, okay.

Therefore, d/dt $\dot{r} \times \vec{h}$ which is the left hand side, this gets reduced to – μ by; going on the previous page, μ/r^3 times r $\dot{r}\vec{r}$– r^2v, μ/r^2, r we are cancelling out, so $\dot{r}$r-rv and rather than writing this v, let us write this as in a more appropriate form, this is $\dot{r}$, okay, if we break the bracket, so this gets reduced to μ $\dot{r}$r divided by r^2+ μ r times $\dot{r}$ divided by; this r will go then, this gets reduced to r.

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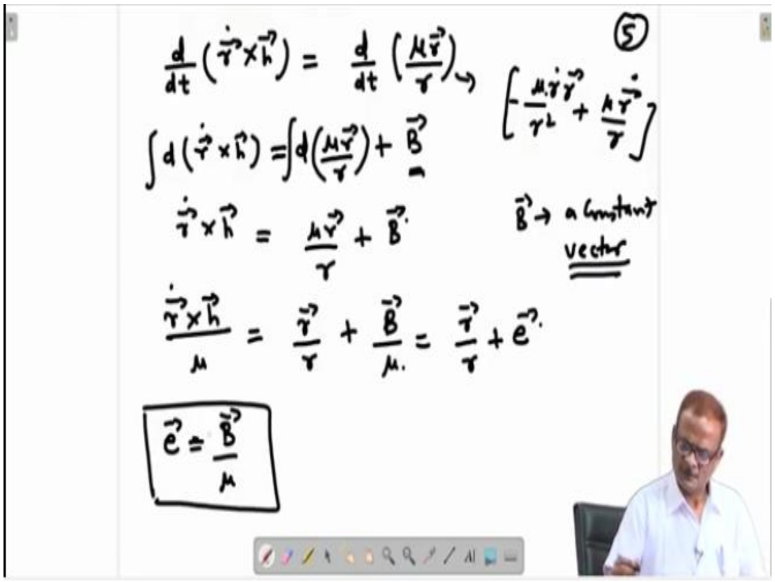

So, the left hand side again we write here d by dt $\dot{r} \times$ h and on the right hand side we have this quantity, so this quantity is nothing but your d/dt $\mu\, \vec{r}$/r, as we can see it by expansion, this is μ/r^2 with minus sign times $\dot{r}\, \vec{r}$ and then plus $\mu\, \dot{r}/$r, this is what we have on the previous page; $\mu\, \dot{r}$r by r^2, we have got here in this format and there is a minus sign also, minus sign is, okay minus sign is absorbed there, so this minus sign is not there, okay.

So, with this now, if we integrate with respect to t, so integrating it this gives us, you can write it like this because both sides are vector, so this constant is going to be a vector and therefore what we get here r×h, this equal to $\mu\, \vec{r}$/r + B, we will divide it and write it in this way, divided by μ r by r + B by μ, 1 μ is missing is here, so we will put μ also, now μ we have divided on both sides, so this is not required here.

So, here B is a constant vector, now why we have written it; it will be shortly clear you know that $\vec{r}$/r this will be an unit vector but still we write it here in this form and this I will write as e, so $\hat{e}$; $\vec{e}$ this equal to $\vec{B}$ by μ and why I am writing like this; it will follow from the other steps I am going to carry out.

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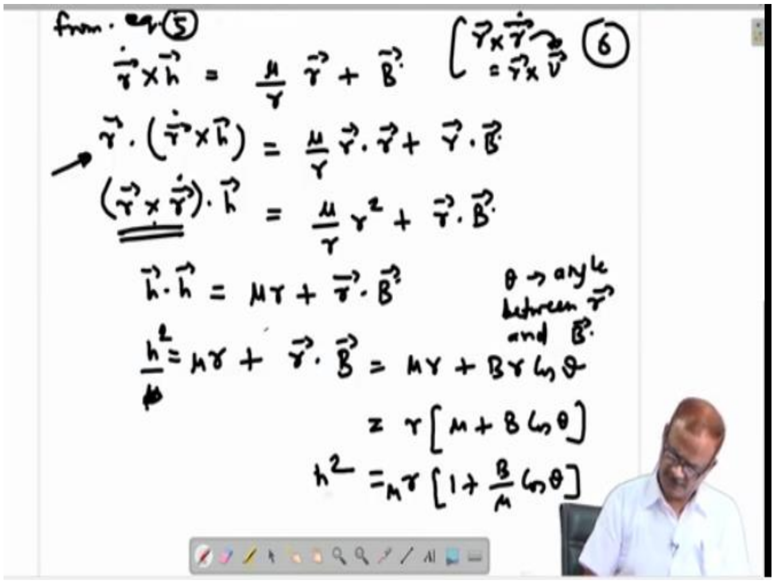

So, we have got here $\dot{\vec{r}} \times$ h, this equal to μ/r, I am taking the original form and working out I just wanted to show you that this quantity this will appear as eccentricity vector. Now here, again we have to do some trick, so numbering of equation I have not done, here it is number (4), this will I number as (5) from equation (5), $\dot{r}\times$h equal to μ/r, now here we are going to work out.

So, we take dot product with respect to r, why I am taking this dot product with respect to r, as we go on the previous page and see here, this is the $\vec{e}$ is appearing, so if I take dot product with this e, so you know that this will get reduced to $r\ e\ cos\ \theta$, okay and if I am able to get a conic section equation from this place, so that means this e equal to B by μ is verified as we will follow, just wait for some time.

So, here this can be written as

$$(\vec{r} \times \dot{\vec{r}}).\vec{h} = \frac{\mu}{r} r^2 + \vec{r}.\vec{B}$$

here you know this quantity is nothing but this is $\vec{r} \times \dot{r}$, this is nothing but $\vec{r} \times \vec{v}$, $\dot{r}$ is nothing but this is $\vec{v}$ and therefore, this is h, so this becomes $\vec{h}.\vec{h}$ = μ r r^2, so here also we will simplify, this is mu $r + r.v$ and this becomes h^2 equal to or h^2 by μ will; on this side, so this is r divided by; now it is okay, okay right now leave it here.

So, $\dot{r}$ this is v, this is μ r + B r cosθ where θ is the angle between r and v, θ, this is the angle between r and B vector, so if we take here r as common, so this will be mu times B cos theta and if we take mu also outside, so this will be 1 + B/ μ cosθ, on the left hand side we will have h^2.

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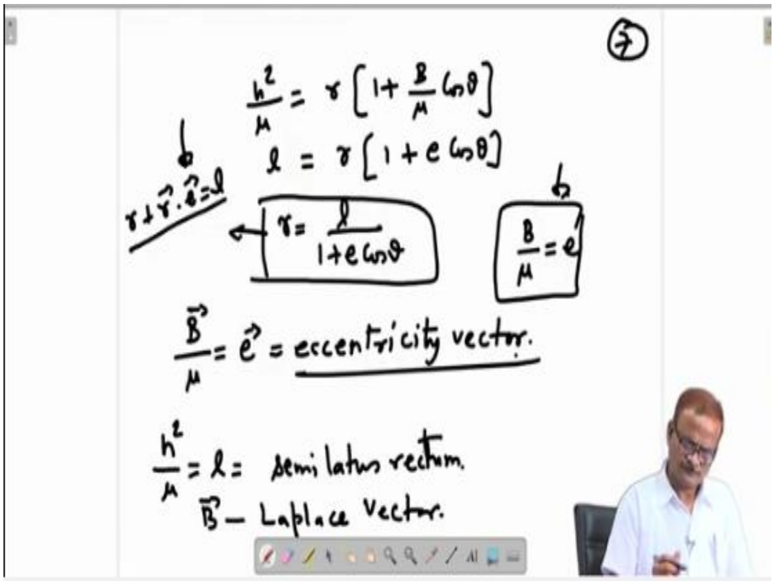

So this equation can be written as h^2 divided by μ, this equal to r (1 + B/μ cosθ) and suppose we write this quantity as e, B/μ is equal to e and here h^2 by μ equal to l okay, so what we get here is the expression for the conic section, so you can see that the same problem can be worked out in numerous space in space flight mechanics, this is the beauty of the basic mechanics, very interesting subject.

Now, here from this place it is clear that v/μ, this equal to e and also if we go on the previous page, v/μ we have written here as e, we take the magnitude of this, so this will be v/μ magnitude, so this will be e magnitude, okay so from here we can see that e is an eccentricity vector, so B by μ this quantity is e, the eccentricity vector and also h^2 by μ equal to l which we have written as semi latus rectum.

So, this $\vec{B}$ we call this as the Laplace vector okay, so we have proved that eccentricity is a vector and which was obvious also from the very basic equation, this once we have written in the format $r + \dot{r}\, e = l$, so that means eccentricity is appearing as a vector and here in fact we have been able to prove that it appears as a vector constant in the while we integrate the equation of motion in a particular way, okay.

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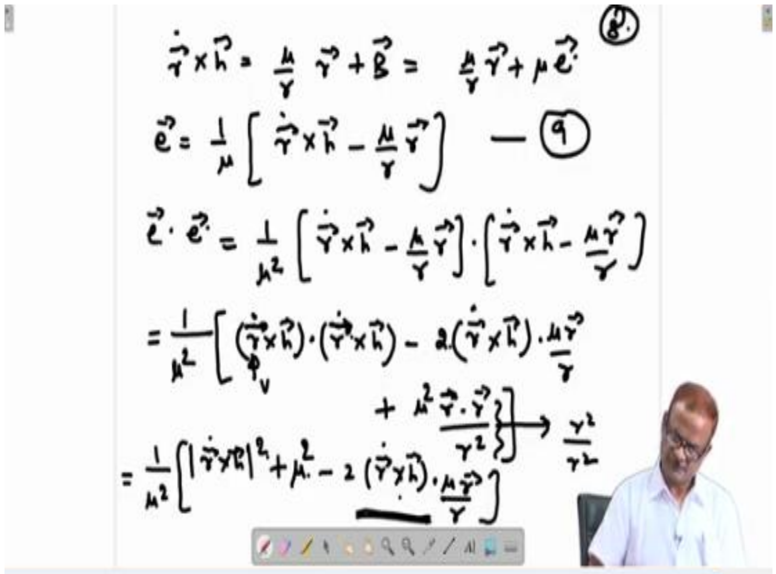

Now, we have to determine the expression for *e*, so we have to find out this $\vec{e}$, we rearrange it like this and this is our 5 and then this we will write as 6, 7, 8, so if we take here this dot product because we have to determine the value of e because this is a dot product, so we can write it this way, now this is v cross; here we are missing 1 vector, this quantity is nothing but your v, v × h, v× h.

So, we need to work it out, so we can simply write this as $\vec{r} \times \vec{h}$ magnitude h^2, μ h^2 we have already written it outside, we have taken it here in this place, so this is fine, I got 1 by μ, let me check, verify it once from this place, dot h, this is μ by r + v μ times v, so this is $-\mu/r$;$1/\mu$, this is okay, so dot product, this is μ^2, this is fine, this quantity is mu times $\dot{r}$ r is nothing but r^2, this quantity is r^2 by r^2.

And therefore, this gets reduced to just μ, so this is plus μ-2$\dot{r} \times \vec{h}$ μ $\vec{r}$/r and this can also be simplified, it is very easy, so we follow in the next step.

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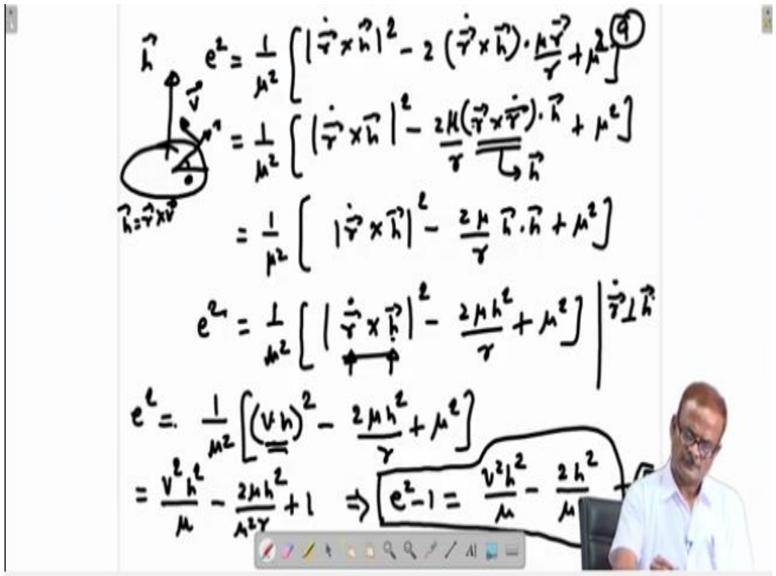

So, the left hand side then becomes e^2 , this equal to $1/\mu^2$ and in bracket, $(\vec{r} \,.\vec{h}\,)^2$; and from here we have minus 2 times, I am copying the previous thing; $\dot{r} \times \vec{h}\,.(\,\mu\,\vec{r}.\,\vec{r}$, this is the dot product here and plus μ, here we have missed out, this is μ^2 so once you multiply it, this μ^2 r^2, this is also μ^2, so this is μ^2, no sorry, not here in this place, the other place we are doing the mistake.

If this is fine, this place it is μ^2 and not here, this μ^2 is here in this place because this part it cancels out, okay so mu square it remains as a μ^2- 2 $\dot{r}\times$ h dot μ r r, this quantity we have already written here, so $\dot{r}$ × h^2 - 2 $\vec{r}$ × h μ/r + μ^2. Now, this we can reshuffle, this can be written as $\vec{r}$ × $\dot{r}$, dot h μ here divided by r + $\mu^2$2, this is μ; so this is 2μ/r.

And this quantity is nothing but your vector h, so this is h; dot product h μ^2 2μ h^2 divided by r + μ^2. Now, we have to evaluate this quantity, so we if we remember that in the orbit, if this is the r vector, this is your θ, v vector will be here and r vector is here in this direction, this is the v vector and perpendicular to this orbit that h vector is located, so h is perpendicular to h equal to r × v, so it is a perpendicular to both v and r.

That means, the angle between $\dot{r}$ and the h vector, it is a 90 degree and therefore that means we can write here $\dot{r}$ is perpendicular to $\vec{h}$ and therefore this angle between will be sine 90 degree, so this we can simply write as 1 by μ^2 v h whole square because sin θ will be the angle between these 2 will be 90 degree, so therefore we can write this as v h^2, so - 2 μ $h^{2\,/}$ r + μ^2.

So, this becomes $v^2 h^2$ by μ^2- 2 μ h^2 / μ^2 r + 1 and this implies

$$e^2 - 1 = \frac{v^2 h^2}{\mu} - \frac{2h^2}{\mu}$$

we have got this expression.

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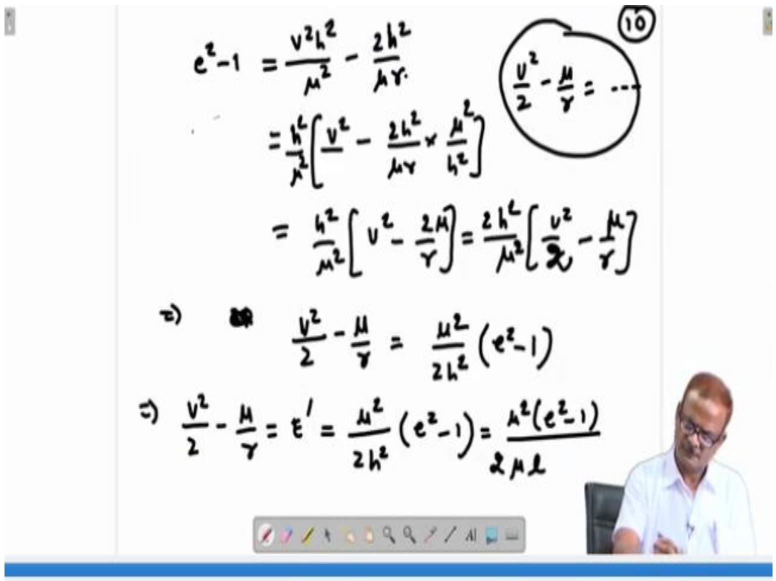

Okay, so we have e^2-1, this quantity is v^2h^2/μ - $2h^2/\mu r$; minus $2h^2/\mu r$, so we rewrite it and so finally, what we want to do; we want to express it in the format $v^2/2 - \mu/r$ equal to something, in this format we want to bring it, so if I try to do this, so this is v^2; so h^2/μ I will take outside and this becomes v^2- 2 h^2 by μ r times μ by h^2.

This is h^2/μ, v^2- h^2 cancels out, μ^2cancels out, this is 2/r and then 2 also we will take it outside, so this is h^2, this is 2 /μ v^2 by 2 – 1/r, somewhere it is dropped out, $2h^2/\mu r$, okay and $v^2 h^2/\mu^2$, here we have dropped out while writing this that square term is dropped out, here this is square term is there, so this square term is here, we can write this as if we are taking out; we will take out h^2/μ^2.

So, if μ^2 is there, so if we take out μ^2 term here, so this recommends v^2 and here this will go as μ^2, okay, so we get here 2μ/r, you have this μ^2 term will be there and this term will be μ/r, so $v^2/2\mu/r$, this is fine, so we can then rearrange it to

$$\frac{v^2}{2} - \frac{\mu}{r} = E' = \frac{\mu^2}{2h^2}(e^2 - 1)$$

Here also this is mu square missing, so this is mu square, so now you can see that this quantity which we have written as; earlier if you remember E', so this is nothing but

$$= \frac{\mu^2}{2hl}(e^2 - 1)$$

so this is another way of working this and from here you can further investigate it, 2 times h^2 equal to μ times l, so we can write it in this way, we have to go to the next page.

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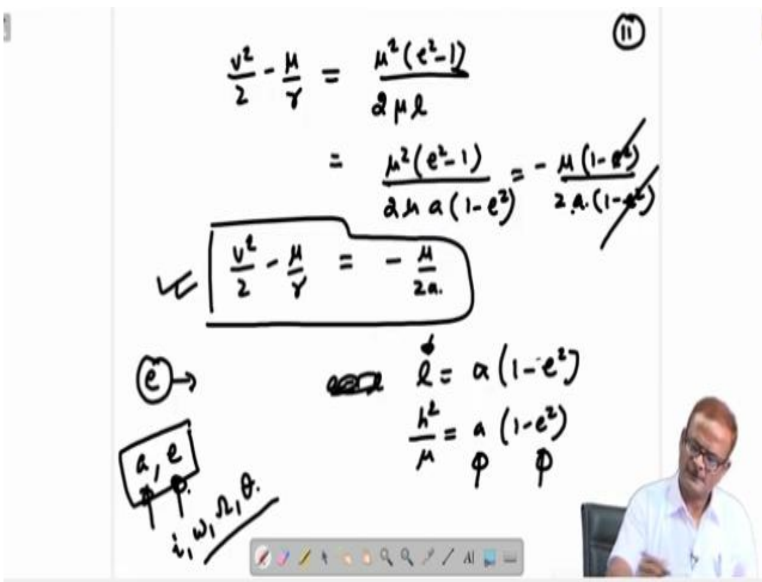

So, from this place

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu^2}{2hl}(e^2 - 1)$$

okay, so right hand side we have $e^2 - 1$, 2μ and l equal to a times $1 - e^2$, so this gets reduced to μ times $1 - e^2$ 2a times $1 - a^2$, so if we cancel out this term and this term, so we get here $-\mu$ by 2μ, so $v^2/2 - \mu/r$, this equal to $-\mu/2a$. This is another approach, we have derived this $v^2/2$ equal to $-\mu/r$, 2a.

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

And you remember here that if your v and r are known, so h will be known to you and μ is known and therefore e is also known from this place, so this expression can also be used, we will write some other expression also, see all these expressions are required while solving different problems at different times, so this is one way we have got the expression for E' equal to $-\mu/2a$.

And here also we have worked out, here see if; here also we have worked out in this place using this E' and using this expression, we developed it and by developing this we came to the conclusion that E' equal to minus $\mu/2a$, so the same thing again we have been able to derive but this we have done in another way, that the so many ways of working the same problem and each of them will be useful while solving certain problem, it is not that if we know one, then it is enough.

We need to know all the ways of working out the problems, so we in this way, once we know all these quantities, so will be able to work out this e also, so the explicit equation for e we are going to write further but once you know that a is related to; l is related to a times $1 - e^2$and some or other your l is becomes known which is nothing but h^2 divided by μ $a(1- e^2)$

So, already a we have determined, so using this expression, we can get e, okay another expression are a, we will do it in the future, so this way we have been able to work out a and e, these 2 parameters are available, the rest remaining are i, ω, Ω and θ these are the remaining 4 we have to work out. So, we will stop here and okay just 1 minute, I will take more.

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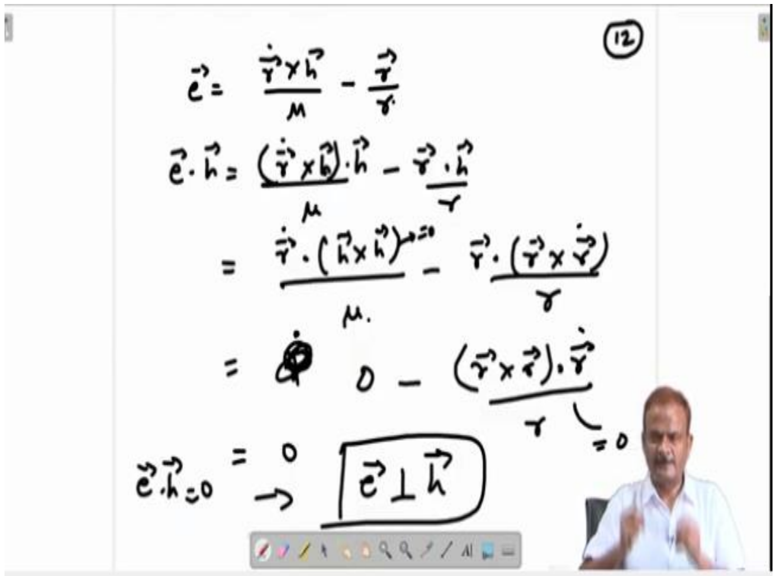

And do one final thing that showing that e and h are also perpendicular to each other, so e vector is $\dot{r}$×h as per our earlier derivation, take dot product with h, so this is $\dot{r}$×h, so this quantity here you can write as $\dot{r}$h × h/μ minus this quantity here becomes r × $\dot{r}$/r, so you can see this quantity is 0, this is 0, so here r, this part gets reduced to 0 and minus here again this can be written as r cross $\dot{r}$/r, so this also gets reduced to 0.

So, therefore this is 0, so e dot h, they are perpendicular this equal to 0, this implies that e and h they are perpendicular to each other, okay, so we will wind up here and we will continue with the orbital parameters because we have to work out the rest 4, so next few lectures, we will wind up this topic, thank you very much for listening.