Space Flight Mechanics

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Lecture – 11
2-Particle (Body) System

Okay, welcome to the lecture no.8. We will continue with the 2 (Body) System, so in the 2 (Body) System what we have done that we got the relative motion equation and then we have reduced it into the form.

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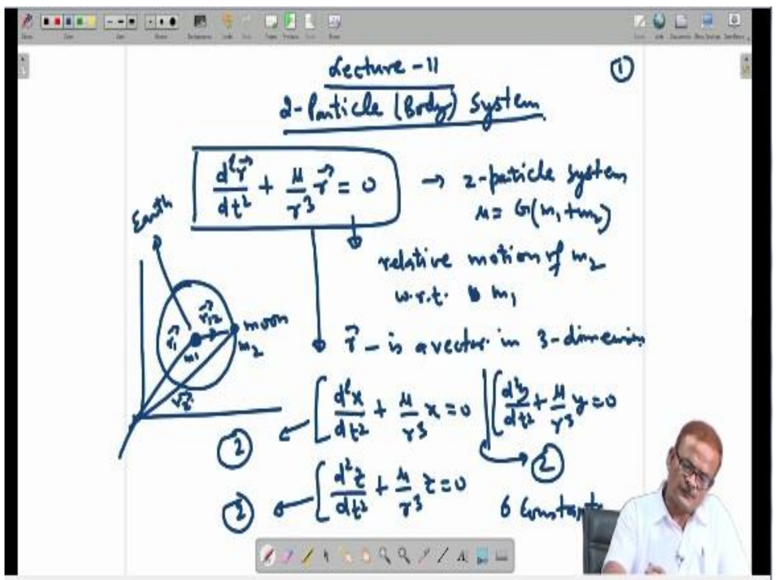

$$\frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3}\vec{r} = 0$$

we have reduced into this form, this for 2 particle system where $\mu = G(m_1 + m_2)$. So here this is a typical case like suppose earth is here and around the earth moon is going on in the orbit, okay so and somewhere inertial frame let us say this is my inertial frame in which I am looking for this motion, this is r_1 and this is r_2 and this r_{12}, okay this is moon and here this is earth.

So as you know in this expression, the relative expression we have got it after writing the motion of these two particles, this is m_1 and this is m_2 so the relative motion we have already written, that is the relative motion of m_2 with respect to m_1. And the same thing is applicable to the earth satellite also or it may be a moon satellite. Okay, so once we have got this equation so in this equation you can remember that r is a vector or 3-dimension vector, this is 3D, 3-demensional or; this is a vector in 3-dimension.

This is a vector, r is a vector in 3-dimension. So as we have discussed earlier we will get here three second order differential equation which are $\mu/r^3\ x = 0$, if we break it up the other one will get as

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = 0$$

and

$$\frac{d^2z}{dt^2} + \frac{\mu}{r^3}z = 0$$

So these are second order differential equations. Okay and this will have two constants, a total of two constants involved, this will also have two constants involved, this will also have two constants involved. So a total of 6 constants are involved, 6 constants. So while we try to solve this equation so we must identify those 6 constants.

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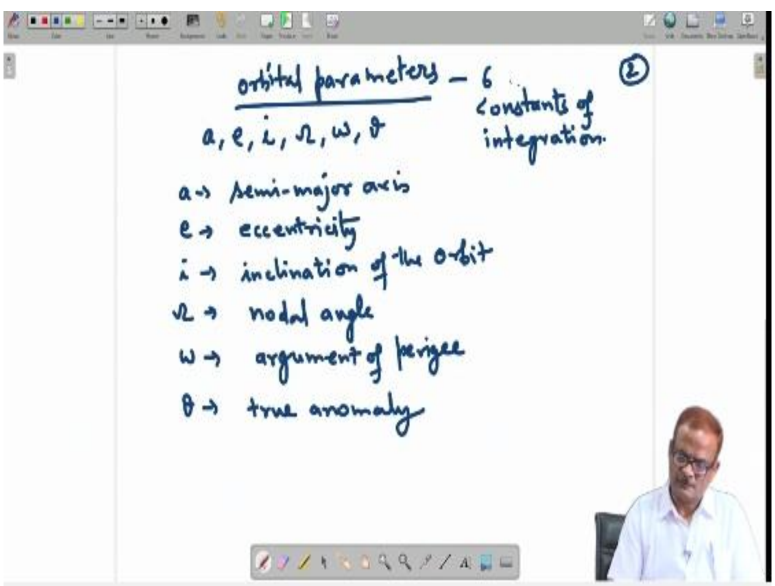

So all these 6 constants they are identified in the form of the orbital parameters. These are the 6 constants of integration which are called the orbital parameters. Now these are $a, e, i, \Omega, \omega, and\ \theta$. what they are called is the semi-major axis, e is called the eccentricity, i is called the inclination of the orbit, capital omega this called the nodal angle, small omega this is called argument of perigee and theta is called the true anomaly.

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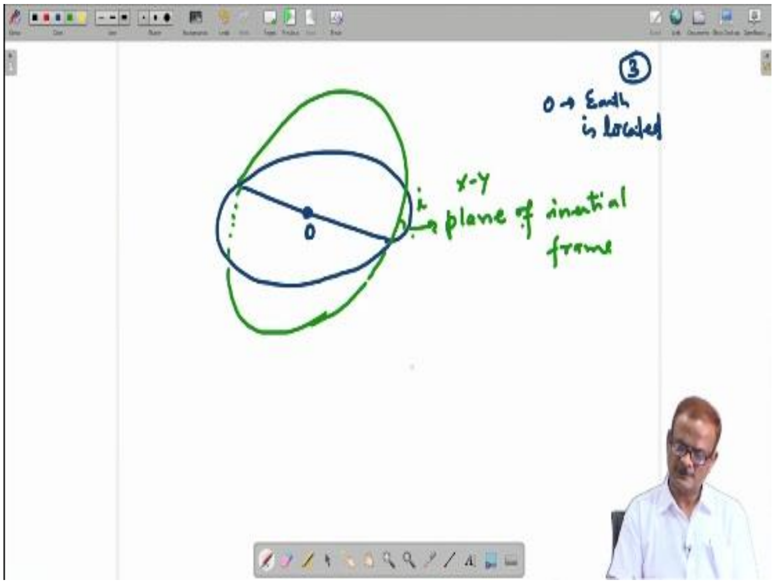

So now assume that earth is located here, so this I will write this as point O, so at O earth is located. And with respect to this one orbit is there which is inclined like this. Say this is the plane of X-Y plane of inertial frame and with respect to this, this orbit is inclined and this angle is i. I will come to the actual picture but just I am trying to show you how the things are working. Okay, so here I have; you can see this file.

So in this file you can assume that this is the X-Y plane, okay which is the inertial frame and if I take the upper cover like this okay so you are creating angle, this angle, I am creating this is called the inclination angle, okay. And in this plane as you can see this is the; if I inclined it like this and draw a orbit like this here, okay so that indicates that your orbit is inclined with the initial plane. So here this blue line which is shown here, this constitute here X-Y plane of the inertial frame and this is your orbit, in a better way we can represent it on the next page.

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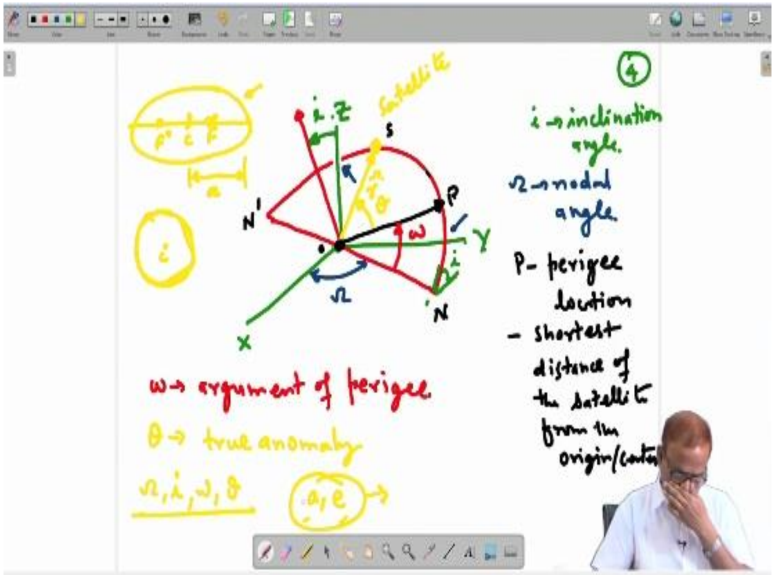

This is the Z direction of inertial frame. This is x direction of inertial frame; this is Y direction of inertial frame. So you can see that here with the, this is the plane of the initial plane of the inertial frame which I have drawn here by this blue line actually, the color has been changed. So this angle is here your angle of inclination *i*. So what you are doing that this red semi-circle or it may be part of some ellipse which is shown inclined.

So actually initially it is align in the X-Y plane and then I am lifting it, so by lifting I am creating this angle *i*, so this Z also it will get inclined and this will come here which I will write as *i*, the angle of inclination, so *i* becomes here inclination angle. Angle from this place to this place, this called Ω which we have written as nodal angle, and here just keep in mind that this red line I have kept here in this place as broken that means the red line going on the backside of the vertical axis which is the Z-axis in this direction. This I have done purposefully.

And here in this place this is coming over this means the Y-axis is lying below the red line. Okay, now let us assume that the satellite is lying somewhere here, it may be moon or it may be artificial satellite, manmade satellite, so satellite is located somewhere here and this is the radius vector to the satellite which we will write as *r*. Then we have; quiet often I have mentioned the periapsis, so this will indicate by p, this we will write as s.

So this line is periapsis, periapsis is the shortest approach of the satellite to the center of attraction. So this is the nearest approach. So p is the perigee location of periapsis location which is the shortest distance of the satellite from the origin/center. So this is your origin o. this will name as N and here this will name N'. So the angle from here to here this we write as ω, so ω we have already indicated this called the argument of perigee. And angle from here to here this we write as θ and θ we have written as true anomaly.

So if you see here we have been able to identify a total of 4 parameters which are Ω, i, ω and θ. Rest *a* and *e* these are related to the orbit which have the semi-major axis and the eccentricity, so semi-major axis in the case if the orbit is elliptical and this is one of the focus, this is another focus f*, okay and this is the center, so distance from here to here as we have done in the conic section this is your semi-major axis.

And eccentricity, *e* already we have defined and again we are going to define, so eccentricity is related to how, say here in the case, there is no eccentricity so the focus will coincide with center *c*, we can write it here as *c* and in that case that gets reduced into the form of a circle. So if eccentricity causes the focus to move away from the center, the higher the eccentricity the more it will show it what moving to what this boundary or on this trajectory, okay.

So a total of 6 parameters are there and these 6 parameters we need to work out also. So in the 2 (Body) problem where a satellite is moving around the earth or moon is moving around the earth or the earth is moving around the sun, in all these cases we try to look for the relative motion, so how the orbit is appearing and around the earth if the satellite is going on so what we will look for, we will look for that where it is located anytime I have to trace the satellite.

See if you satellite at the launching time, if you do not know in which direction it has been sent and in which orbits it is moving, so it will be lost, okay. Sitting in one place and using your radar you will not be able to crash it, because it is a like discovering needle on the surface of the; beneath the surface of the sea, that means on the bottom of the sea, the water it is full of water and on the bottom a needle is there and that needle you have to locate. It is become so difficult.

So orbit determination gives you in which orbit your satellite is going on so that you can orient your radar in a particular direction and whenever satellite passes over your location, so you will be able to upload the data, you will be able to give certain command, you will able to download certain data and so on. So from that point of view this orbit is very important, without this no satellite can operate.

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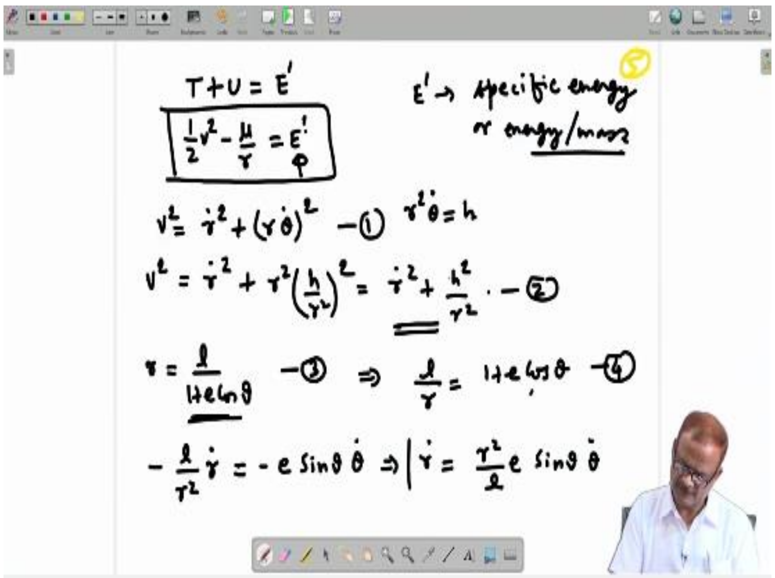

Okay so with this we proceed further. So already if you remember that we have proved

$$T + U = E'$$

where t was 1/2 v^2 and - μ/r this equal to E' and this relation we have used sometimes back. If we go on the previous pages, see this relationship we have derived, okay. This relationship can be derived, if we solve the right hand side. What will be the value of the right hand side?

So this we are going to do. And this is called, E' is called specific energy because m is not present here, specific energy or energy per unit mass and we have to solve this. If we solve it we will get that equation we have utilized earlier. So this is our objective right now to work it out. So we start with this and also we have the relationship, $r^2\,\dot{\theta} = h$, so here we will replace this, this is $r^2\,\dot{\theta} = h$, whole square, so this is our relation number 1.

So if you remember we had this one on the right hand side we had certain quantity and plus one constant *c* which we were utilizing. Now we will find out what is the value of the *r*, we will evaluate and $\dot{\theta}$ and therefore we will be able to evaluate and if we insert here in this place so we will be able to evaluate this E'. So

$$r = \frac{l}{1 + e\cos\theta}$$

we reorganize it and write it as $1/r = 1 + e\,cos\,\theta$. Why we have done this?

Because then the differentiation will be easy in the sense that we do have to take the differentiation of this quantity which is in the denominator, we have bring it here in this place. So you will see how easy this becomes, $-l/r^2$ we differentiate with respect to t, so $l/r^2\dot{r}$ then this taken from the right hand side e sinθ times $\dot{\theta}$ and there is a minus sign here, so this implies $\dot{r}$ equal to; minus, minus sign will go r^2 divided by l, e sinθtimes $\dot{\theta}$.

Okay. And rest r is known to us in this format, $\dot{\theta}$ is, again r/l is there so we can do some simplification, let us go the next page and then we will think of doing the simplification, this is $\dot{\theta}$, so this is equation number 5. Now we need to insert here in this equation.

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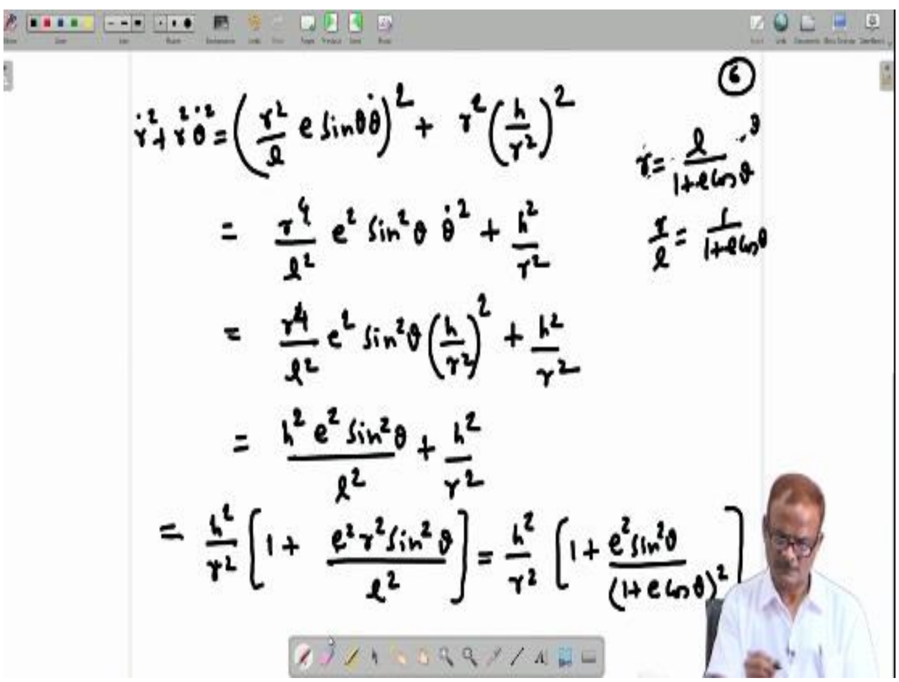

So $\dot{r}$ from the previous page, this r^2/l, r^2/l times e sinθ, $\dot{\theta}$, this is whole square, this is

$$\dot{r}^2 + r^2\,\dot{\theta} = \left(\frac{r^2}{l}\,e\sin\theta\;\dot{\theta}\right)^2 + r^2\left(\frac{h}{r^2}\right)^2$$

what is missing? Sin θ times $\dot{\theta}$, h^2/r^2, h^2; this is r^2, here $\dot{\theta}$ also we need to eliminate r^4, $\dot{\theta}$ is h/r^2 so here this becomes $e^2 \sin^2\theta$, h^2/r^4, that cancels out leaving us with l^2.

Okay, so h^2/r^2 we take it outside then this gets reduced to $1 + e^2 r^2 \sin^2\theta / l^2$, h^2/r^2 coming out so 1+ h^2 has gone here, so $e^2 r^2 \sin^2\theta / l^2$ which is okay, h^2 / r^2 1 +; now if we know that

$$\frac{r}{l} = \frac{1}{1 + e\cos\theta}$$

or let me write the equation in proper way and from there we will write it.

$$r = \frac{l}{1 + e\cos\theta}$$

so r/l is 1/1+e cos θ not; so we utilize it here in this place so this is $e^2 \sin^2\theta$ divided by, so if we insert this, so this becomes $(1+e\cos\theta)^2$.

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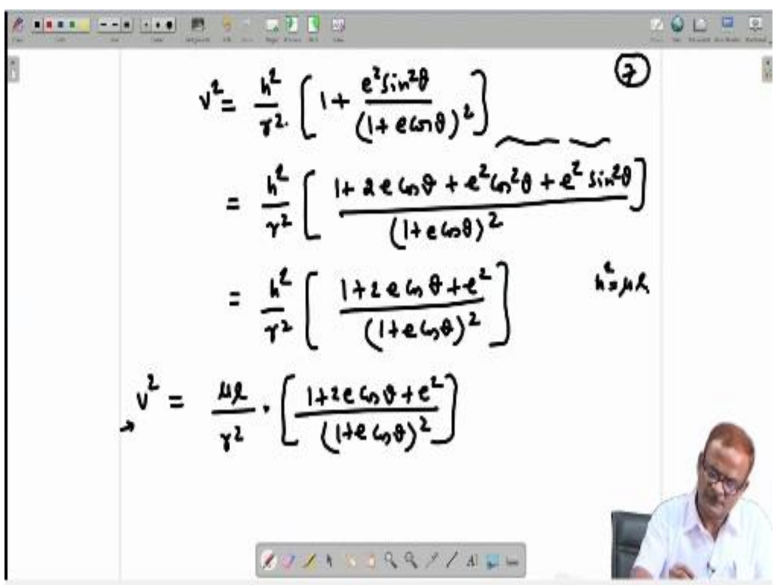

So our v^2 then gets reduced

$$v^2 = \frac{h^2}{r^2}\left(1 + \frac{e^2\sin^2\theta}{(1 + e\cos\theta)^2}\right)$$

And we are aware that $h^2 = \mu \times l$ so we utilize this here as the r^2 times $1+2e\cos\theta + e^2$ divided by $(1+e\cos\theta)^2$.

Now we will do little simplification here in this place. So what we will do, we will try to eliminate this r. We have got this; okay we do not need to eliminate this r because finally we have to write the equation; this expression we are writing in terms of r only, so we will not eliminate that. We keep it in this format. Now the other part we have – μ /r. So we will go to the next page.

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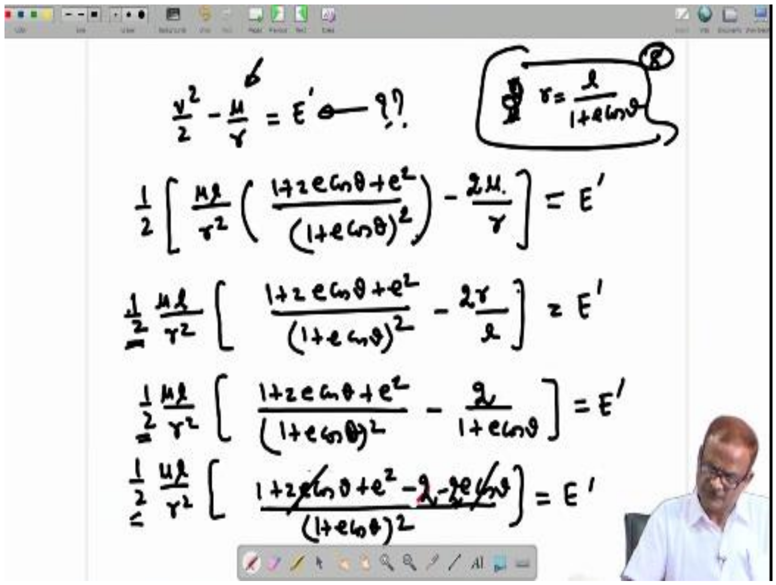

So we were looking for

$$\frac{v^2}{2} - \frac{\mu}{r} = E'$$

And this is the quantity which we are trying to evaluate. So we will write here 1/2 and v^2 from the previous page then we fetch μ /l r^2, μ l/r l^2, μ l/r^2 and times 1+2e cos θ e^2, 2e cost θ times e^2 divided by $(1+e \cos \theta)^2 - \mu$ /r.

This equal to E'. We take out μl/r^2 of the bracket so we get here $(½ \text{ e cos } \theta \text{ } e^2 \cos\theta)^2$ and μ has gone from this place, l we are taking here so the l also we are taking outside in this place so l we have to write here and r only one 1 r is here, here we have the r^2 so we must have another r here in this place, okay so this is the quantity here. If we take it inside so this is μ r, r cancels out so yes we get our so μ l, l cancels out, this is fine, so you can check it we get this quantity.

So 1/2 μ l/r^2 and then we know again utilize this relationship r/l or the

$$r = \frac{l}{1 + e\cos\theta}$$

okay so from there r/l will be 1/ e cosθ, so we get here 2e cos θ e^2 divided by e cosθ, e cost θ whole square – r/l so this is 1/1+ e cos θ, this equal to E'. We need to expand it whole square so 1+2e cos θ + e^2 – 1 – e cosθ = E'.

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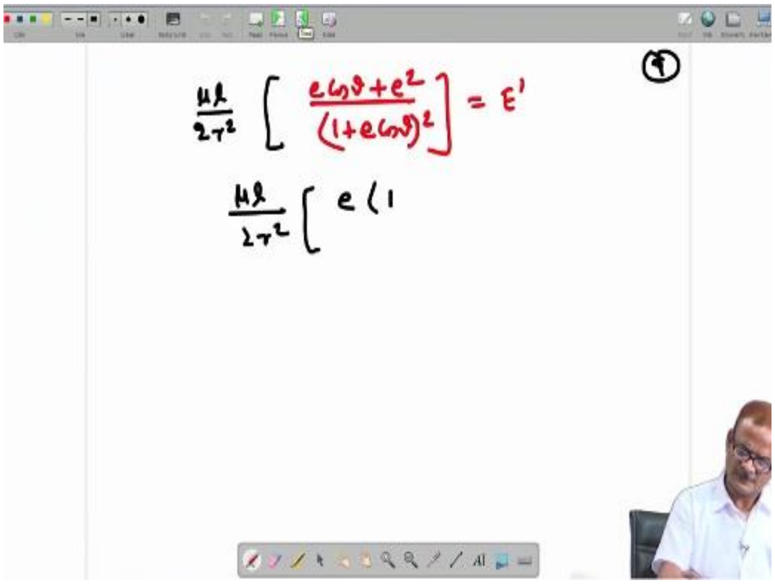

Now this one, this one, this one will cancel out and here this part, this part we have to subtract so this will be e cos θ + e^2, e cosθ + e^2 divided by 1 + e cos θ whole square. Okay. This equal to E'. So μ l/2 r^2, we take it to outside common so this is e cosθ, somewhere; we have missed out one term here, this two we have taken it outside, okay. So there will be a term here which will come as 2, so that gets canceled out and so this is 2r.

Here also this 2 should appear. Okay, because here this is only μ/r we forward this factor, so this 2, so this becomes 2 here and this also gets into the format 2e, okay. So we need to correct it. This is 1. We will write freshly on the next page. So here the cancelation take place, this 1 and this 1 then cancels out. This is 2 times 1+e cosθ and e cos θ is, so 2 and 2e cos θ gets cancel out and here we get e^2 – 1. So e^2 – 1 so in the next page we have to rub it out.

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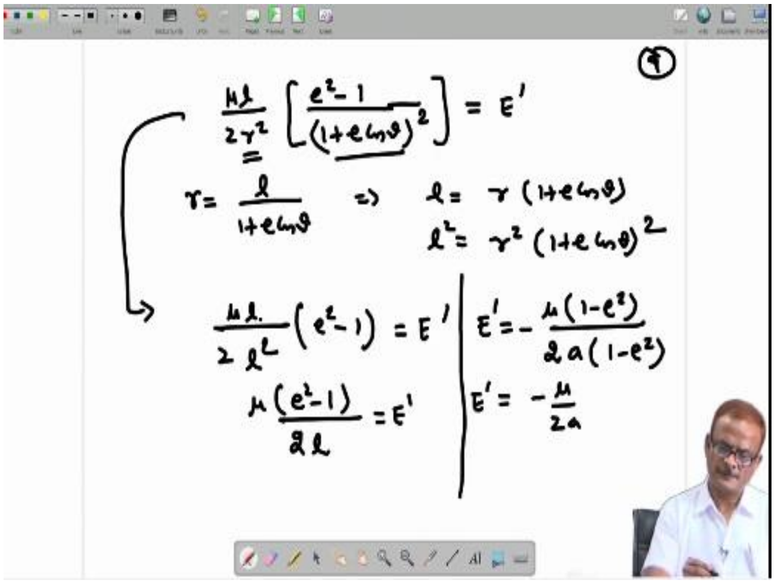

So from the previous page this is μ, μ l/2r^2 and in the bracket this is

$$\frac{\mu\, l}{2r^2}\ \frac{e^2-1}{(1+ecos\theta)^2} = E'$$

Okay. And whatever the simplification we can do at this stage we will do. So already r we have written as by 1+e cos θ so this implies $l\ =\ r\ (\ 1 + e\ cos\ \theta)$ and $l^2 = r^2\ (1+e\ cos\theta)^2$.

So if we look here in this one, so this gets reduced to 2, this factor and this factor together that makes it $l^2e^2 - 1$ = E'. So this gets reduced to $e^2 - 1$ μl, l cancels out, this is $2l\ =\ E'$. So we are close to our result. E prime, now we will pull out this, we will write this as e^2 and this is 2l= a(1-e^2), the result we have already derived. So we use this, so this gets reduced to μ/2a, so E' = - μ/2a.

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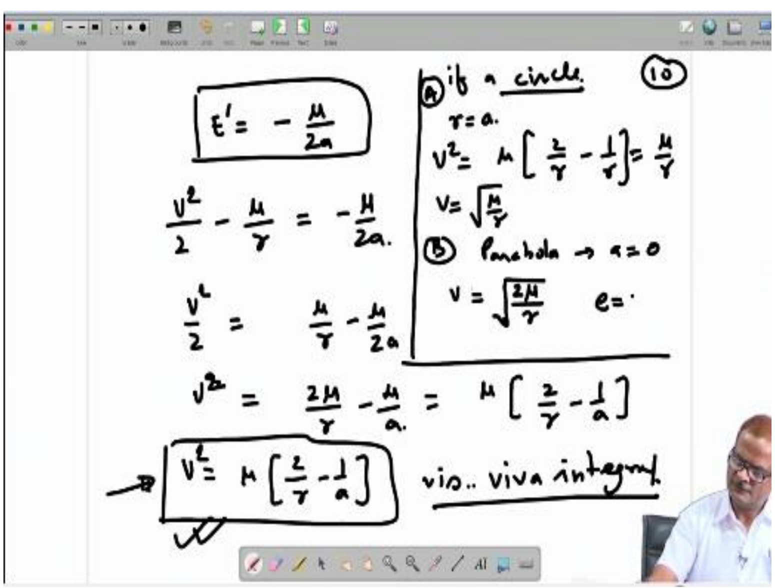

So thus we have

$$E' = -\mu/2a$$

So whatever equation is now,

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

and if we reorganize it, so $v^2/2$ this will be μ r – μ/2a, we take out μ/2 as common so this becomes 2r – 2/r and here this factor, okay we write in a fresh step here. This is v^2, this equal to 2 μ/r – μ/a and here if we take out μ outside, so this is 2/r – 1/a so this will be a square = μ (2/r – 1/a)

This name is given as this. And this result we have utilized earlier. And how we have reached to this place? We have started from this place, here v^2. So while solving the problem earlier this was related to this issue where we had to determine r dot square, we utilized the result here directly at some state. So without that, without this it should not possible to work out problem. Okay so this completes this particular part. Next; and this is very important.

Let me discuss about this particular expression here. Say, if a circle, if the orbit is a circle so then I think in that case r = a, this is case number a, so

$$v^2 \; \mu \left(\frac{2}{r} - \frac{1}{r}\right) = \frac{\mu}{r}$$

And

$$v = \sqrt{\frac{\mu}{r}}$$

. B, a case of parabola. For this case a = 0 and therefore v^2 this becomes 2 μ/r or v = 2 $\sqrt{(\mu/r)}$.

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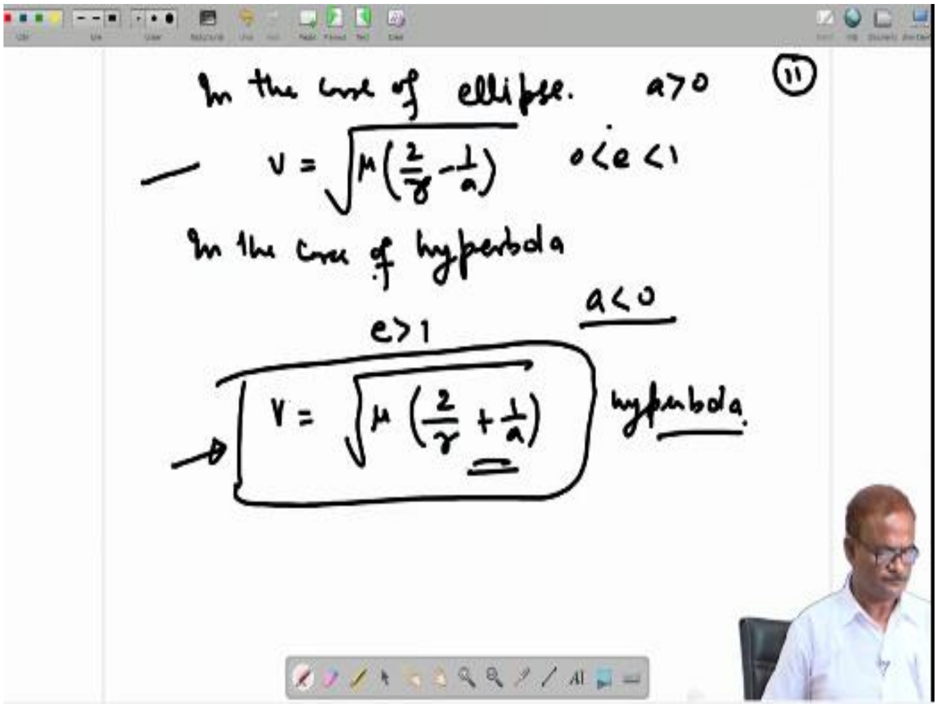

In the case of ellipse, so this expression $v^2 = \mu$ (2/r – 1/a) under root it remains as it is where eccentricity here in this case lies between 0 and 1. In the case of parabola its eccentricity is 1, here in this its case eccentricity is 0. In the case of hyperbola, where e > 1 so v^2 as given by 2μ times 2/r and a in that case is negative so that makes it 1/a. So this is for the case of hyperbola. And later on you will find this equation very useful in solving many problems.

$$v = \sqrt{\mu\left(\frac{2}{r} + \frac{1}{a}\right)}$$

This is v square again. These are the common errors which scripts in while writing, this is *v*, let me check on the previous page also, this is okay, here this is okay, this is okay. Okay so we have completed this part. In the case of parabola a = ∞, what we have written, see these are the mistakes which taking place after getting tired a = ∞ and therefore if you put here in this part a = ∞ this part vanishes and you get to μ/r.

If a = 0 that becomes infinity, it is a absurd. Okay. So a = ∞ in this case in the circle r=a, in the case of hyperbola a is negative; in the case of parabola; this ellipse is positive. Here a is less than 0 and therefore we write it with plus sign. Okay so we conclude this lecture here and continuing the next one. Thank you for listening.