**Satellite Attitude Dynamics and Control**

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**Lecture – 09**
**Rotation (Contd.)**

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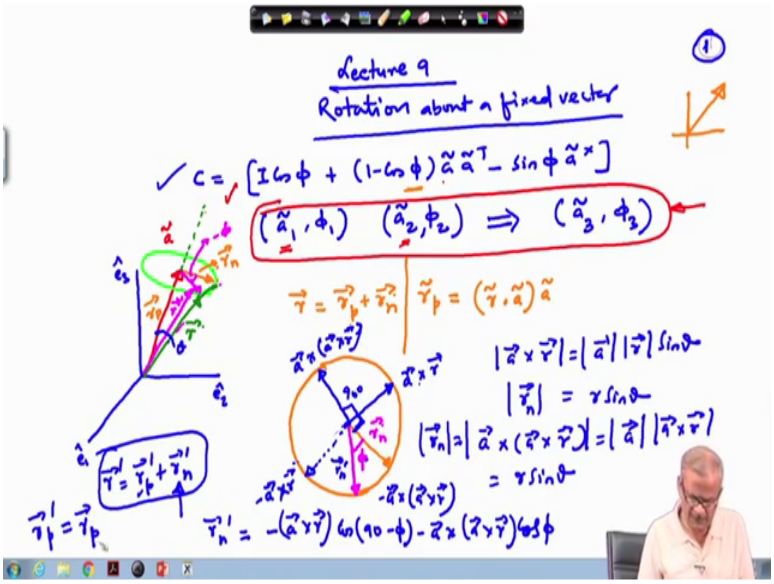


Welcome to the 9th lecture on Satellite Attitude Dynamics and Control. So, today we will discuss about the rotation about a fixed vector, already we have been discussing about this. So, as you can recalled at we have written the rotation matrix C as I cos phi plus 1 minus cos phi a tilde a tilde transpose minus sign phi a tilde cross ok.

And then, we saw that if we are looking for two consecutive rotation composite of two consecutive rotations, so we are the first rotation is given by a 1 tilde, phi 1, and the second rotation by a 2 tilde, phi 2 ok. So, composite of these two rotations, the equivalent of these two can be given by so this implies the whole rotation should be about one single lapses which is given by a tilde 3, where a tilde 3 is again by unit vector. All this a 1, a 2, these are the unit vector. And this is by rotation by phi 3 angle.

So, these two rotations phi 1 and phi 2 about a 1 and a 2 respectively are combined together. They combined together at equivalent to a tilde 3 rotation about this vector by angle phi 3. And this we have obtained some expression for this. So, we will get back to that. But, before this we I told you that you will be discussing about indeed that this rotation matrix is a rotation about the vector a tilde by phi that we have to check geometrically, so that we have not still done.

So, today we are going to do that say this is our e 1, e 2, e 3 cap, this is the basis vector. And this is the vector a tilde ok. So, a tilde is the unit vector remember. So, magnitude I am showing it in general like this, but magnitude may it has to be taken of the unit magnitude, but direction will be governed by whatever say depending on what kind of rotation, you have given; so either we go from C to a and phi or either from phi to C.

So, here our objective is to given two rotations phi 1 and phi 2 about a 1 and a 2. And what will be the equivalent of that? So, this we have already done ok, this part we have derived. And I will look back again into this. Today, I am going to verify this that this expression is indeed, and equation for rotation about a fixed vector a tilde.

So, let us say this vector a tilde is here ok. And rotation about this vector is given by so this is your r vector, which is being rotated about this a tilde vector. So, we consider that this is say long this direction a tilde vector may be of unit magnitude. So, obviously a tilde is not of this magnitude, but we are just showing it like this that a tilde is along this direction ok.

Now, we get this is my vector, so I can break it into two portions. One along the one along this direction and one along this direction means, one along the a vector and another perpendicular to the a vector. So, we can write here r equal to this will indicate by r normal, and this part we will indicate by r parallel. So, r will be equal to r parallel plus r normal.

Now, this matrix this is rotation matrix. So, this is basically it shows the rotation of the frame by phi angle about the Eigen axis. So, here in this case a tilde this is indicating the Eigen axis, a tilde is the unit vector. So, this happens to our Eigen vector Eigen axis about which this frame is being rotated.

Now, instead of considering this frame that this frame is being rotated, consider that just opposite of this, that this vector there is a vector r, which is being rotated from this place to this place by minus phi. So, if the frame is rotated, already we have seen that if we have one vector like this and we rotate the frame by theta, so that is equivalent to rotation of the vector in the opposite direction by minus theta keeping the frame fixed. So, the same principal is applied here. So, here after breaking these two, now we write it in a proper way.

So, let us make a circle here. So, this is the component where which is being shown as r n, so this is our r n vector ok. Now, the component of vector r along a direction, so r tilde dot a tilde. This is the component of the r vector along the a direction, where this is indicated by r p. So, this is your r p tilde and because it is along the a direction. So, we have to multiply it by vector a ok.

So, now r time if we rotate this say if we rotate it from here to here by minus phi this angle is minus phi, we are rotating in the clockwise direction. So, same rotates in the anti-clockwise direction about this vector by phi, so that is equivalent to rotation of this vector in the opposite direction by angle minus phi.

So, here we can indicate this. This is the minus phi angle. So, I will just write the magnitude here ok, it has rotated from this place to this place. So, this is your r n vector and this is the rotated r n vector. So, this is equivalent to that this r vector is there and this vector r, you rotate from this place to this place. So, it comes from here to here. Now, this is the vector r. So, this is you r prime so r prime is here. So, it has rotated from this place to this place by angle minus phi that is in the clockwise direction about the a tilde vector.

Now, few things we can observe that a vector is along this direction, r vector is along this direction ok. So, now a cross r in which direction it is going to lie, a is along this direction and r is along this directions. So, if you put this r n from this point, we will see that it will perpendicular to a cross r will perpendicular to both a and r. So, r is in this plane. Here r is forming by this plane. This particular line this green line and this is the orange line, so it is a perpendicular to this. So, means it will perpendicular to this vector r n, and we will show this, this is your a cross r a tilde or either a cross r, if there a is unit vector.

Now, why it is on this circle, there is a reason for this. If you look into the magnitude of this a cross r vector, so this will be a magnitude times r magnitude and sin theta and this quantity is one, so therefore this is r sin theta. And what is your r sin theta if you look into this, so this angle from here to here this angle is your theta ok, from green line to the orange line this angle is theta. So, r sin theta is nothing but your projection of this r vector along this direction that is the magnitude of r p vector this, the magnitude of the r p vector sorry this the r n 1. So, this is projection here along this direction this is r n. So, this angle is theta, and therefore this is here this quantity.

So, a cross r it is magnitude remains equal to the magnitude of r n, therefore it lies on the circle. Now, again if we take a cross a cross r means, we write it like this. So, you will see that a vector is along the normal direction, it is a coming out like this it is a coming out here out of the page ok. And a vector is along this direction; so a cross a cross r taking the right hand role ok, so it goes here. Now, this is perpendicular 90 degree, this angle is 90 degree, we are so 90 degree angle, we will show like this.

So, therefore perpendicular to this will be again here in this place. This vector will indicate by a cross a cross r. Again the magnitude of this vector a cross a cross r, we will see that this is a cross times a cross r magnitude. So, this quantity is one. And already we have see that this quantity is r sin theta. So, this gets reduced to r sin theta means, this is also equal to the magnitude of r n so this is also the magnitude of r n. So, it lies here in this direction.

So, opposite of this vector here in along this direction, this is minus a cross r. And similarly, opposite of this from here to here, this will lie along this direction; and that we will write as minus with minus sin, we have to write minus a cross a cross r. So, this is 90 degree this is 90 degree just opposite of this. So, it will lie somewhere along this line.

Now, this is the projection of the vector. So, if this is r prime, and so this is r prime normal this is r prime normal means, r prime again can be broken along two directions; one along the direction and one perpendicular to this, which will we are showing here as a circle. So, let us say this vector being so near. So, r prime this we can write as r prime parallel plus r prime normal. So, here this is nothing but your r prime normal.

Now, this vector it can be so this vector of a minus a cross r, we have already shown here. And this is our r prime n. So, now the new vector the new vector consists of this one. If we can define this r prime n, so our job will be done. So, for defining r prime n, we see that this is component of this vector along this direction, and then component of this vector along this direction.

So, we can write here r n prime this equal to minus a cross r along this direction, which is cos 90 minus phi and component of the vector along this direction. And we have to take the component here. This vector in the opposite direction and component along this direction, so this is minus a cross a cross r times cross phi ok. So, this is your r prime n, it is a normal component. And if we combined with this, this r p prime ok, which is the parallel along this direction. So, I will ultimately get the final result. So, your r p prime r p prime this is nothing but your r p, so we just need to add this.

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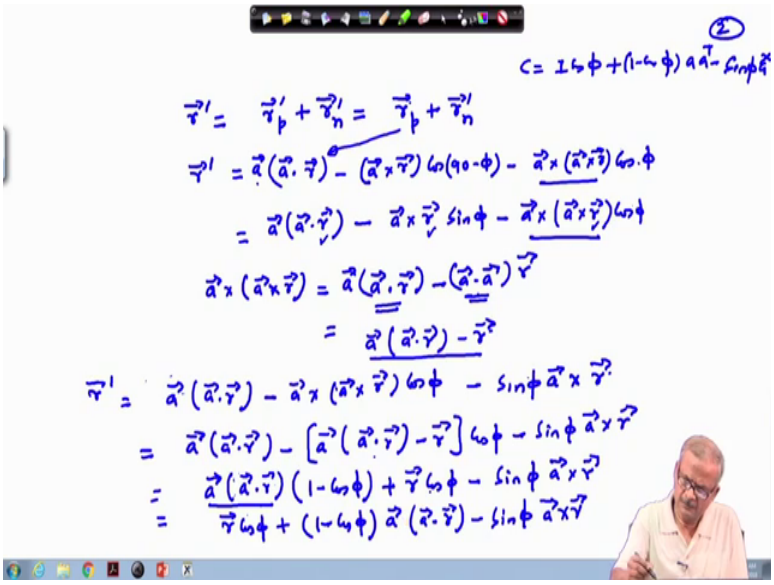

So, final audition: however vectors r prime will look. This is r prime p plus r prime n, and this is equal to r p plus r prime n. And r p is the component of vector r along the a direction, so this already we have written either you write in vector notation or matrix notation it does not matter. So, here we are using the vector notation. So, this part and this part already we have written, so we just copy that. So, this is a cross r cos 90 minus phi and minus a cross a cross r sin phi, so this is our rotated vector.

Now, we have to get this into the matrix format to see that everything is correct ok. So, one more a step, we will simply this as this is sin phi. So, this is a cross r sin phi, and this we need to work out here sorry, this part is you cos phi, we have written cos phi by mistake here this is cos phi. So, this is our cos phi. Now, we need to rearrange it to get the solution, what we are looking for ok.

So, first of all we will explore this is what this quantity is a cross a cross r ok. We need to expand, and look into this exactly what it is indicating. So, if we expanded, so this turns out to be a dot r a dot r a minus a dot a r. Now, a dot r this quantity what this is we have to look into a dot a, obviously we know this quantity is equal to one, because the vector a is having magnitude one. So, writing this here.

Now, a dot r this is the angle between this is the projection. So, here what is the angle between these two vectors between the a and the r vector that we have to decide. So, what we do that we write in a way where ultimately, it comes in this format what we have been looking for C equal to I cos phi plus 1 minus cos phi a times a transpose minus sin phi times across, this is what we are looking for.

So, we will try to reduce this vector into this vector into this form. So, ultimately this what is happening that there is a matrix, which is operating on this vector r ok. And that rotates and gives it a changes it to the vector r prime. So, ultimately our objective is to reduce it to this format. So, what this quantity is just one, so this is a times a dot r minus r.

So, r prime is our you will write this as this part we are writing just taking sin ahead of this, so this is sin a cross and r; so whatever strategy will be to take out this r out of the bracket ok. So, if we are able to do this, my problem will be solved. So, copying from this place a times a dot r minus this quantity ok, it will come here in this place. So, a dot r and plus this minus r minus r times this cos phi cos phi.

Now, what we see that here something is common between these two terms ok. So, we write this as a times a we take and this is a scalar, this is the dot product. So, this is a scalar quantity, so that becomes 1 minus a times a dot r times cos phi so this is we take it outside the bracket. This is a dot r times 1 minus cos phi, and here we are just left with plus r cos phi minus sin phi a cross r. This quantity as we will show that if we write this as we will do little arrangement will put this from here in this place. So, this is r cross phi plus this quantity, thereafter 1 minus cos phi a times a dot r, and there after this quantity sin phi a cross r.

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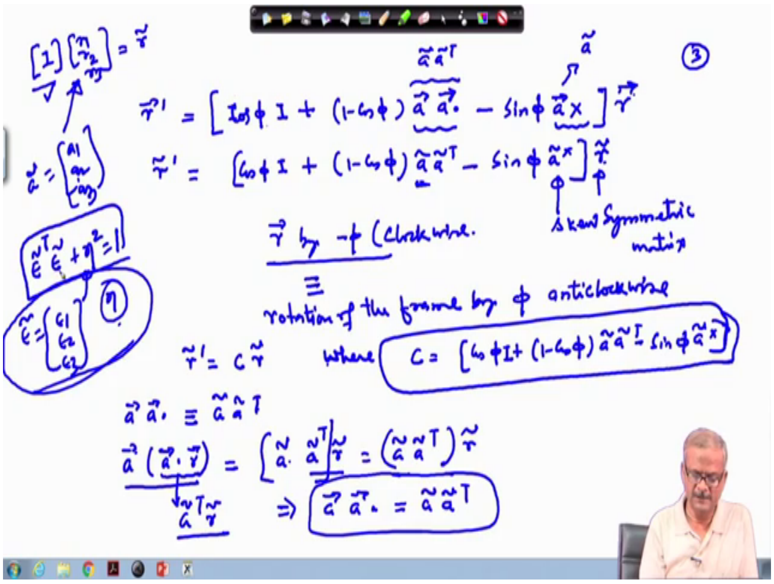


So, our r prime that becomes r cos phi r cos phi, then 1 minus cos phi, so what we are going to do now that write it in a converted to a matrix format ok. So, for doing this so what our strategy will be just take it r outside, let us write the r out of the bracket. So, we write r here in this place ok.

Now, look here in this place this is r cos phi, this is a vector ok; and if I take it outside the bracket as here in this place, so I must put here cos phi cos phi times identity matrix. So, this is a vector which operated once it operates on I ok, here this is a I matrix. And you have here r 1, r 2, r 3, so when any vector is multiplied by the identity matrix is remains as the it remains intact. And therefore either we write it here in this format or either in the matrix notation that is like in this way, it does not matter. So, we have taken it outside.

The next term once we go for this terms, so this is 1 minus cos phi and a dot r; so here 1 minus cos phi a times a dot r. So, here we will write this as dot, this is here dot written. And this quantity as we will show, this is equal to a tilde a tilde transpose. r we have taken outside, so this is operating on this r. And thereafter, this particular term a cross r, so r is taken outside. So, this remains minus sin phi a cross. So, we next get minus sin phi a cross so this is a cross.

And finally, obviously we will instead of writing this a, and this arrow rather than writing we write as a tilde, which is indicating the vector here in this format ok. So, we have this cos phi I plus 1 minus cos phi, this quantity we will show as it is equal to a tilde a tilde transpose minus sin phi, and this is nothing but your a tilde cross. This is a skew symmetric matrix. And this is operating on here r, so we can indicate this as the r tilde, and this is r tilde prime.

So, this vector it operates on this vector and it converts these two r prime means what this is indicating, this is indicating the rotation of vector r by minus phi rotation of vector r by minus phi ok. And this is equivalent to rotation of the frame rotation of the frame by phi, which is anti-clockwise. And this is clockwise.

So, this indicates that r tilde prime, this can be written as C times r tilde and where C equal to cos phi plus 1 minus cos phi times I 1 minus cos phi a tilde a tilde transpose minus sin phi a tilde. So, this is what we have been looking for only thing remain remaining as that we need to look into this. a times a dot this is equivalent to this is an identity basically, so this is a tilde times a tilde trans.

So, let us look into this here this way. If I put it this has the r, so this is nothing but a tilde, then this is a scalar ok; so we can write this as a tilde transpose times r tilde ok. Now, if we combine these together. So this becomes a tilde times a tilde transpose, and this is so ultimately this is a vector ok. And here this itself is a scalar ok, but if we write it in a way, where this part this part we have broken has a tilde transpose times r tilde, which is the inner product that is a dot product.

And then separated out this ok, so once we separate out it gets into this format. So, this implies that a times a dot this equal to a tilde times a tilde transpose. So, this is what is has been used in the figure shown. So, this proves that indeed the C matrix is a rotation matrix, which rotates in any vector r about the fixed vector a, which is in this case the Eigen axis.

Now, from here we go into the Euler's theorem, which we have been working with what we have this is basically part of Euler's this is Euler's theorem itself, but what we are going to do? We are going into the Euler's parameter now.

So, we will convert this, so that this indicates the Euler parameter rather than the a vector, which is a vector this is three dimensional one a 1, a 2, a 3 while the we write in the Euler parameter terms. So, Euler parameter it is a four variable system in which as I have told you that if I indicate this Euler parameter by E tilde, so E tilde transpose E tilde plus eta squared this will be 1. So, these are the Euler parameters, epsilon tilde, this equal to epsilon 1, epsilon 2, epsilon 3. These are three variables involved, and this is or the parameter and this is one parameter.

So, we are going to look into this. So, this is showing that this is a four parameter system, but actually only three independent coordinates ok, because it is a dependence shown by this equation, where epsilon is related to eta is related to this quantity. So, eta can be expressed in terms of epsilon 1, epsilon 2, epsilon 3, but it has got it is certain advantage which we are going to discuss.

Thank you for listening. We will continue in the next lecture of; thank you again.