Satellite Attitude Dynamics and Control
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Lecture – 06
Kinematics of Rotation (Contd.)

Welcome to the 6 lecture today we will discuss about the Euler's theorem and what does it state? So, we will write it instead of a speaking it here.

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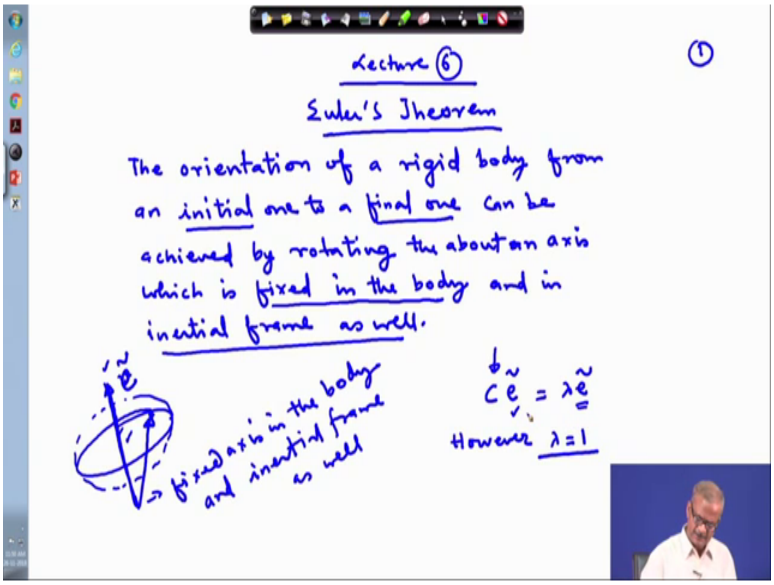

Now, this theorem is very important because you can see that the if we are representing the rotation in terms of the psi theta phi that is a in terms of the Euler angles.

So, that becomes an affair involved with the sin cosine all this trigonometric quantities. But, if we want to eliminate this trigonometric quantities so, for that these we have to look into the Euler parameters and equivalently it is also called as the quaternions. But the Euler parameters and quaternions they are restrictly not the same because the quaternions are developed from the complex notation while the Euler parameters it is a purely it is a developed from the matrix notation.

So, these 2 are totally different things, but they are equivalent and quite often in literature they are used without any making any distinction. So, we continue with the Euler theorem. So, already we have looked into the rotation part. So, how the rotation is represented now Euler theorem it states that let us first look into 1 example.

So, can we have camera here. So, if see if we have earlier looked into that if this is a box and this I want to represent it like this finally, rotate in this way. So, this we have done in 2 ways like giving 1 rotation like this and then 1 rotation like this or either from here to here and then like this ok. Now we can have very complex rotation so, we have given some 3 rotations about the different axis and the finally, from 1 configuration let us say we have come to this consideration.

So, Euler theorem what does it state that there is an axis fixed in the body itself. So, this is a body and it is a fixed in the body and also in the inertial space. So, if I say this is a vector in my finger it is a vector in the inertial space and this is passing through this body and this is also fixed in this body and if I rotate this body about this axis this is fixed in the body remember.

So, this will my finger will pass through and it will be represented like this will go across this from inside ok. So, this axis is fixed both in the inertial space here also and in this body itself. So, in the body the axis it is not changing and if I rotate about this let us say this axis is fixed like this and I if I rotate it like this. So, I can go to different orientation. So, this 3 rotations that we have given it can be represented by a single rotation which we call as the Eigen axis rotation and this is all about the, this is what the Euler theorem it states.

So, let us write that theorem, the orientation of a rigid body from an initial one to a final one means from initial one to a final one we are going by giving 3 rotation about the 3 different axis ok. So, this can be achieved; can be achieved by rotating body about an axis which is fixed in the body and in inertial frame as well.

So, it is a fixed in the inertial frame are also fixed in the body and in the inertial frame as well. So, let us say that which axis is here represented by some nu tilde or maybe if we can represent as e tilde. So, this is the fixed axis, fixed axis in the body and inertial frame as well ok. So, if I give rotation and this is my body ok. So, if I rotate this body let us say we make it 3 dimensional or something like this.

So, if we rotate this body and say if there is a vector going from this place to this place. So, this vector is e vector this is fixed in the body and also in the inertial space and this is a vector which is pointing to any particle in this body ok. So, as we rotate about this axis we can see that this axis does not get effected this will not suited. So, this simply implies that if c is the matrix rotation which is indicating rotation about this eigenaxis then c times e tilde must be equal to lambda times e tilde means let us say only showing the that the e tilde vector which is not rotating it can only get magnified by certain magnitude but it is a direction is not changed.

But here for c 2 be any rotation matrix already we have looked into that it should satisfy certain properties and we will look into that also here in this place. So however, lambda need to be 1 ok; however, lambda equal to 1. So, if this property is satisfied then our job is done.

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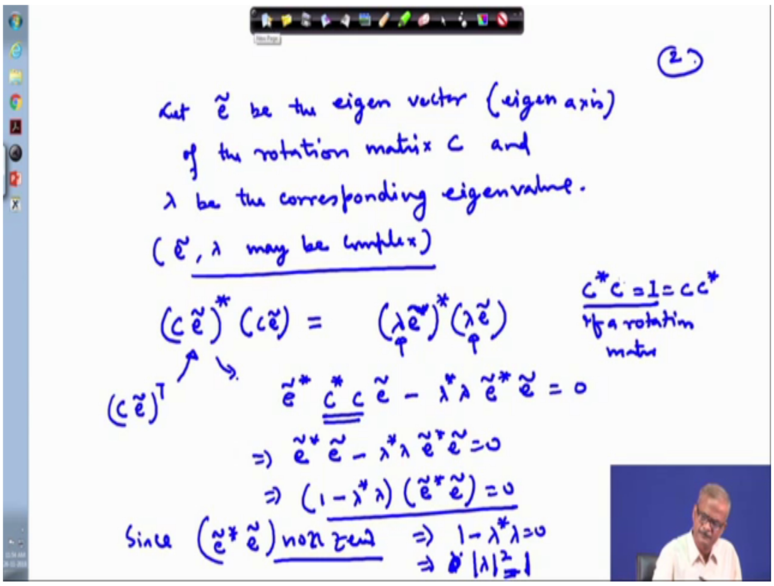

So, let us to state it formally what we call this as the eigenaxis rotation matrix c and lambda with the corresponding Eigen value. So, here e tilde and lambda maybe complex also. So, if it is so, then transpose we have to replace by Hermitian operator or up we will replace this let us write it this way c e tilde ce tilde c operates on the e tilde and if e tilde happens to be e tilde and lambda they have they happen to be the complex one. So, in that case we will replace it by the Hermitian conjugate; that means, instead of writing ce tilde transpose we are writing it this way. This must be equal to now, this is operating on this e vector and therefore, this is lambda e from the previous page what we have written lambda e tilde.

So, this can be rearranged as from this place this will be this is a scalar, it may be complex or it can be non complex. So, lambda a star lambda times e tilde a star e tilde ok. So, if this is a rotation matrix. So, we must have this is equal to I if a rotation matrix and this is c.

Since e tilde star this is non zero. So, this quantity is non zero this vector this is the Eigen axis we are assuming and therefore, this will be nonzero and this implies that 1 minus lambda star lambda will be equal to 0.

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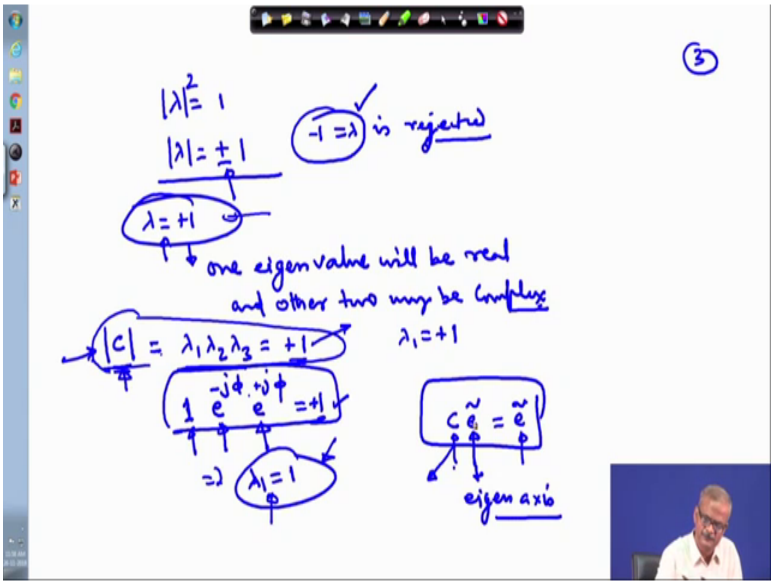

And this implies lambda lambda magnitude a square this will be equal to 1 or lambda equal to lambda square magnitude we have written is equal to 1 and therefore, lambda magnitude is equal to plus minus 1 ok.

But as we have discussed earlier that the minus 1 this Eigenvector. So, Eigen value that we are getting this will correspond to the left handed triad so, we reject it. So, the minus part now lambda lambda equal to minus 1 this we are rejecting. So, with this lambda equal to plus 1 then we proceed now as we have stated that lambda for a rotation matrix it is a fact that at least 1 value will be real 1 Eigen value will be real and other 2 maybe complex.

If it so happens so we can use the property of or determinant of a matrix c equal to c determinant this will be equal to lambda 1 lambda 2 times lambda 3 and already we have looked into that 1 of the Eigen value is 1. So, let us say so, this quantity is plus 1 and 1 of because this is a rotation matrix. So, this must have already we have looked into that this quantity should be plus 1.

Thereafter, also we have lambda equal to plus 1 1 1 of the Eigen value. So, let us say lambda 1 is plus 1. So, we will have here 1 the other 2 are complex so, complex conjugate. So, this will be e times j phi and e times there will be complex. So, we will represent this as e times j phi and e times plus j phi and this will be equal to 1 ok. So, this satisfies; obviously, we can say that this is satisfied lambda 1 this is the property of the determinant of a matrix and if we take the other 2 as the complex conjugate and 1 as the real 1 one Eigen value so, we get this 1.

So, this is satisfied so, this implies that lambda 1 should be equal to 1 as we have rejected here this in the minus part. So, this should be equal to plus 1 this quantity is plus 1 because the determinant of this will be representing as plus 1 for a rotation matrix. So, the Eigen value also this must be plus 1, has already we have stated that we will reject it because this will be not related to the right hand triad we are discussing about.

Thus what we see that c times e tilde this is equal to e tilde means the vector e on which the matrix operates it does not change that vector it remains intact means this becomes the Eigen axis which is fixed in the body and also in the inertial space and this is the rotation matrix corresponding to this Eigen axis.

Now, the question arises how to go about showing this c matrix, how the rotation matrix will be represented? So, that is another issue which will come to sometimes afterwards.

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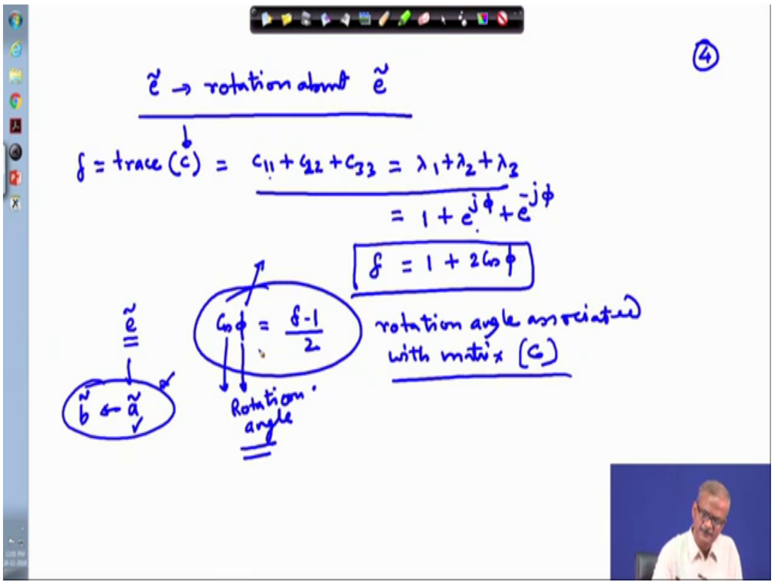

Now, this rotation about the e tilde, rotation about e tilde will be given by using the properties of a matrix. So, we know that the trace of c means the sum of the diagonal elements means c 11 plus c 22 plus c 33.

This will be equal to lambda 1 plus lambda 2 plus lambda 3 this is the property of the trace of a matrix where here in this case c is the rotation matrix and c 1 1 c 2 2 c 3 3 are the diagonal elements lambda 1 lambda 2 lambda 3 they are the Eigen values. So, therefore, lambda 1 this equal to 1 this e to the power j theta and plus e to the power minus phi and minus phi here phi is the rotation angle ok.

So, this can be written as 1 plus we know that this quantity will be equal to cos phi. So, trace c if we let us say indicate this by delta. So, this becomes delta equal to and then the cos phi equal to delta minus 1 divided by 2, this gives you the rotation angle associated with matrix c. So, this is the angle by which the rotation matrix c will rotate about the Eigen axis. So, you are given 3 general angular displacement.

So, that will correspond to 1 single displacement of magnitude phi about the Eigen axis e tilde. So, instead of using e tilde later on we will continue with using a tilde because it will be much more convenient to represent a tilde. So, if we are converting into other frame. So, of after the rotation of the frame it can be represented as b tilde and it is of much simpler to work with. So, this is our phi is the rotation angle, this is the rotation angle. And later on we will look into that indeed this phi happens to whether rotation angle.

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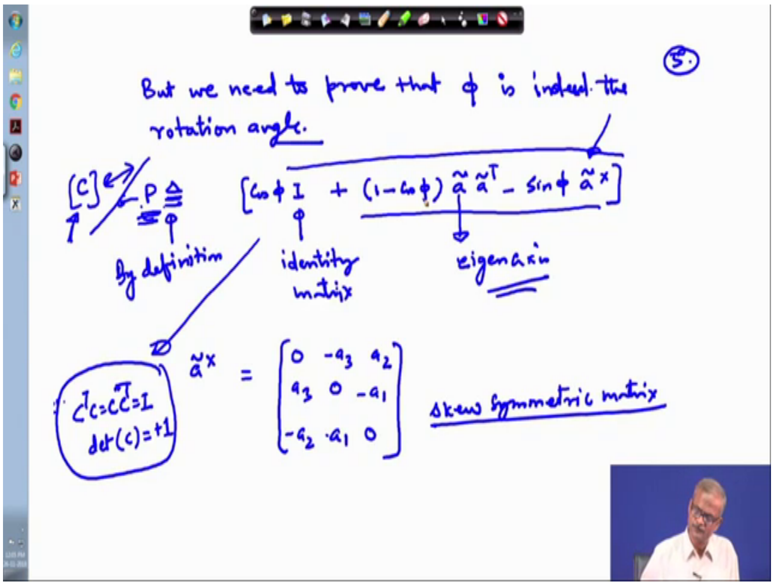

But, we need to prove that phi is indeed the rotation angle. So, without going into the details let us considered at we have a matrix P, this means by definition which we are defining as cos phi times I where I is the identity matrix 1 minus cos phi times a tilde times a tilde transpose minus sin phi times a tilde cross ok.

As earlier defined a tilde cross this represents this is the skew symmetric matrix and this is your minus a 3 a 2 a 1 a 1 on this side this will be minus a 2 and here this is a 3. So, this is your a skew symmetric matrix. So, here right now we have taken it for granted that this is the combination which is representing matrix P and we need to prove that this P is happens to be a rotation matrix which rotates about the eigenaxis a tilde, this is the Eigen axis.

So, other way if you can look into that if P is given if your rotation matrix is given. So, how you will find phi which is the rotation angle about the eigenaxis and the eigenaxis itself another way of looking into the same problem that given this a tilde which is the Eigen axis and rotation about that Eigen axis by phi. So, get the value of get this matrix P. So, P is nothing, but your c matrix that we have been representing. So, we will come to this right now we will use this p notation because first we need to prove that P is indeed the rotation matrix.

So, already we have looked into the rotation matrix properties. So, therefore, if this is representing rotation about the Eigen axis by angle phi so therefore, this p matrix most satisfy the properties of the c matrix, where c transpose c time c transpose this is equal to I and also the determinant of c this is equal to plus 1. So, this must be satisfied now if this matrix is satisfying this properties then this will indeed the rotation matrix, but we need to work it out. And obviously, whatever is represented here this whole thing it can be shown by a very simple geometric construction that this is indeed a rotation about a tilde by angle phi which will come of course, during course of time.

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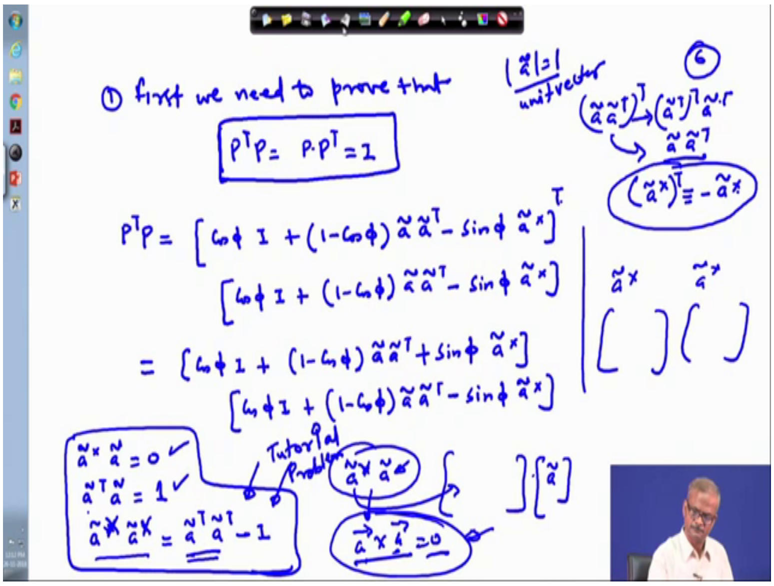

So, first we need to prove that P transpose P equal to P P transpose equal to phi this is the first requirement that we have to do. So, P already we have assumed of particular form of P so P transpose P.

This we can write as cos phi this whole transpose and this is to be multiplied by. So, if as we proceed so, mathematics involved is very complex and very lengthy and therefore, we cannot afford to do it in this lecture class and we will do the same as problems in the tutorial class. So, those some of the things will be said to you as problems and you need to work out if you are not able to do it then the solution we can look into the solution.

So, if taking the transpose of the 1st one that gets reduce to cos phi I I transpose equal to I 1 minus cos phi a tilde a tilde transpose if we take transpose of this and this will give you a tilde transpose whole transpose times a tilde transpose. So, this becomes a tilde times a tilde transpose. So, it yields the same thing minus sin phi this is a scalar. So, there is a nothing to worry about and a tilde cross transpose this will be equal to or we can write this as a tilde cross with minus sign that you can check because it is a skew symmetric matrix. So, if you take the transpose as seen on the previous page.

If you take the transpose of this so, minus sin will appear here the cos sin will appear. So, the and then if you take the minus sin ultimately we will get the a cross itself. So, from there you can just look into this is very simple and therefore, this is a cross transpose. So, we need to put here plus a tilde cross and then this is to be multiplied by cos phi I cos phi minus sin phi times a tilde cross. And if you multiply it and expand this and few properties you need to use so, first we have to expand it and then multiply it.

And the properties needs to be used is a tilde cross times a tilde this will be equal to 0 and then a tilde transpose a tilde this equal to 1. Because, we are assuming this a tilde magnitude to be equal to 1 which is say unit vector and a tilde cross times a tilde cross this will be equal to a tilde transpose times a tilde transpose minus I.

So, here the 1st one is very simple a tilde cross times a tilde. So, this simply implies that you have the skew symmetric matrix this one here and this you are operating by on by a tilde and other way if you look into the vector notation. So, this is nothing, but a cross a and a cross a is; obviously, 0. So, either you do it this way or either do it this way because this is giving you this is the skew symmetric matrix operated on why this a vector that gives you a vector and also this gives you a vector, but this vector of course, this turns out to be 0.

This is very obvious this is a inner product and because this magnitude is 1. So, therefore, we get here 1 and this part if you write this two is skew symmetric matrix like you have the a tilde cross write the skew symmetric matrix for this and a tilde cross write the skew symmetric matrix take the product. And then if you simplify it and take the turns outside break it into two parts so you will get this so, this I will leave as a tutorial problem to do so, this is your tutorial problem ok. Some practice is required for it is a good to do this practice ok.

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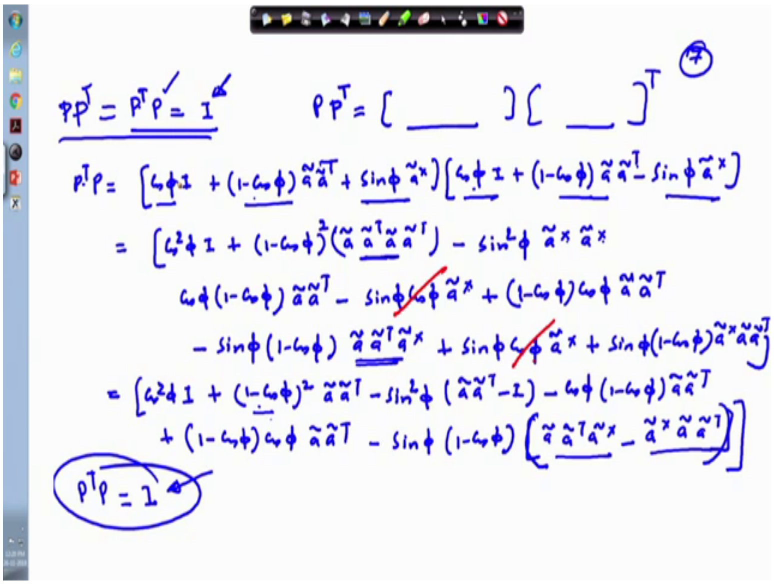

So, we have written earlier P transpose P or pp transpose whatever way you write. So, P transpose P you can expand it and once you expand it you will get this planned cancel out the terms you will get I or either way whatever a way you work or either you are start with P P times here P transpose and the same thing you have to write the matrix here this matrix and this matrix whether transpose expanded and there after workout.

So, if what we get there this as the quantity as I. So, for you I will try to work out at least one of this, I will have to write the whole matrix here only then we can proceed. So, this is P transpose P is equal to cos phi I a tilde times a tilde transpose minus. So, this becomes a tilde cross cos phi I. So, one by one take the product here. So, if we take the product the first term is cos square phi I ok.

Similarly, the other terms we can multiply this with this. So, 1 minus cos phi times a tilde transpose a tilde times a tilde transpose ok. And here this gives us minus sin square phi a tilde cross times a tilde cross ok. So, these are the corresponding three terms we have multiplied then we need to do the cross term also. So, that will be cos phi times 1 minus cos phi times a tilde a tilde transpose this term and this term.

Similarly, this term needs to be multiplied by this one. So, this is sin phi times cos phi times a tilde cross thereafter this term needs to be multiplied by this. So, that becomes 1 minus cos phi times cos phi a tilde times a tilde transpose and this one this one multiplied together this we have already multiplied the last one this remains. So, this comes with minus sin phi times 1 minus cos phi a tilde times a tilde transpose times a tilde cross.

We have to take care that you are putting it in proper order; order should not be destroyed ok. So, if lastly we have to work with this one. So, this is sin phi times cos phi a tilde cross multiplying this together this together that gives us plus sin phi times 1 minus cos phi a tilde cross times a tilde times a tilde transpose.

And the third term we have already multiplied which is written here in this place. Now, the properties we have stated on the previous page we can use them and shorten this all this terms.

Now, this is cos square phi I this term is 1 a tilde transpose a tilde this is 1. So, that we can write as here we are missing this square term this and this multiplied together this is a square. So, 1 minus cos phi a square a tilde times a tilde transpose, this already we have stated that this quantity is a tilde a tilde transpose minus I. So, we replace it by this particular 1 and then we are left with this term. So, cos phi times 1 minus cos phi transpose a tilde cross ok. So, here we erase it because this term, this is sin phi cos phi a tilde cross.

So, this term is with plus sin and this term is with minus sin. So, they will cancel out. So, we simply eliminate that term there are few more terms which can be eliminated. So, we have noted till this extend cos phi 1 minus cos phi a tilde this term goes then plus 1 minus cos phi times cos phi and then we have this term here minus sin phi 1 minus cos phi we have sin phi this is plus phi 1 minus cos phi what we have to check whether these are cancelling or not.

So, if finally, we have minus sin phi times 1 minus cos phi, if we take this as common. So, we will have here a tilde times a tilde transpose a tilde cross minus a tilde cross a tilde times a tilde transpose. So, if we work it out and we can show that this gets reduce twice for our job is done. Now, what we need to do that we need to expand this term so; obviously, this is a teenagers job and it is left to you at this a stage that you prove that this quantity P transpose P this is equal to I.

Where you need to cancel out some of the terms, by expanding this by writing it in a proper format and the basic mathematics involved already I have a stated you that use those properties to work it out ok. So, we stop it here and we will continue in the next lecture and if you can work it out yourself that will be a very big credit for you because you need to put it in a proper format and this most turn out to be equal to I.

So, this is one of the property that we are showing that indeed this P matrix P transpose P and equal to this will be P times P transpose this will be equal to I if we prove this. So, one part is proved the other part will remain. So, we continue in the next lecture for the time being.

Thank you very much.