Satellite Attitude Dynamics and Control
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Lecture - 53
Gyrostat (Contd.)

Welcome to the lecture number 53, we have been discussing about the gyrostat and we derive the equation of motion and that was in more of a generalise format. So, today we will take up that generalised rotational dynamics equation and then linearize it, to get the gyrostat linearize equation of motion and that is the equation which is used for the control purpose locally. Once the system is distort from its reference condition or the equilibrium conditions.

So, from there whatever the small disturbance is there so that can be controlled using your linear control system ok. And also as we know f as a hold at this whole system is non-linear and if we want to control the non-linear system so in that case we have to design the non-linear control. And then we can there is no bound on the disturbance, disturbance can be as large as possible and if your control non-linear control is capable of tackling the problem, then the system can be brought to the reference condition again. So, we start with our derived equation of motion.

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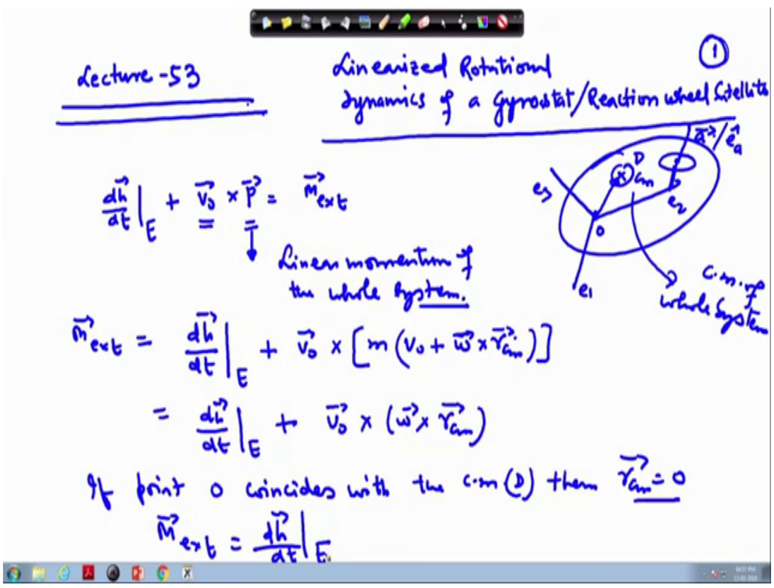

So, last time we have worked out. So, we had here the body in which this is the point o e 1 e 2 and e 3 that is gyrostat whose axis is here and a or a unit vector is there either written here in this or this format.

So, here in this case v 0 is the velocity of this point and p is the linear momentum of the whole system ok. So, for this particular case plus v 0 as we have observed that, this linear momentum of the system if the centre of mass of the system is located here, this is the centre of mass of the whole system for the whole system.

So, in that case this velocity of the centre of mass will be v 0 plus omega cross r c m ok. I have be multiplied by m so that becomes the linear momentum of the system and as a joule this gets reduced to this we have discussed in the last class. So, v 0 cross omega cross r c m, and if point o coincides with the centre of mass let us say this point is we write this as some point D; centre of mass at point D then r c m will become equal to 0 and therefore, m external gets reduced to your usual form of the rotational dynamics.

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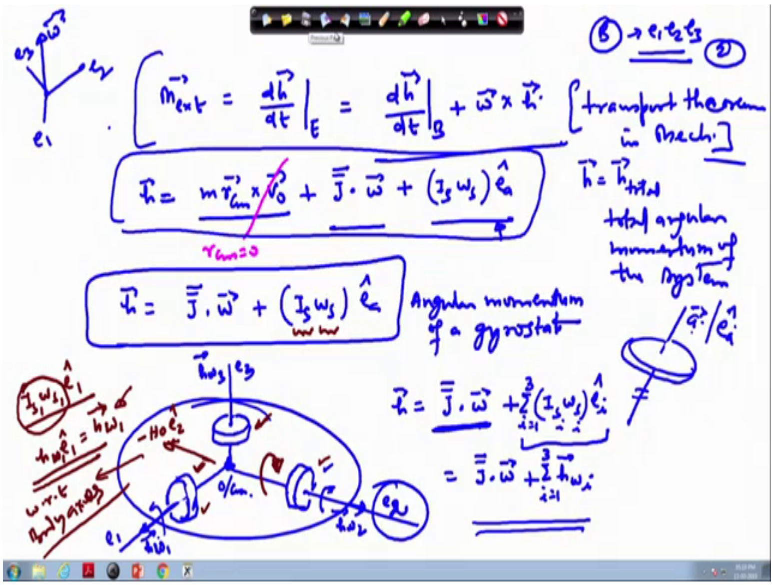

This we can write with respect to the body frame means the here the B frame is e 1 e 2 e 3 which is fixed at point o. So, in the case once the o coincide with the centre of mass so this is the situation. So, this is your so the transport theorem in mechanics transportation theorem or transport theorem in mechanics. We have derived this earlier also ok. So, here in this place h is the total; h this is the h total ok. And that is a total angular momentum of the system.

So, h 0 we have written as m times r centre of mass cross v 0 so; obviously, this term will not be present because r c m equal to 0 here. So, this term will drop out and rest we have j double dot omega where omega is the angular velocity of the frame e 1 e 2 e 3; e 1 e 2 e 3 is angular velocity is omega and then we have written I s times omega s times e a cap ok.

So, last time we derived all this things and. So, re proceed from this place and this I do not remember that whether I have done this part or not, but perhaps I have done this. So, for the time being I will assume that this is there and later on I will come to this issue ok. So, our h then gets reduced to e a cap; where e a cap is nothing, but the unit vector along the axis of the gyrostat this is rotational wheel the wheel which is rotating. So, it is along the its axis the a vector we have taken. So, or either we have written as e a cap this is the unit vector.

So this is the angular momentum of a gyrostat, if you have such multiple wheels means say now you have this body and at the centre of mass of this body this is point o slash centre of mass. So, one wheel you are positioning here in this place another wheel you are positioning in this place, another wheel you are positioning in this place ok.

So, this is your wheel ok. If and for simplicity we can assume that this wheels are located along the 3 body axis ok. So, here a was written. So, this a can be assume like its a around the e 2 vector or e 3 vector or either e 1 vector ok. So, we put 2 other wheels. So, with that this system looks like this, you have this 3 wheels and we will assume that this wheel is rotating this wheel is rotating right from the beginning one of the wheel. So, we will assume that the wheel along this is rotating and we will give a bias momentum to this rotation we call this as the bias momentum. So, I will come to that later on let us look into this equation first.

So, if you have this kind of situation where instead of 1 wheel you have 3 wheels. So, the same equation you will write as h equal to. So, what we are doing? This is the angular momentum of the whole body about point o including the wheels as such you just jam the wheels, wheels are not rotating. So, this becomes a mass lambda mass sort of thing. So, we are considering this to be a rigid body. So, this is very simple equation I time this inertia add it for the whole system and dot omega and plus the angular momentum of the wheel at the respective positions. So, here this is the wheel number 1. So, it will have the angular momentum along this direction, if it is rotating this way if it is rotating this way. So, it will have angular momentum along this direction.

So, we can write this as the shape h 1, h w 1, h w 2 and we can write this as h w 3 means this these are the angular momentum of the wheel 1, 2 and 3. So, this indicates nothing, but h w i summation i equal to 1 to 3. So, you can see that without much difficulty if once we have got the main equation. So, we got the main equation and we derive this equation for the h and this is for the force ok. So, from here easily we have got for the whole system without much trouble.

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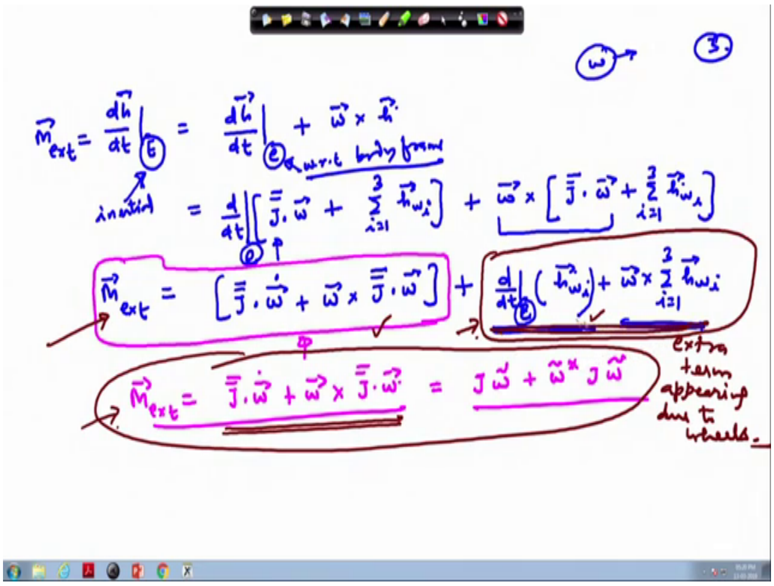

Now we can expand the equation. So, we will have d h by d t E equal to d h by d t with respect to the body frame or here in this case we have the e frame. So, I will right as with e notation this is with respect to inertial frame and omega cross h, then insert those values h equal to J double bar omega cross h which will be. So, why we are converting it from the inertial frame to the body frame? This is in the inertial frame and this is with respect to the body frame ok. Reason is very simple that the omega that you measure its a directly measurable in the inertial space.

Sitting on the earth we do not measure the angular velocity of the satellite. The instruments are on board the satellite and those instruments the red gyros they are measuring the angular velocity of the satellite. So, its a natural to describe this kind of system in the body frame itself, in terms of body frame. So, this is your M external the torque acting on the system ok.

So, the right hand side now in the body frame this differentiation is with respect to the body frame therefore, in the body frame the inertia move this a inertia term it will not change and then we can write this as, plus we can pick up the term from this place and right here omega cross J double bar dot omega, we club it together and then the other terms we can write as d by d t this is with respect to small e.

Now, you can see that this part, this as usual this is your the normal if while we work with the rigid body dynamics. So, we derive this equation M external what we have written as the Euler's dynamical equation M external equal to J plus omega cross this will derived ok. The same thing we have written in other format also like instead of writing in terms of the inertia (Refer Time: 16:03) we can write in terms of the inertia matrix and there we have written it like this, times omega tilde cross times J time omega tilde.

So, what we can see that this term is extra addition, this is the extra term appearing due to wheels. This term is as usual the external torque applied on the system and this is your basic Euler's dynamical equation. So, this part this is your extra part. So, now, we need to we have done this part we need to work out this part. So, d by d t so you can do it on the next page.

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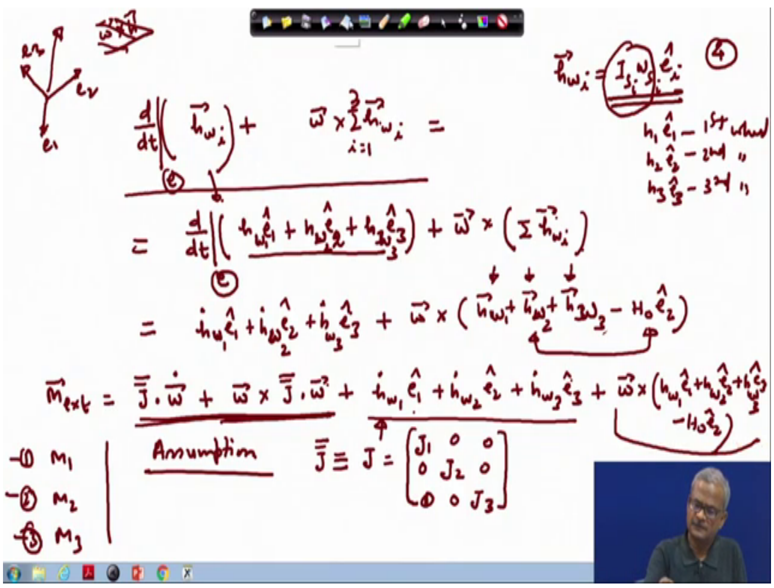

Summation omega cross which (Refer Time: 17:39) oh here, this term is separate we should remove this part and this part is here written separately this term is separate this term is separate, this is one term and this is one term.

So, we have this term here and the other term is omega cross h w i. So, this is omega cross omega cross h w i and i were is from 1 to 3. So, this is the term that we need to work out, now h w i this term we need to expand and look into. So, if you look here in this part you are in this place. So, i s times omega s times e a cap.

If it is along the first direction so, instead of writing here e a cap I can replace this as e 1 cap and this is along the first direction. So, first direction a spin along the first direction like this we can write and this is the term which we have written as h 1 ok. And h 1 multiplied by e 1 cap we are writing as h 1 for we have given this notation h w 1 d because this is for the wheel this is h w 1, w stands for the wheel ok. So, following this notation we should understand that here h w i this is nothing, but i w times or I s times omega s times depending on the direction e i cap this is your h w i.

So, once we differentiate this quantity here. So, we pick up this and we have to calculate this expand this quantities. So, this quantity if we expand it so, you can see that s w i can be written as h 1 and let us name this quantity for this is for the I t s wheel. So, we will name this as h 1 times e 1 cap for the first wheel, s 2 times e 2 cap for the second wheel, s 3 time e 3 cap for the third wheel. So, this is for the first wheel, for the second wheel and for the third wheel.

So, following that notation so this becomes h 1 e 1 cap, and this differentiation stands with respect to the body axis s 2 times a 2 cap omega cross; this summation of the all the terms. So, for summation h w i let us write for the time being like this. So, now, we have to look into this place so; obviously, in the body frame this e 1 is not changing ok. If you are looking taking derivation this is e 1 e 2 and e 3 and if you are taking the derivative of this term with respect to this frame itself.

So, in this frame h 1 will change. So, we will get h 1 dot e 1 cap plus h 2 dot e 2 cap plus h 3 dot e 3 cap. However, your e 1 e 2 e 3 these are the 6 vector in this frame and this basically these are the unit vector. So, this vectors are not changing with time. So, we need not take the derivative of this, but if the same thing we are doing with respect to the inertial frame.

So, we have to take that term like writing omega cross h that extra term appears if it is with respect to capital E. But here in this case this will not appear because this is with respect to the body frame. So, this quantity plus omega cross this quantity which is h 1 plus h 2 plus h 3 all these are for wheels h 2 deserved for the wheels I have missed here. So, omega 1, omega 2 this is h wheel 3 all if the all these places h w 1, h w 2, h w 3. Carrying this subscript it is a little difficult, but for clarity I am putting it here ok. So, all these terms can now be combined.

Now, as I told you in the beginning that we also put some bias momentum ok. So, bias momentum I told that this wheel will be given a bias momentum and let us say that bias momentum is given in the negative direction of h 2 e 2. So, that will be negative direction of this one. So, that will point here in this direction. So, this is h 0 e 2 with minus sign this become, suppose this is the bias means, these are not rotating at that time this is rotating and this is rotating here in the opposite direction h 0 minus e 2. Already I have written this plus minus if you remember, that I have written there the bias momentum as plus minus h 0 times the corresponding the unit vector in that direction. So, here I am assuming that its a only pointing toward the negative direction of the e 2.

So, if we do that. So, in that case we will have one more term here in this place which will appear as minus H 0 times e 2 cap and this is associated with this wheel. So, right in the beginning your wheels are suppose not rotating, only this wheel is rotating and that is having the angular momentum H 0 e 2 with respect to the body axis.

So, this is with respect to the body axis system, then for if the satellite is getting diverted from its equilibrium state or the differences state because of the external disturbance. So, will you will speed of this wheels this one wheel this is second this is the second one and this is the third one ok. So, as you speed up ok. So, this terms will then appear; all this 3 terms will appear and this modifies your equation of motion.

So once we now combine the all the terms. So, what we have written that in external, this equal to J dot times omega dot plus omega cross J double dot sorry J times J dot omega ok. This was the basic term involved and there after this term is there which we have now expanded. So, this expended term we can put here from this place, this becomes h w 1 dot e 1 cap plus h w 2 times e 2 cap e 3 cap and plus omega cross this quantity here, which is h w 1 times e 1 cap h w 2 times e 2 cap plus h w 3 times h 3 cap minus h 0 e 2 cap.

And then we can get the term separated out along the 3 body axis; this is along the first body axis, this is along the second body axis and along the third body axis. So, here what we will assume that we will assume that J double bar its equivalent of the matrix notation, this we will write as I 1 I 2 I 3 assuming that the other terms are 0 the off diagonal terms are 0 or either maybe for better representation we can keep this as J 1 J 2 and this as J 3. So, indicating it is a diagonal matrix.

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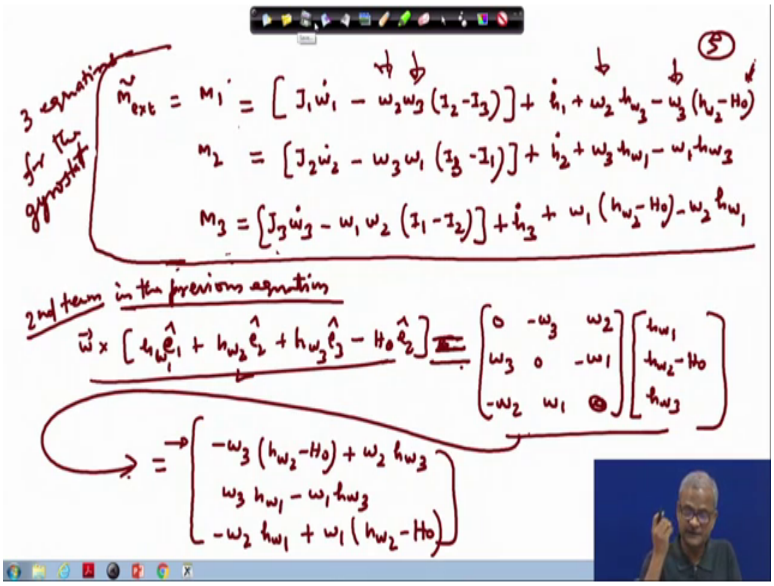

So, we will have here M external, in matrix notation M 1 M 2 and M 3 and this we can write as follows. So, the first term basically we can separate out like this we can work out all the term separately. So, the first term it appears from here this times omega dot. So, the first term will be J 1 times omega 1 dot plus and if you remember from your attitude dynamics. So, this part will simply breakup as minus this you will writing in a cyclic way I 2 minus I 3. So, this is the part from this place to this place, there after we have to use this. So, that part goes as.

So, we write here separately, this and plus h 1 dot which is appearing from this particular one this is corresponding to the e 1 direction and then from this place we have to work out the corresponding value. So, this we need to do ok. Similarly the other part will be J 2 times omega 2 dot minus omega 3 omega 1 I 1, I 3 minus I 1 plus h 2 dot and plus other terms that we need to write. Similarly this will be omega 3 dot minus omega 1 omega 2 times omega 1 minus omega 2, this is the easy way of remembering this as I have told you earlier this is s 3 dot now rest of the terms we need to insert here in this place whatever they are. So, those terms are related to your omega cross this.

So, omega cross here h 1 or h 1 times e 1 cap or h w 1 times e 1 cap h w 2 times e 2 cap and h w 3 times e 3 cap minus h 0 times e 2 cap. So, this two terms will be combined together and if we combine so we can write this as. So, we are looking at this term the second term, this is the second term in the previous equation previous equation ok.

So, this term then it can be reduced to omega cross already you know that this will be written as minus omega 3, then omega 2, omega 3 0 this will form this skew symmetric matrix minus omega 1, minus omega 2, omega 1 and then 0 here in this place and this will corporate on this vector. So, this vector is h w 1, then h w 2 minus h 0 and h w 3 ok. So, this part is equivalent to this in the matrix format ok. So, from this place it is easy to write it like this and therefore, I am writing this way. So, this becomes equal to minus omega 3 times h w 2 minus h 0 plus omega 2 times h w 3.

The second term will be omega 3 times h w 1 minus omega 1 times h w 3 ok. And then the last term will be minus omega 2 times, h w 1 then plus w 1 times h w 2 minus h 0. So, these are the 3 terms and this 3 terms will appear in this place and we can write them as this term we can write first omega 2, h w 3 minus the other this term then we can pick up this particular term ok. So, that becomes minus omega 3 times h w 2 minus H 0 ok.

So, this completes the external torque along the first direction, similarly the other term I can take up which is omega 3 times h w 1, omega 3 times h w 1 and minus omega 1 times h w 3. So, you consider the sequence; here for omega 1 you had here omega 2 and this is omega 3. So, same way here its appearing omega 2 omega 3 and there is a minus sign here and the corresponding term h term gets exchange omega h w omega 3 comes here and h w omega h w 2 goes here and h w 3 comes here in this place, and this is the only extra term which is due to the bias momentum which is appearing here in this place.

So, similarly the third term we can pick up. So, here first term is omega 1. So, we will pick up this and write here omega 1 times h w 2 minus H 0 and then minus omega 2 times h w 1. So, these are the 3 equations and this is a non-linear equation ok. So, 3 equations for the gyro stat. So, how many degree of freedom this system have? This system has 3 plus 3 total 6 degree of freedom; why?

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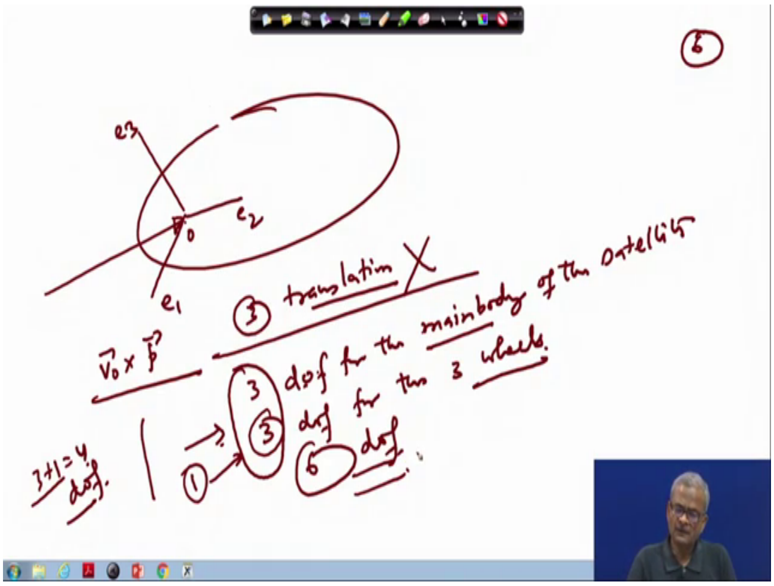

See, if your point o is not coinciding with the centre of mass of the system, then this translational motion for this enters into the equation rho v cross p ok. So, 3 becomes for translational which has got deleted because of the centre of mass coincide with the point o. So, this will get dropped out.

So, rest we are left with then the 3 rotational dynamics equation for the main body ok. So, 3 degree of freedom for the main body of this satellite, and 3 degree of freedom for the 3 wheels ok. So, total 3 into 3 total 6 degree of freedom system this is. So, here in this case if gyrostat, if this gyrostat has only one wheel. So, in that case this 3 will be replaced by 1 means then it becomes 3 plus 1 a total 4 degree of freedom system, but here currently this is 6 degree of freedom system. So, we have completed this equation of motion and ok. Next we can discuss about the linearization so that we will do in the next lecture. So, we will continue in the next lecture.

Thank you very much.