Satellite Attitude Dynamics and Control
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Lecture – 52
Gyrostat (Contd.)

Welcome to the lecture number 52. So, we are going to start with the Gyrostat rotational dynamics.

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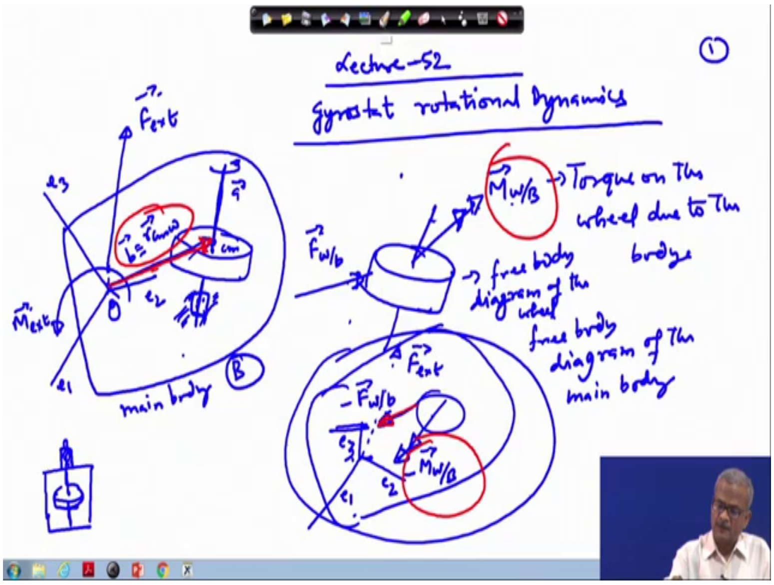

So, now let us consider that we have this main body which we have written as B and then we have the reaction wheel here. Centre of mass is located let us say somewhere here this is your centre of mass of the wheel and as earlier we have e 1, e 2 and e 3 these are the body axis this is point 2 from here to here, this is b vector which is nothing, but r centre of mass of the wheel. Now let us assume that at the point o a moment is acting which is M external ok, in addition we will have the force acting on this system.

So, let us so the force acting on the system by F external. So, somewhere F external force is acting on the system and here there is a wheel. So, on this wheel there is no external force, but the force on this can arise from the motor let us say that there is a motor attached to this or in general case the motor can be also in general if you look on a gyroscope.

So, I will see that, if this is the rotor axis of the gyroscope; electrical gyroscope. So, on one of the axis there will be a motor ok. So, this motor will be rotating this rotor. So, similarly suppose there is a motor here and this motor; obviously, this needs to be attached to this main body and this will be there is a rotor axis is; axel is attached to this. So, as this rotates so, there will be a torque applied on this.

So, on this wheel we will have a torque applied which will be in some arbitrary direction, let us show this in some arbitrary direction. So, we will show it by a vector M w B means this is the torque on the wheel; torque on the wheel due to the body. So, because of this; so, from Newton's third law and equal and opposite reaction will also act on the main body. So, on the body also if we remove this wheel from this place; so, let us show it like this, this is the body and this wheel is emptied from this place .

So, here in this place you will have a torque acting which will be just opposite of this and this will be M w B with a minus sign and torque is often shown by if you are showing it by a vector, so, it can be shown like this also. Here I have shown this like M external I could have equally directed a vector like the M external here in this point.

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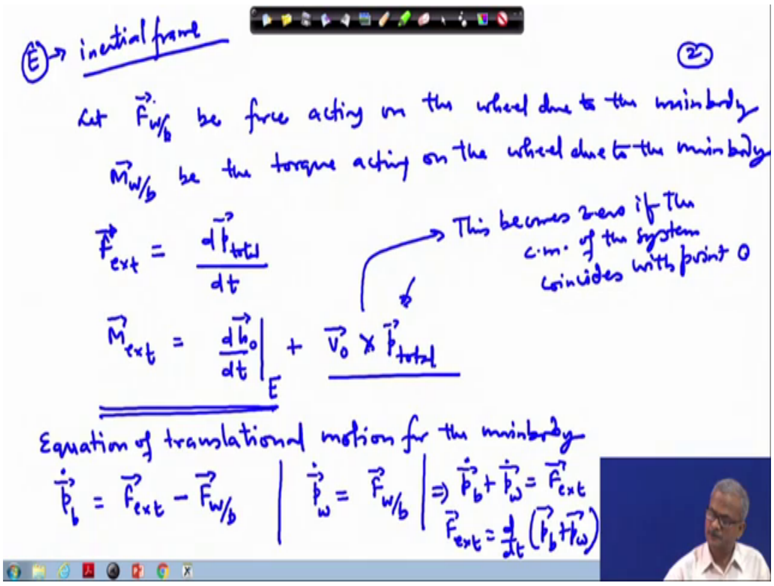

So, let F w B be the force acting on the wheel. So, similarly we have there will be a force acting on the wheel. so, because of wheel is in contact with the body. So, a net force will act on this wheel which we can write as F w b and as a result in the opposite direction we can show F w b with a minus sign this will be the force acting on the main body ok.

So, finally, what we are going to get, we are going to derive this equation f external is the force acting on the main body in inside which your wheels are also accommodated. So, this will be given by dp total where p total is the linear momentum which we have already derived by dt and the other equation M external will be given by d h by d t and this is about point O because we are writing to the whole thing about this point O.

So, this is point O, but with respect to the inertial frame this E stands for inertial frame plus velocity of the point o cross p total. In the case the centre of mass coincides o coincide with the centre of mass then this term will drop out. So, this will become zero. So, this becomes zero if the centre of mass of the system coincides with point O and then we recover the normal equation which is given here.

So, equation of translational motion for the main body we can write this as p b dot this will be equal to f external, the external force acting on the system and minus F w b. This is the part here this is acting on the body F w b with a minus sign. So, the total force and one force here say if we have shown here in this place f external is acting here. So, at this point this is your e 3 direction, e 1 direction, e 2 direction. So, somewhere along this direction your F external force is acting. So, we need to show that.

So, F external and this force this here it is shown like minus F w b. So, this we have to write. So, this the beauty of the vector notation that we do not have to worry later on we can change all the directions along the body axis. And similarly, the equation for the wheel then we can write as p dot wheel this will be whatever the forces acting on the wheel. So, this is nothing but F w b and if we add both of them. So, this implies p dot b plus p dot w this equal to F external; that means, F external this equal to d by dt p b plus p w p total.

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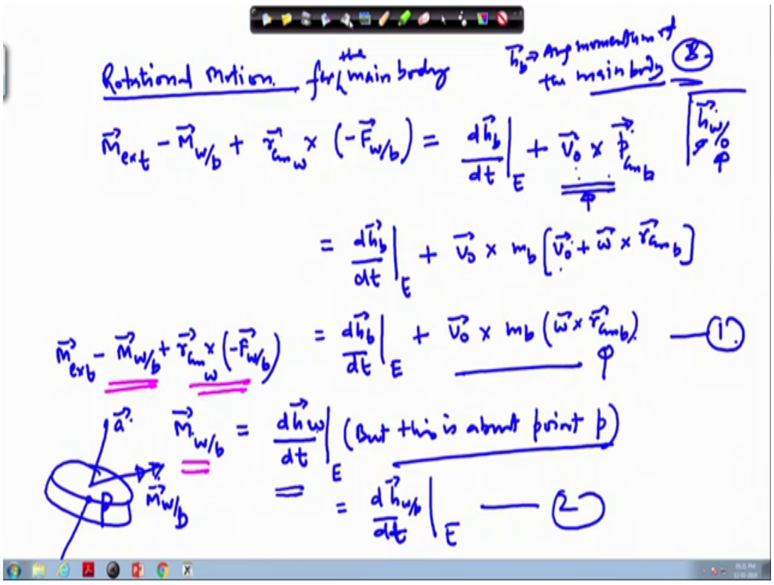

Now, rotational motion equation, what are the torque acting on the main body. So, this is M external minus M w b this part minus M w B is acting here, M w B is acting on the wheel with minus sign it is acting on the main body ok. And then also a force due to this reaction this is the term, if the force you are applying on the main bodies applying a force F w b on the wheel then opposite an opposite one will be applied on the main body also. So, this is the distance from this point to this point, this is the radius vector which we have written like this. So, we need to take the cross product of this force along with this vector to get the torque.

So, this term we will have then r c m wheel cross minus F w b, this we need to add ok; so, is the total torque acting on the system. And as a result of this what will happen, so we are what in as a consequence what we are doing we have made one free body diagram. So, this is the free body diagram of the main body and this is the free body diagram of the wheel. So, if you proceed like this, your system can in fact, your system can becomes quite complex ok. So, if your system is very complex. So, in that case how you are going to model it ok.

So, taking approaching the problem if I have a started only with a very simple model and done this then it would have been very difficult to do this. But now here, we will see that how the system evolves without understanding the real physics behind the system we could have solve the problem ok, but then we would have miss the physics involved which will be very useful if you do a more complex system. So, it will if you take the space station.

So, many spacecraft will come to that and dock some will move here and their sum will rotates something can be done, space station itself it will rotate that basic platform. So, modelling that kind of system and on that say inside the satellite or on the space at this, on the international space station itself say the control movement gyro is there. So, that gyro you can move to rotate the whole platform your own system is there.

So, this way the system becomes very complex more over the flexibility in the system is there it is not that the system dynamics exchanges a lot if the system is flexible we are working with a rigid model, so this our elementary cause ok. So, if the system is flexible then the equation will be much more complex and whatever the result you have got for the rigid system it may not be valid at all for the flexible system.

So, for that we need to take into account and work out from the scratch the whole thing. So, here in this case the this is the free body using the free body diagram we are writing the equation of motion for the this is the equation of rotational motion for the for main body. So, if b is the angular velocity of the body, so dh b by dt and this is h b is the angular momentum not angular velocity, angular momentum of the main body this is excluding the wheel and this we have to take the derivative with respect to the inertial frame.

So, all the torque acting on the system and on the right hand side then the rate of change of the angular momentum and together with this so, we will have the extra term v 0 cross the corresponding centre of mass of the main body. So, this equation already we have derived. So, in the past we have worked out this equation where this extra term appears. So, we are using that information here in this place ok. So, if you remember that once we were working with the rigid body dynamics. So, there was an extra term and in the case the centre of mass of this body it coincides with the o coincide with the centre of mass of the body so, this term will simply dropout ok.

So, therefore, this can be written as the right hand side can be written as dh b by dt with respect to the frame e and v 0 cross, what this p 0 term is? p 0 is nothing but m the mass of the main body. So, we can write here outside itself main body mass times velocity of the centre of mass will be given by r centre of mass of the main body v 0 cross v 0 this will dropout, so this term will drop out and writing here m b omega cross r cm b.

So, you can see that if r cm b here if this turns out to be 0, so this whole term will drop out; means if the point o coincide with the centre of mass. So in that case this term will not be present so this is your equation for the main body and this we have got by drawing the free body diagram for the main body, this is our equation number 1. Now we write the equation of motion for the wheel. So, for the wheel we have the external torque acting on the wheel is only due to the main body which we have written like this.

So, therefore, this will be dh w by dt and remember this is with respect to the inertial frame, but this is about its own axis or the this is about the point we have written in the point as, this point we have not named here; this point we have not named, this point here we have written as p.

So, we will continue with the notation p this is about the, because we have separated out the system remember, we are considering now the free body diagram and this is the point p; P is located here, a vector is here in this direction. On this, then a torque is applied and this torque we are showing like M w b ok.

So, under the action of this torque we are writing this equation. And there is a difference if I write here h w o so; that means, this is the angular momentum of the wheel about the point o ok. While, this is the angular momentum of the wheel, but this is about the point p itself ok. So, we could also write it like this w p by dt and differentiation is with respect to the inertial frame, so this is a second equation.

So now, add this 2 equations so if we add, so we can see that the common terms, this term which is with minus sign and this is plus sign they will cancel out and this term we can take it on the right hand side ok.

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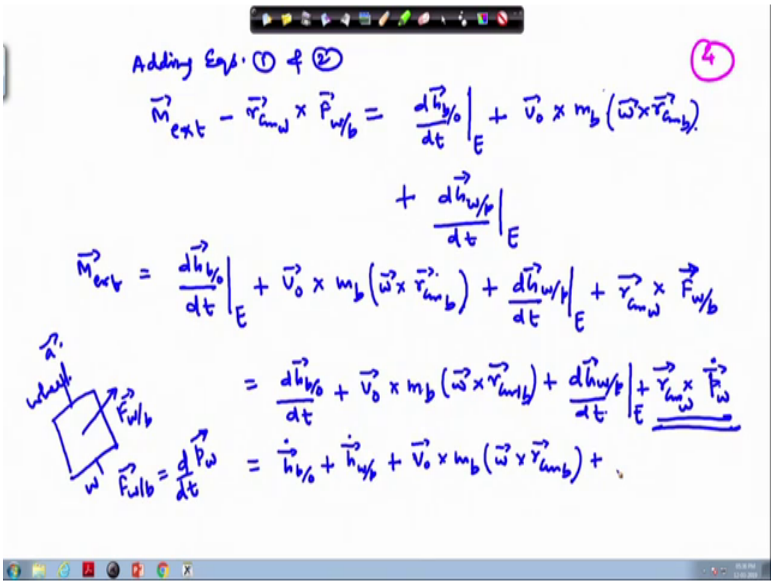

So, if we do this manipulation. So, we get the equation here as M external adding equation 1 and 2 adding equations 1 and 2 we get M external minus r c m w; r c m w cross f w b and on the right hand side then we will have these two terms and plus this term. So, dh b by dt and this is about point o, remember this we are writing about point o.

So, we can put a notation here as o this we are doing about point o. So, this is with respect to E and then we have v 0 cross m b times r cm b sorry, the here we have m b times omega cross r c m b, m b times omega cross r cm b and plus the extra term for the wheel which is about point p, but the differentiation is with respect to E, this we can take it on the right hand side. So, M external this becomes dt this with respect to E plus and plus this term r cm wheel cross F w b and because we have made the free body diagram of the system means on this wheel the force F w b this is acting ok.

So, F w b we can directly write this as d by dt p w where p w is the linear momentum of this wheel, this is the wheel it is a a direction for this. It was inclined on the right hand side in the previous figure just bring anywhere it does not matter, we just we are carrying the concept here.

So, plus r cm of the wheel cross this is p w dot. And p w dot we can insert here in this place reorganize the whole thing. So, we need to reorganize the whole thing, so I will not carry this d by dt term we will simply write it like this and h w about p, this is about p and all of them are with respect to the E frame.

So, we can carry a notation here maybe E or we can drop it out altogether, this E considering that this is with respect to the E frame means with respect to the inertial frame that will simplify little bit our working we do not have to carry, so many subscript and superscripts. So, this term we need to expand and right here in this place. So, what we do instead of writing one more step let me make it compact here in this place and write it this way.

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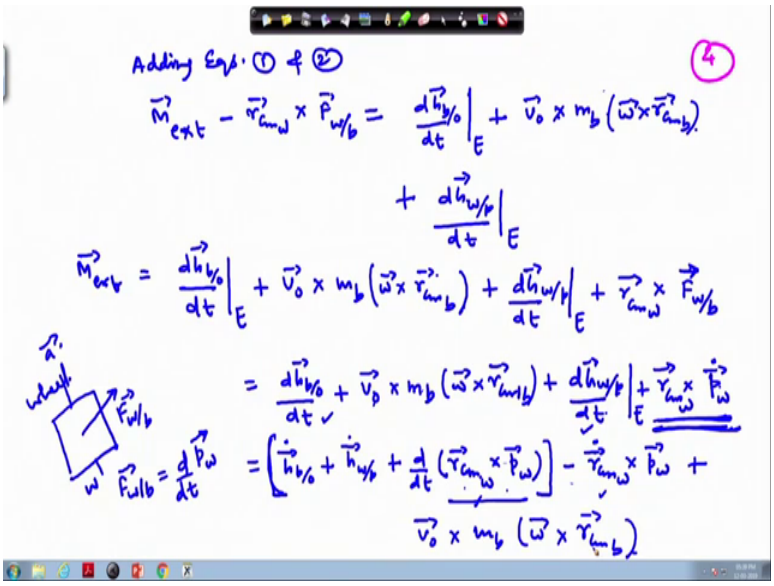

We take this term remove this dot and put it here in this format, but there will be a; if we differentiate this we can see that r c m cross p w dot will be produced and one more term will be there which term is not in this place ok. So, for that we will compensate here ok. So, this term we have taken into account, this term we have taken into account and together with this then we have taken a term from this place and written like this ok.

So, from this place we need to write the corresponding term here r cm w dot cross p w because if we differentiate we will get a term like this with plus sign ok. So, that term will get cancelled by this term and we will recover this sum here. So, this term has been written using these 2 terms this and this and then of course, we have to write the other terms which is plus v 0 cross m b times omega cross r cm b ok.

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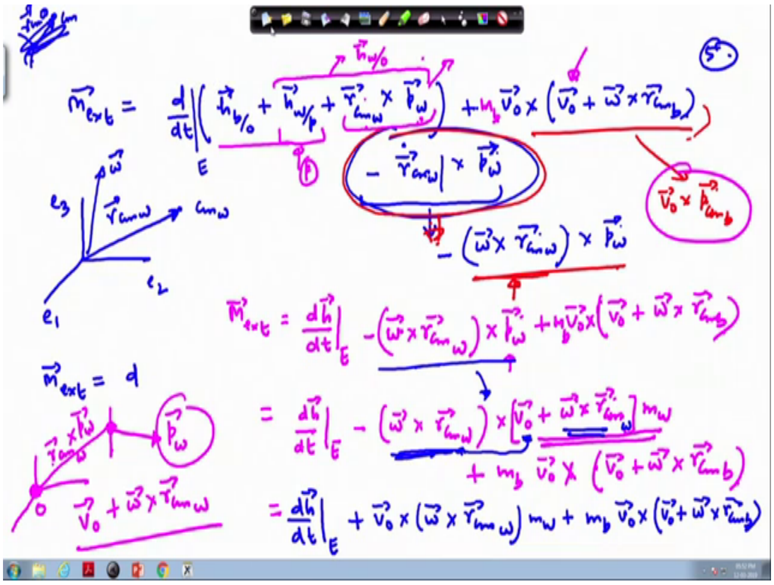

Now, we reorganize the whole thing, then the M external this we can write as let us write this term here this d by dt is there, so we are going to write this as d by dt this is h b about point o plus h wheel about point p and plus this part r cm w; r cm w which is nothing but b we have used a notation and times p w. And the other terms then we write v 0 times this term. So, of course, if you see here this term; this was v 0 plus omega cross r cm b we can write like this.

So, this is the second term we can write as v 0 cross v 0 plus omega cross r cm b ok. This particular term we can write as this r dot cm, this is with respect to the inertial frame remember, while we are differentiating, so this differentiation was with respect to the inertial frame. And therefore, this can be written as omega cross inertial frame taking this differentiation, so your r cm w is the location of the centre of mass of the wheel this is r cm w.

Here this is the body axis e 1, e 2 and e 3 which is fixed in the main body somewhere this is the centre of mass of the wheel and this vector we are writing as r cm w. So, the rate of change of this vector it will depend on the angular velocity of the main body which is rotating at omega. So, this becomes omega cross r cm w and cross p w and this is with a minus sign here. So, this particular part we have written here only this part, this is continue from this place, so this is with minus sign here so, this particular part here we are working. Now we can look here that this term it can be written as v 0 cross p centre of mass of the main body.

So, for this we need to work out here, so we reorganize it in this state so, M external then this will become. Now you can also you can recognize this term, that this term is nothing but moment of angular momentum of the wheel about the point o, this is the angular momentum of the wheel about the point p and this part, this is the centre of mass located at a distance r cm w and this is the linear momentum. So, multiplied by this radius vector once this is getting multiplied so, we get the angular momentum term.

So, this gives you the angular momentum of the wheel; that means, simply what we are doing, that we have the point o here and the wheel centre of mass is here in this place. So, angular momentum about it's this point itself and plus this r cm vector and cross the linear momentum of this point. So, linear momentum of this point let us say we have written this as p w ok so r cross this quantity p w.

So, that gives you the h w o; so if we add this so this part we can write as M external this equal to dh by dt and this is differentiate term with respect to point; with respect to the e frame and rest this two things we have to combine to get a single term. So anyway this term drops out and this can be written in this way also and in other way also we can write.

So now we combine the terms here, this is minus omega cross r cm wheel plus p w and plus v 0 cross anyway this v 0 term it drops out once you get this cross product so it does not matter and somewhere the mass term we have dropped out, the equation; while writing the equation of motion this m b was there. So, m b we have written here this is and in this part M external m b is here present this is fine, m b is here and this term we have taken just I am checking that I have not dropped here the terms, so m b is here. So, this h p 0, h w p, d by dt and outside v 0 plus m b ok, so we can write here m b term. This was dropped out, so we need to include this term here this is m b. And mass is involved here in this place for the wheel, so this term is ok.

Therefore, r cm will cross, now this will become linear momentum of the wheel this we can write as v 0 plus omega cross centre; this is the centre of mass of the wheel. So, omega cross r cm and this multiplied by m wheel. So, this gives you linear momentum of the wheel, so here the whole mass is concentrated.

So, r cm cross p w this p w will be nothing but the velocity of this point. So, the velocity of this point will depend on omega cross r cm wheel and to that you have to add the velocity of this point o, so we have to add here v o. So, this is what we have done and multiplied by this. And then of course, this term is there m b v 0 cross v 0 plus omega cross r cm b, now we can add up the terms.

So, if we combine the terms, so we will get a particular solution, this is omega cross r cm wheel this part we have written; this part we have written here in this place. So, p w so omega cross r cm wheel this part is ok. So, if we combine so dh by dt E and taking this part here.

So, this if we break this part this particular one, so we can write this as this particular term we are taking and multiplying with this so I will change the order. So, this becomes v 0 cross omega cross r cm wheel and this term and this term they will cancel out because cross product of this omega r cm w this is omega r cm w, so both of them are the same term.

So, cross product of the same vector this will get vanished and therefore, we get here m w and this term is here m b plus v 0 cross. So, if we see that just of flip of things has simplified the whole thing. So, in the next step we wind up this.

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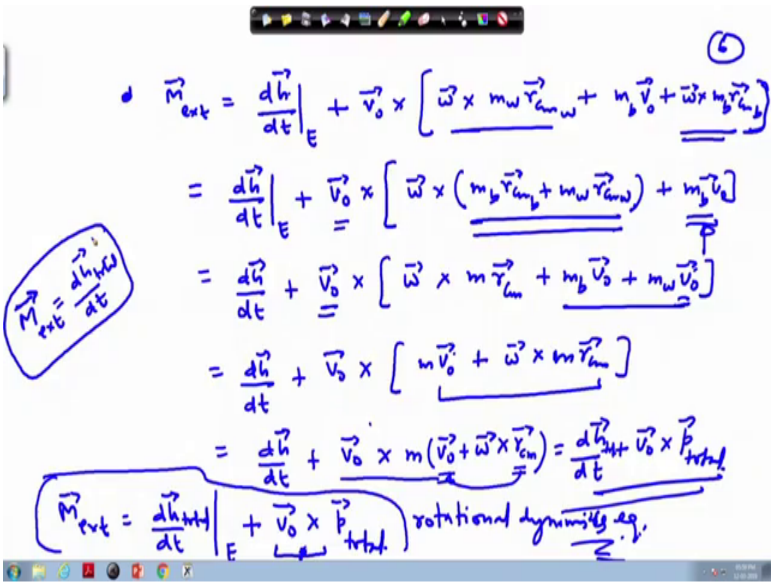

So, M external then gets reduces this equation to dh by dt and then combine this term. So, we can see that m w is here, m b is here, v 0 is here, v 0 is here, so v 0 we can take it outside. So, we put v 0 outside and cross then omega cross m w times r cm w.

So, we are writing this omega cross m w times r cm w plus now take this term so v 0 cross already we have taken outside the brackets, so this term remains. So, here we get m b and this is v 0, so next step we get here m b times m b v 0 and plus the other term which is v 0 is taken out side. So, omega cross m b times r cm b; so omega times m b times r cm b.

This two terms we can take together m body times r cm body plus m w times r cm w. This term is a Fourier term which we have added this will get deleted here. Let us write this as v 0 cross omega cross and this term is nothing but your total mass times r cm for the whole system. So, adding this term has not made any difference to the our equation, but it gives us some benefit also.

So, here perhaps I have missed out one term it seems, m b times m w times v 0. What we will do, say if I write here like this, does it make any difference to the system because if you say that v 0 cross v 0 this will cancel out ok. So, adding this it does not make any difference.

So, I can write this as v 0 cross this term can be combined together and can be written as m times v 0 and plus omega cross m times r cm. So, this is dh by dt plus v 0 cross m we can take it outside and this is the velocity of the centre of mass of the whole system. So, this becomes m time's v 0 plus omega cross r cm.

So, this is dh by dt plus v 0 cross p total, this is h total and this is p total. So, what we have got M external this is what I wanted to prove, dh by dt this is total with respect to the E frame that is the inertial frame and v cross p total. So, this gives us the equation of motion rotational dynamics equation basically; rotational dynamics equation. And this equation is of immense importance in the case we do not require this. So, this just gets dropped out as we have seen that here this term this is v 0 cross.

So, in this final equation where we have written this is v 0 cross, if your centre of mass is coinciding with the point o, this r cm will be equal to 0, this is r centre of mass. So, r cm will be equal to 0 and this term then drops out v 0 cross v 0 then will be 0. And we will recover the basic equation which you have used during your whole b tech studies and also you have done this same thing in the basic physics during 11th (Refer Slide Time: 49:08). So, this is the difference that it makes, and it becomes important when you are not choosing your reference point as the centre of mass of the system.

So, at your level you may not require it, but you should know that there is some extra; there is one extra term which is appearing in this equation if the centre of mass it's not coinciding with the reference point and the same equation can be written also in somewhat different way. So, we will look into that equation some other day. So far today we wind up this lecture here ok.

Thank you very much.