Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture - 50
Reaction Wheel / Gyrostat (Contd.)

Welcome to the 15th lecture. So, we have been discussing about the reaction wheels and in that contest we have derived the linear momentum equation. Now we will go for the angular momentum equation and work out the details.

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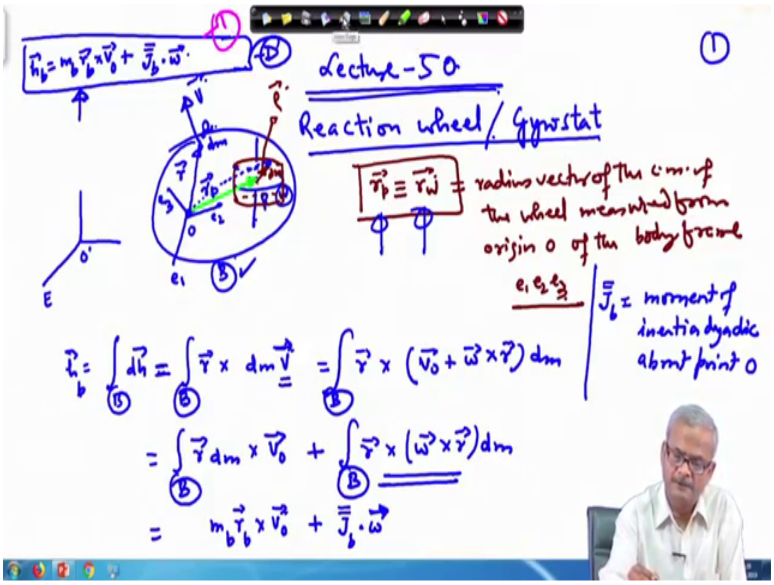

So, we have, this is the universal system the body frame fixed here and wheel is located here this we have tagged as body and this we have tagged as wheel and this we have written as r w, this we have written as r p here.

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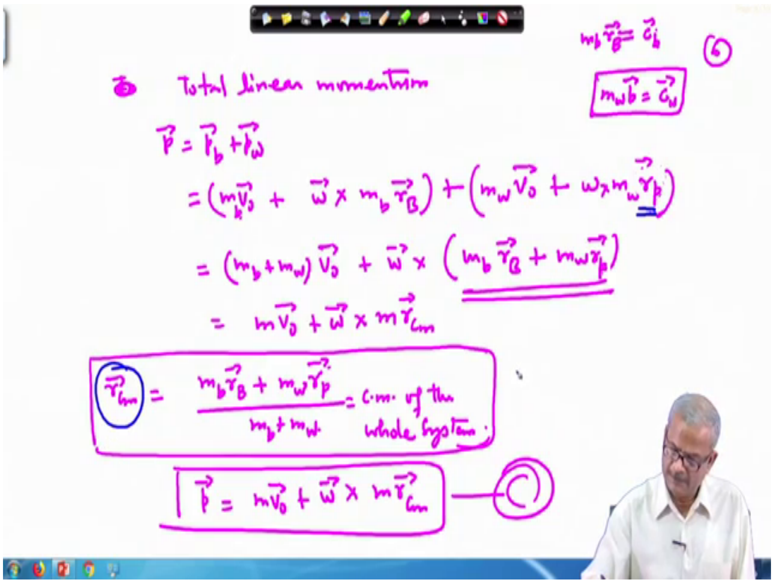

So r p, r p or r w this point we have written as P. So, this is your basically r p. you can show it by some other this is r p and then your wheel is thick. So, this is not just a plate thin plate so we need to draw it appropriately. So, your wheel is thick. So, from this point to any other point so here d m mass is there. And this vector we have written as the rho vector and r p also we can write as r w. So, this is the radius vector of the centre of mass of the wheel measured from origin o of the body frame e 1, e 2, e 3. So, we are going to now calculate the angular momentum for these 2 entities; the main body and the wheel separately.

So, let us assume that this is a mass d m on the main body and r is the radius vector to this point. So, what will be the angular momentum of this? If V is the velocity of this point and we can name this point, this point we are named as P. So, let us name this as q. So, this is an arbitrary point. So, r cross angular momentum is defined as r cross d m times d m V. And if we integrate it; so, we get h and we integrate it over main body and what this quantity V is? V is, obviously V 0 plus omega cross r and this type of integration we have done numerous times. So, I hope that you are now by this time conversant with this, this integration over the body cross V 0 and plus r cross omega cross r d m this is integration over the body. So, this gives us r times d m is the location of the centre of mass of the so, this is m body plus r centre of mass of the body, cross V 0 and plus this quantity as we know this is the angular momentum, this part can be written as in terms of moment of inertia ok. So, this is the moment of inertia dyadic, this is for the body dot omega.

So, I am not expanding and then working and doing so on. So, because we have done so many times that you should by now remember it. So, here J b is the moment of inertia about the point o ok. So, we should write here also the moment of inertia dyadic about point O. And here we will write h b. So, we should write here as a conclusion; h b equal to m b r b cross V 0 plus J double bar b dot omega. This is our main result.

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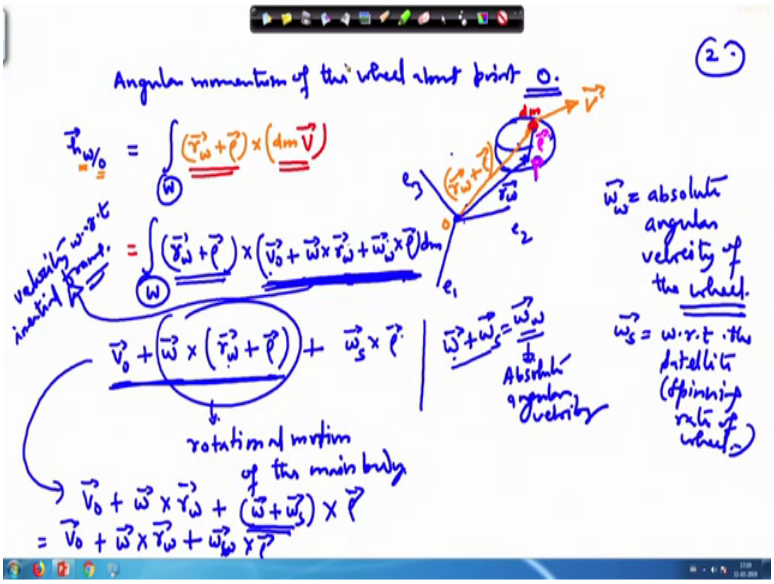

Next we look into the angular momentum of the wheel of the wheel about point O. So, this we will write as angular momentum of the wheel with respect to point o or about point o. This tag is necessary in order to differentiate as we will see later on. So, that becomes we have to integrate it over the wheel ok. And any point in this wheel which is shown here as I am showing it by this one. So, that we have to draw radius vector from this point to this point.

So, I will do it on the next page say if this is your body frame e 1, e 2 and e 3 and somewhere your wheel is there. So, the centre of mass of the wheel to this location you have the r wheel the radius vector as we are writing, r wheel or r p. This is the point P and then any elementary mass we are taking out here in this place and to this the radius vector is rho. So, we write it here on this side, rho is the corresponding radius vector this is point P ok. And this is the distance is r wheel this is the centre of mass of the wheel location plus rho this is your vector here. So, this mass is located by the vector r w plus rho with respect to point o. So, angular momentum of the wheel about the point o this is given by r w plus rho cross d m times velocity of this point. So, let us say the velocity vector is directed here.

So, this is velocity of point let us term this as q r say this is, finding some notation. Ok I will not name this. We will fall short of the notation. So, this is your point mass here which is d m. So, d m v, this is your linear momentum of this mass and r w this is the radius vector to this point. So, this gives you the angular momentum. Now we need to expand this. So, r w plus rho cross V equal to V 0 plus omega cross r w. So, this is the velocity of this point V 0 and what will be the velocity of this point, because this body is rigid so, at what angular speed this body is rotating, so add to that and then at what rate this wheel is spinning? So, wheel angular speed is w, w, w, we are with we are writing here for this rho d m. So, this becomes your ok.

So, w w is your absolute angular velocity of the wheel.This we can also understand from the point of view that V 0 plus omega cross r w plus rho. So, this is this part arises because of the rotational motion of the, of the main bodyand then your satellite itself is spinning, this wheel that itself is a spinning ok. So, if we have the spinning rate of wheel we write as w s ok; which is this is w s is with respect to the satellite. This spinning rate of wheel; so, w s cross rho. Now if we break it up this particular part. So, this becomes V 0 plus omega cross r w r r w is where r w is the distance to this point which we have already written, which appears here in this place also in this place and plus omega plus omega s cross rho we can write it like this.

So, this becomes V 0 plus omega cross r w plus this quantity is absolute angular velocity. Omega is the angular velocity of the satellite to this if we add the spin of the wheel with respect to the satellite. So, that becomes the angular velocity of the wheel with respect to the initial frame ok. So, this becomes the absolute angular velocity. So, therefore, this becomes w w cross rho. This is what we have written. So, this is the distance to this point and then the corresponding velocity in the inertial frame; only thing that we are taking about this point. So, this is the velocity in the with respect to the universal frame.

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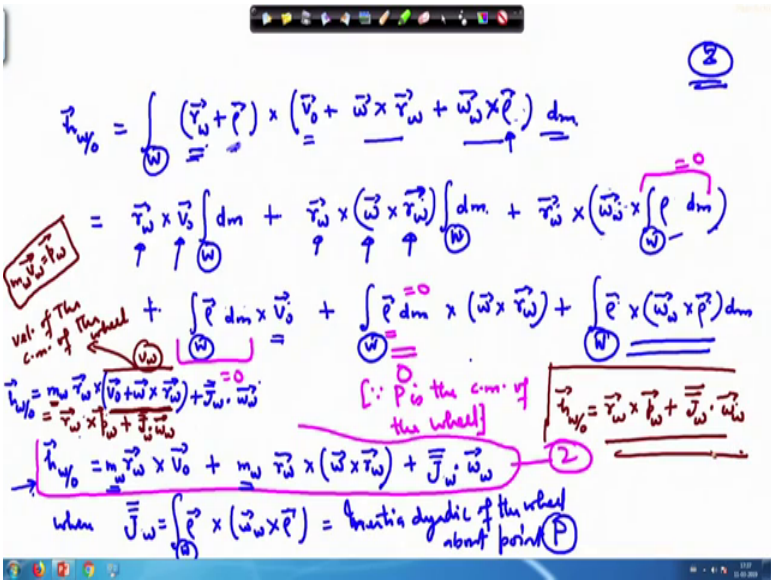

Velocity with respect to the; so, omega wheel with about point o; h about angular momentum of the wheel about point o this we have written as I will refresh on this page, r w plus rho cross V 0 plus omega cross r w plus rho w d m. Now, for working out this we need to expand it and write the whole thing. So, one by one we can expand and work it out. So, the first one we get as r w cross V 0,.ok this term with respect to this term we are expanding. You can say that this term is a not dependent on mass, this term is not dependent on wheel mass and all the terms here they are not dependent on wheel mass and therefore, they are taken outside the integration sign.

Now with respect to this rho we are going to integrate. This rho we are going to use it. So, with respect this rho will be multiplying all the terms. So, this becomes rho d m cross V 0, V 0 is not dependent on the integration over the wheel. So, this comes outside here ok. This is rho here, notation we have used for this is no rho double rho w, but rho we have used. So, this part I have corrected here. So, rho cross;. so, total of 6 terms we are having. Now we can see your some of the terms that drop out because, we have chosen P as the centre of mass of the wheel. So, therefore, this term is 0, and from P we are measuring this rho vector and therefore, this will drop out so this quantity is 0 here and then this quantity similarly this becomes 0; here in this place this quantity becomes equal to 0. So, these terms will simply drop out. And what we get here; h w o this equal to r w cross.

So, we can write here mw times rho cross V 0 plus here again this integration is m w this term drops out, out this term drops out, this term also drops out. So, 3 terms to where in a sequence they are getting dropped out and the last term this term is nothing but where this is the quantity rho cross inertia dyadic of the wheel about point P. So, had all this equations the, one equation we have got is into this is equation number 1 and this is equation number 2. We need to add it to get the total angular momentum.

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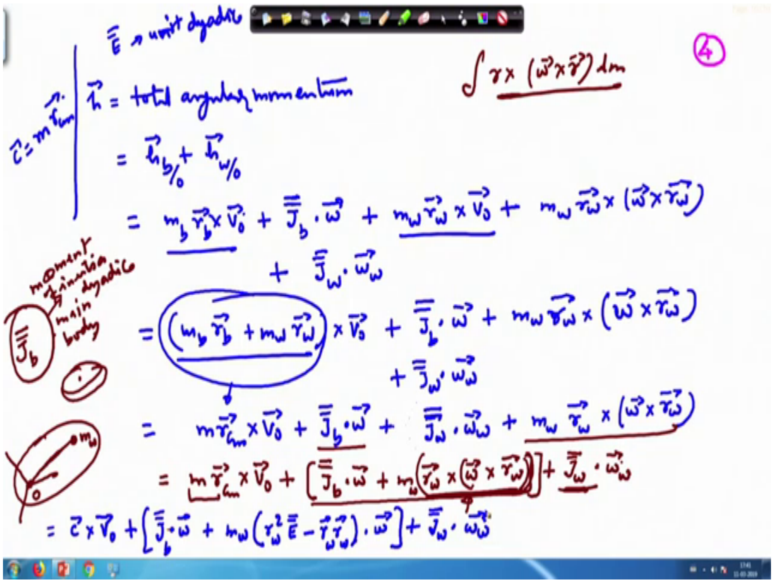

So, h there is a nothing, but the total angular momentum and this can be written as h main body plus and this also is about the point o. So, maybe we can go back and put this point o. So, this is also about the point o, this is about the point o, and also the wheel about the point o. So, this is the total angular momentum we just need to add that and then put all these quantities here m b r b V 0. m b r b cross V 0 plus J double bar. Here this quantity we are going to write this currently at J ok. So, here also we will write as J. The reason will become clears once we advance. This is the inertia the dyadic of the wheel. So, this is for the body times omega. This is what they J b dot omega and plus the other part h w 0. So, we have to copy all this things m r V 0, m w r w, m w r w cross V 0 plus m w r w m w r w cross omega cross r w and plus the third term here J double bar w time . So, this 2 terms can be combined together, see if this is derivation is bit longer if we have chosen the point o to be the centre of mass itself it would have been much shorter ok. But then it deprives us of the what happens if the point o does not coincide with the centre of mass and quite often this is the situation it is not always that your orginalize at the centre of mass of the whole system. There may be difficulty with the fabrication of the system itself that you are not able to conceive your center of mass at that point from there you are doing the measurements.

So, in that case this equation what we are deriving here they will be applicable. This is the total mass m and then the centre of mass of the whole system. So, that we have written as we have used at notations from there this is the r c m and ok. So,, this becomes m times r cm. This is the, this gives you this quantity which is here cross V 0 plus ok. This equation what we have written here, this we can little bit refine it and this will be required later on so, we can also rewrite this as h w o this equal to m w, m w in these 2 places they are available. So, we can take this as common. So, m w, r w is also a common. So, r w goes outside and cross V 0 plus omega cross r w plus J double bar w times omega w. which we can re write as m w, r w, w r w we keep it here cross m w this is the velocity of the centre of mass of the wheel. So, this is the velocity of the centre of mass of the wheel and therefore, we can write this quantity here as, velocity of the centre of mass of the wheel and together with we multiple pi by m w. So, the quantity m w times v w we will write as p w.

So, here we write this as p w, this mw times this quantity here this is your V w. So, m w times V w is p w. So, we are writing here and plus J double bar times omega w ok. So, this expression will be especially helpful later on. So, here all here I am writing it finally,r w cross p w plus omega w. We combined this term and this term and this particular term here m times r c m we can write us for gravity we can write as C . So, this quantity you can see that this is the moment of inertia of the moment of inertia dyadic of the J double b bar, J b double bar. So, this is indicating moment of inertia dyadic of the main body, moment of inertia dyadica and to this your then adding this particular term.

So, this has the same configuration as say the r times omega cross r dm. Once we write it like this so, this has got the same configuration. So, this also indicates moment of inertia times omega but here it is for the here we have missed out that w term this is w subscript is there. So, you have the main body here. So, main body moment of inertia about this point and then this point mass also has moment of inertia about this point o. This point mass is what here the wheel whose mass is m w. So, m w has if it is concentrated here and then you are taking this. And so, we can write this as we can we will write up on the; we will have to expand it little bit and then write it. So, let us, so, here this becomes C cross V 0 where, C we are defining as m times r c m plus J double bar dot this is body, omega plus m w and this we can writer as r w a square E double bar. Where E double bar is the unit dyadic as per our earlier notation unit dyadic minus r w r w dot omega plus.

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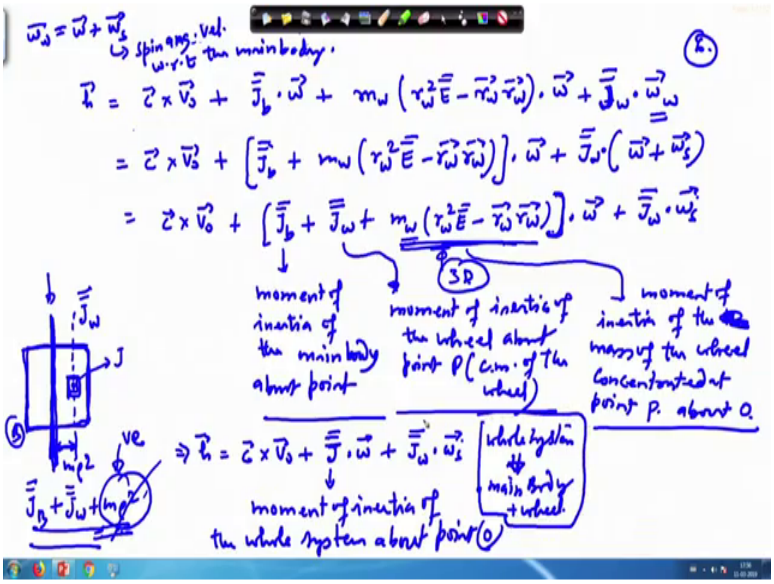

So, this we have copied from the previous page. Now, we can write this as C V 0 plus J b double bar, and omega this is absolute angular velocity of the wheel. So, this will be equal to the angular velocity of the main body and with respect to this then the wheel is spinning. So, your omega w this equal to omega plus omega s; so; this is spin rate with respect to the, omega s is the spin rate where this is a spin velocity or the spin angular velocity, a spin angular velocity with respect to the main body.   This term we can pick up from this place. This dot is here, and we can write J w and plus m w times r w a square this is the unit dyadic minus r w; this part is moment of inertia of the main body about point o and moment of inertia of the wheel about point P which is the centre of mass of the wheel, the wheel and then this acts as a point mass.

So, this is the inertia of the point mass. Inertia moment of inertia of the mass of the wheel concentrated at point P. The inertia of the mass of the wheel concentrated point P about o. So, this three together means you have say some body is there and the in the parallel axis theorem what we do? We have one body here and inside this there is another body. So, what we do that, first we find out the mass of the moment of inertia of the main body about this; say line then we find out the moment of inertia about this small body about this line and then we consider this to be the point mass and then calculate the moment of inertia about this by writing m rho square.

And at the further of this is the main body b. So, we write as J B and plus here it is a inertia about this line is suppose J w double bar. So, we are adding this J w double bar and then to this m rho square you are adding. So, this part we have to put it in a vector format because it is in 3 dimension. So, this is in 3 dimension, 3 D. While here it is written only for 1 dimension ok. So, this is a simplified model. So, this is not applicable here, but this is the way so, this is something acting like a parallax theorem. So therefore, this part the quantity in the bracket is the moment of inertia of the whole body about the point o. So, this gets reduced to this implies h can be written as C cross V 0 plus J double bar dot omega plus. So, this is the moment of inertia of the whole system. Moment of inertia of the whole system about point o whole system means it is a whole system implies, this implies main body plus wheel. So, we will stop here and then we will continue in the next lecture.

Thank you very much.