Satellite Attitude Dynamic and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture – 48
Gyroscopic Motion (Contd.)

Welcome to the 48 lecture. So, we have been discussing about the gyroscopic problem. So, in that case, again the few days back I have shown you one YouTube video in which you have seen that earth is precising about certain axis.

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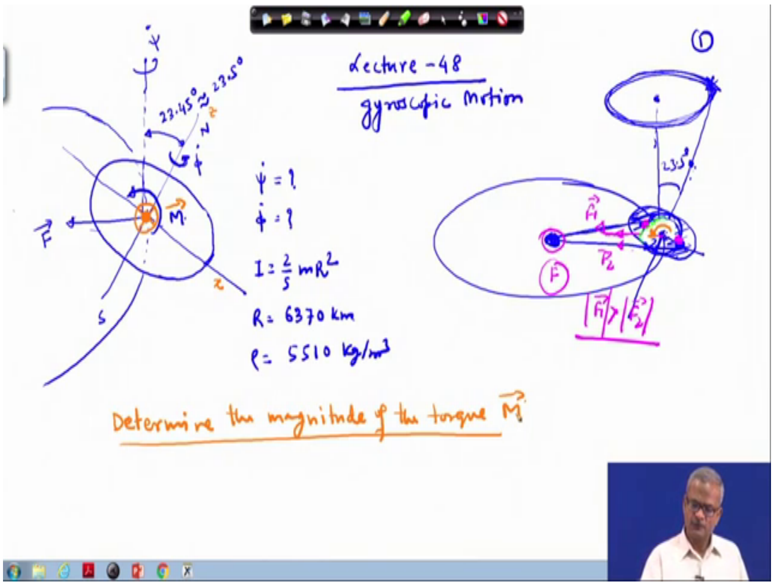

The axis which is lying toward the this is perpendicular; so what is happening actually that earth is moving toward the sun ok, and it is inclined on its axis. So, let us say that this is earth and it is inclined like this. And this is the polar axis of the earth. So, this axis of the earth, it is a pointing toward the Polaris at the current stage. So, basically it is a going in a path you say there is a circular path in which this axis is moving and this angle is approximately 23.5 degree and we have gone through this video, so this is the same problem.

Now, this earth is not perfectly spherical ok, instead of this it is oblique, it is something like this. So, already you know about the gravity gradient problem. So, you know that on this half of the earth due to sun, there will be larger amount of force as compared to this half. So, here the centre of mass of this, centre of mass of this one ok; one centre of mass lying here, another centre of mass lying here. So, this F 1 and this is F 2 magnitude wise, this will be greater than the F 2.

And therefore, this not only due to the sun, not only there is a force towards the sun, which is we can write as F the resultant force which is responsible for going in this circle. The other planetary masses right now we are ignoring ok, those things if you have to account, so you have to do the numerical calculation through theory you cannot do, solve such kind of problem.

Now, because of this gravity gradient ok, here the gravitational force is different and here at this point is a different. So, a net torque is acting the torque which we can show like this ok, it is acting here in this direction; it is acting like this, not visible. This torque is acting about so we can assume that the at this point if you see, so through this the y-axis is passing. So, it is a going here into the page, so y at y-axis is passing through this. So, this is your x-axis, this is the z-axis of the earth, now we can solve this problem.

So, what is the question here, the question is to determine the magnitude of the torque M. So, we have to determine this torque which is along the y-axis, which is very obvious here in this case. So, to work out this problem, we go back to our earlier lecture and from there we can look into the corresponding equation we have derived.

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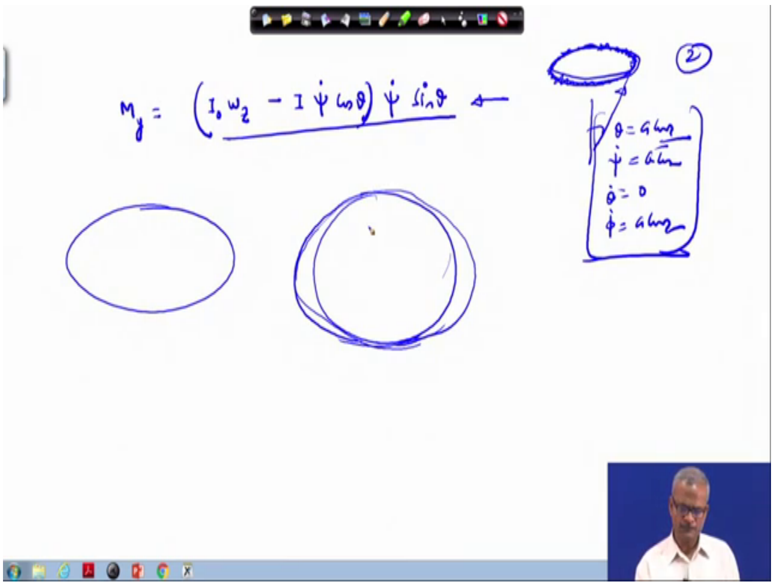

So, if you remember we have written for the M y, the expression I 0 times omega z I times psi dot cos theta. You can go back and look into the lecture, so there is a small bobbling of the say if once the earth polar axis is rotating in this precession this circle, so there is a small bobbling here is the a small notation is there ok, so that notation is being ignored. So, you can consider theta to be a constant, psi dot is also a constant, theta dot becomes 0 and phi dot this is a constant. So, this case already we have worked out, so for that case only this expression we derived.

Now, insert we need to insert all these values, though the earth is oblate this is of this shape, it is not here say this is your sphere. So, the small rate of oblateness is there. So, you can say that this is appearing in the form an at ellipse. If we cut a plane pass a plane through the North Pole and the south pole, so we can say that this is in the form of an ellipse. So, let us rub it out. So, here what we assume that this oblate, eccentricity of the such kind of ellipse ok, it is not required here ok, basically it is a earths shape is ellipsoidal shape we call it the ellipsoidal shape.

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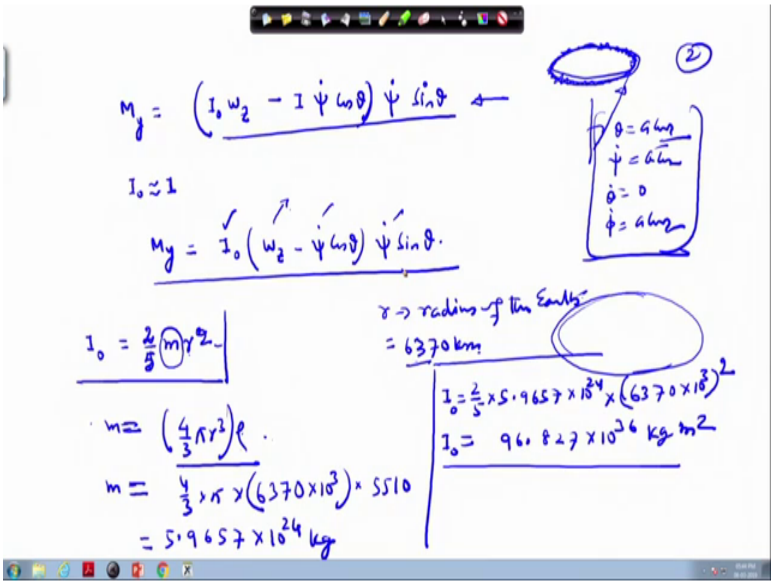

So, what we assume here that I 0 this is nearly equal to I ok so that the whole case get simplified. So, therefore M y will be replaced by I 0 minus omega here omega z minus psi dot cos theta times psi dot sin theta.

Now, this quantities we can compute. So, I 0 is 2 by 5 mass of the earth times r, 2 by 5 m r square or ok, because it is a solid if it is hollow, then the result will be different 2 by 3 in that case we take. So, we put all these values here. So, this will be 2 by 5 and m will be now assuming this to be a though this is this is a steroid sorry, this is a ellipsoid. So, but a still we will assume this to be a sphere and calculate its mass, so that will be 4 by 3 pi r cube times rho which is the density and then you have of course the radius of the earth ok, which is r is the radius of the earth ah.

So, r square here r square term will be not there, this goes into I 0. So, we put insert all these values 2 by 5 times 4 by 3 times pi and r is 6400 kilometers around. So, 6374 is a right value, but here we write its 6370 kilometer, so this is 10 to the power 3 ok, so this value is 6370 kilometer. So, we are converting this into the meter here and the density is given to you, so which is 5510 kg per meter cubic. So, from here this is we are first we calculate the mass of we remove this, this portion we will remove and we calculate the mass here.

Let us calculate the mass first and thereafter we will calculate inertia. So, this quantity turns out to be 5.9657 into 10 to the power 24 kg, so this is the mass of the earth. Then we compute this quantity I, this will be 2 by 5 times, mass is here 5.9657 into 10 to the power 24 and then r, r is 6370 into 10 to the power 3 meter. So, we square it and this gives you 96.827 into 10 to the power 36 kg meter square, so this is your I 0. So, the moment of inertia is known here ok, omega z and psi dot all these values we need to insert, theta is known to us.

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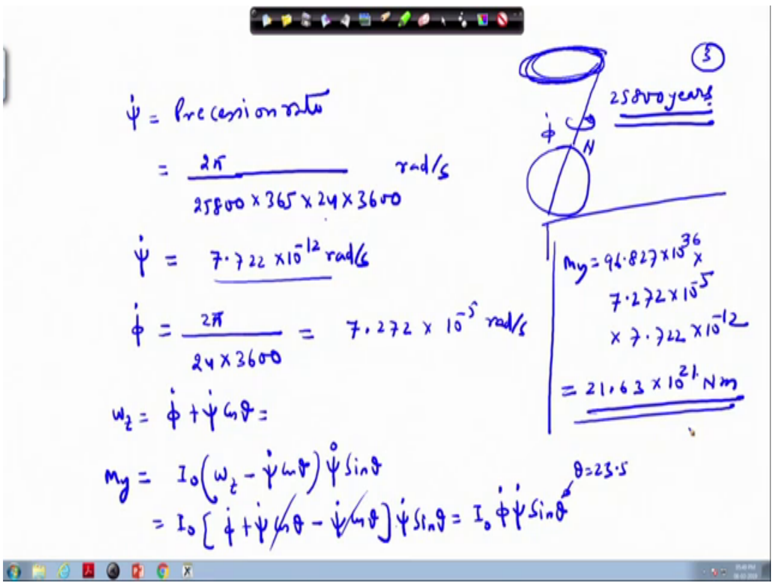

So, now psi dot this is the precision rate. So, it is a known that the polar axis of the earth, it is a going in this circle this is a north pole of the earth. So, it is a going in the circle in 25800 years. So, it will complete the circle in 25800 years, so that quantity is known to us. So, psi dot is 2 pi, this makes 2 pi angle. So, 25800 into 365 we convert into day and then into hours and then into seconds minus 12 radian per second; so this is psi dot.

And then phi dot accordingly we calculate on that this axis, we have to compute the phi dot. So, earth is rotating by 2 pi angle approximately in 24 hours ok, so that value we write here 24 into so approximate value I am writing, it is a not exactly we are writing, 272 this is phi dot we have got. So, omega z is nothing but phi dot plus psi dot cos theta, we can insert these values here.

So, going back on the previous page this expression we pick up and write on the next page, it is a M y I 0 times omega z minus psi dot cos theta cos theta, and then multiplied by psi dot sin theta psi dot sin theta. So, omega z is the quantity here, phi dot plus psi dot cos theta and from there then the psi dot cos theta get subtracted, so these two cancel out leaving us.

And of course, we have to multiply this term also, so psi dot sin theta. So, this gives I 0 times phi dot times psi dot sin theta inert all the values which we have computed here. So, M y this will be I 0, we have computed here 96.827, 96.827 10 to the power 36 ok, times phi dot which is available here 7.272 into 10 to the power minus 5. And similarly, the psi dot is available to us so that gives us 21.63 into 10 to the power 21.

Now, of course you need to insert this value theta equal to 23.5 degree approximately, so it Newton meter so this is the torque about the y-axis enormous torque. So, these are the problems related to your gyroscopic motion of the earth. Now, we are so based on the same principle in fact, gyroscope we try to develop and based on the same principle we have done that. We do some more problems.

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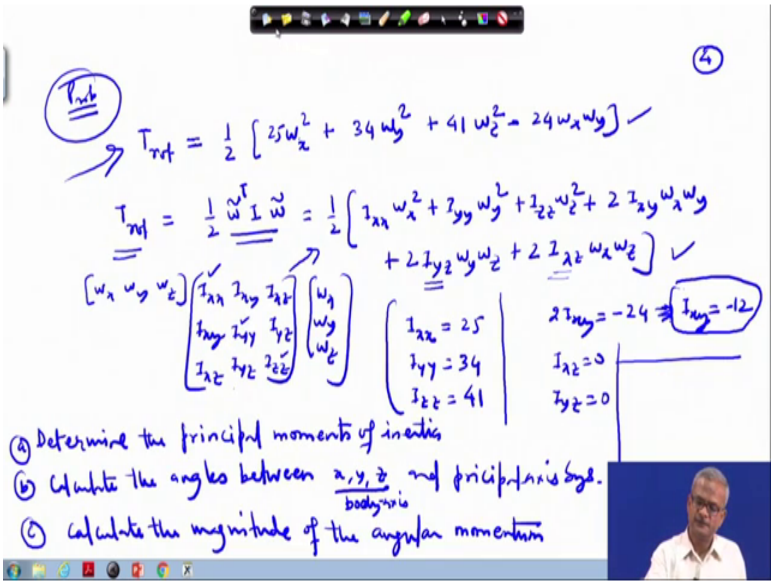

Say the rotational kinetic energy, it is a given to us 1 by 2 omega x square plus 34 omega y square. So, the rotational energy of a rigid body it is a given by 1 by 2 omega tilde I times omega tilde here there is transpose. And this quantity becomes equal to I x x times omega x square I y y times omega y square I z z times omega z square plus 2 x y times omega x omega y plus 2. If you expand this ok, you have to write it in this format omega x omega y omega z, write here I x x I x y I x z fill out up to this point ok; here also write omega x omega y omega z multiply it.

So, this will (Refer Time: 18:20) this, 2 times I x z times omega x omega z. So, this is the rotational energy of the system. Now, we are in this case, this is the problem we are discussing what we need to do, question is a determine the principal moments of inertia, calculate the angles between so this is the body axis for the case which is described here, body axis system and the principal axis.

Calculate the magnitude of the angular momentum. So, what we can if we compare this equation and this equations. So, what we have observe from this place that I x x this quantity will be equal to 25, I y y this will be equal to 34, and I z z will be equal to 41. So, this are the diagonals elements here ok, this, this and this. What about the off diagonal terms? So, in the off diagonal terms if you look for the I x y, so here this quantity is there. So, 2 times I x y if you compare this, this is minus 24, so this implies I x y this equal to minus 12.

Then we can compute the other quantities I x z there is no term related to I x z also I y z. So, I x z this equal to 0 and I y z similarly this is set to 0. So, this completes your description of the inertia matrix. So, your inertia matrix now it looks like ok, we will do in the next page.

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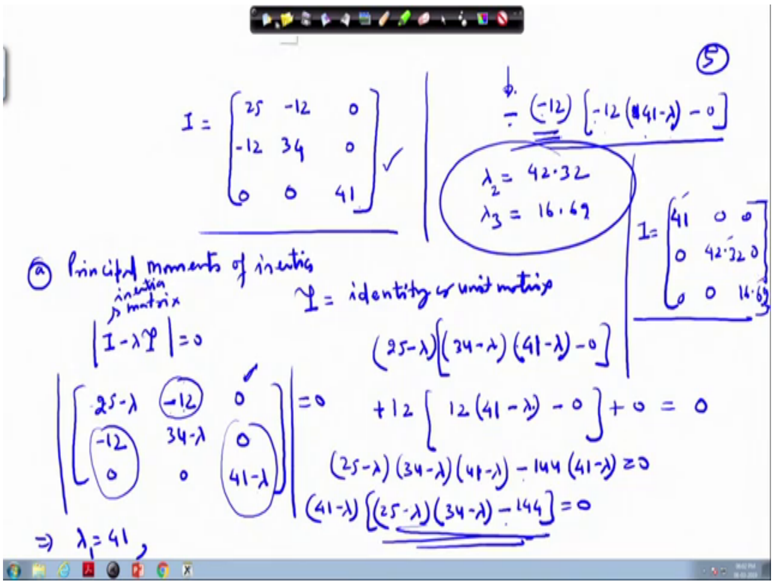

So, the inertia matrix then becomes the first element was 24, then 34 and this was 41, I x y we have got as minus 12, the others are 0. So, this is the inertia matrix from there we have to calculate the principal moment of inertia, principal moments of inertia this is the first problem and how do we do this. The basically, you have to find out the Eigen values of that while discussing the rigid body attitude dynamics. So, I have discuss that in details, so just as an example I am solving it here.

So, we have to solve it for where this is an identity or unit matrix. So, this is inertia matrix inertia matrix. So, if you insert these values, so this becomes if you subtract it, so 25 minus lambda this will be minus 12 0, this remains 0, 34 minus lambda and this remains 0 0 0 41 minus 41 minus lambda. So, we have to take determinant of this, this and set it to 0 and solve this. So, this will yield us 25 minus 4 times 34 minus lambda times 41 minus lambda. Here minus 12, it is missing. So, we need to put here minus 12.

Then we to the other term this term related to this will be 0. So, we have to work out only for this, so this is minus 12. So, minus dot makes it plus ok, I hope you are aware of the determinant I am presuming it ok, so that becomes 12 times 41 minus lambda minus

0. And the other term will be 0 and this equal to 0, we have to set, so this is 25th; this is 41. So, this is 25 minus lambda 34 minus lambda plus 144 times 41 minus lambda 34 minus lambda plus 144 this equal to 0.

And if we solve this, so lambda equal to 41 one is the very obvious result. The other one, this polynomial you need to expand and workout. So, the other solutions I did it again I have not done any manual work in this, I just used MATLAB to work out all these solutions. This sign will check it the sign, this is minus 12 minus 12 times minus 12 into 41 minus lambda, this 41 minus lambda, so here it should be a minus sign; this will be a minus sign.

This is, here the odd term the row and the column that makes it a first row and the second column, this is 3. So, correspondingly the minus sign here, then we take this element which is minus 12. And then we choose the corresponding minor here, so this is minus 12 times this one minus 12, this row and this row the corresponding one. So, minus 12 times 41 minus lambda and 0 times 0, so this goes here, so that makes it minus plus and here minus, so this is total minus. So, there we should have minus here in this place.

And then we need to solve this part, so by solving I have got this part once we work it out. So, for that lambda turns out to be 42.32, so let us say this is lambda 2, and lambda 3 is 16.69 and this we can say that this is lambda 1. So, these are the three principal moments of inertia. The quantities we have got earlier, these are not principal moments of inertia. These are just the diagonal terms here ok, but not the principal moments of inertia.

Principal moment of inertia we have taken, once the off diagonal terms are 0. So, according to that scheme, so once we put this matrix. So, now, the I is then becomes so this is 41, there are 3 terms here; so, 42.32 and 16.69 and other terms are 0. So, this constitutes now gives you the principal moments of inertia this three terms. And we also need to find out the second problem was calculate the angles between the x, y, z axis and the principal axis directions.

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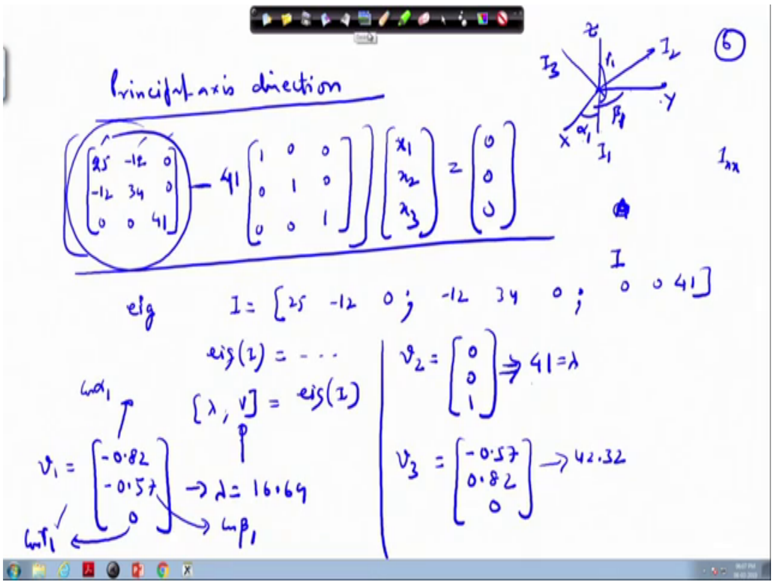

Principal axis direction is say related to the Eigen vector. So, we need to solve it for the Eigen vector. So, each of the principal moment of inertia need to be picked up and the corresponding value we need to work out. So, what you need to do that here there was the first term we have written as 25, this was 25, then 34 41 minus 12 0 0 0 and here minus 12 minus lambda. So, pick up the value let us say you are picking up the 41. So, 41 lambda times I**.** So, unit matrix here and let us say that the corresponding angles you have the current axis in which this moment of inertia this terms are described. So, let us say this is x, y and z and according to the same scheme you are written I x x and so on. And thereafter let us assume that this is your I 1, I 2, I 3 the principal axis direction.

So, the corresponding on angle for this is alpha 1 this is beta 1 and this angle from the z axis this makes this is gamma 1. Similarly, for the y axis and z axis. So, from this is for the I 1, I 2 similarly you can orient with respect to x y z axis. So, this alpha 1, beta 1, gamma 1 that gives you the Eigen vector. So, here let us say that the corresponding Eigen vector you are writing as x 1, x 2 and x 3 then you need to solve for this. So, this also I have not done any manual calculation you just need to find out the Eigen values in the MATLAB once you have this matrix. So, this matrix if you pick up say if let us say this is the a matrix or this is the I matrix already we have defined. So, in the MATLAB you can give the command like the 25 minus 12 0 minus 12 34 0 and this is semicolon here 0 0 41.

If you give Eigen I. So, this will list you the Eigen values the three Eigen values we have written on the previous page. So, this 2 here and 1 here in this place. So, lambda one is 41. So, these are the three Eigen values that you will get using this commands instead if you write here in this format say, if lambda v format and write it like this. So, this also gives you the corresponding Eigen vector. So, here in this case using the same command, I have found the Eigen vector to be minus 0.82 and it gives you MATLAB gives you the normalized Eigen vector means it is have magnitude of this vector will be equal to 1 and this is corresponding to lambda equal to 16.69.

Similarly, the second Eigen vector this is 0 0 1 and this is corresponding to 41 and the third Eigen vector this is not equal to it is a corresponding to. So, this is equal to lambda this equal to lambda and then the third one is minus 0.57 then 0.82 and 0 and this is corresponding to 42.32. So, these are your three eigenvectors which define the principal axis and already, I have stated you that this will be defining cos alpha this will be defining cos beta and if this is for the first axis then we will write this as like and this is cos gamma 1. So, once you solve it then you get the corresponding direction of the Eigen that axis.

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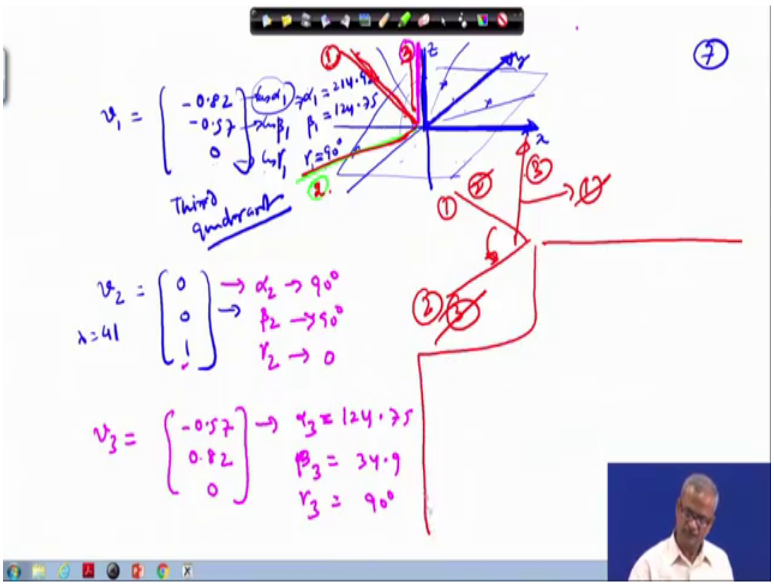

So, here in this case thus nu 1 equal to minus 0.82. So, this is somewhere located in if you see if we have this x axis here, y axis and z axis vertically up. So, we can see that this quantity is negative, this quantity is negative. And this is corresponding to cos alpha

1 this is corresponding to cos beta 1 and this is corresponding to cos gamma 1. So, this is the location of the first axis so where it is a located. So, this quantity cos alpha is negative. So, this point is located in third quadrant. Similarly, the other one so third quadrant ones it is located in the third quadrant minus, you can see that the first, second and this is the first quadrant, second quadrant and the third quadrant will be here. So, accordingly we can find out the angles.

So, the corresponding angles then turn out to be this is the this implies alpha 1 equal to 214.92 and beta 1 this is 124.75, because you are measuring from this the first one you are cos alpha you are measuring from here, cos beta you are measuring from here, cos gamma I will measure from this axes. So, cos gamma is 0 means it is a 90 degree angle. So, cos gamma so gamma 1 is 90 degree. So, 90 degree you are measuring where it will be. So, all the points which are lying in the x y plane will any vector in the x y plane you take along this direction, you take along this direction take along this directions. So, all of them in the x y plane.

This is the x y plane here. So, in this plane all the vectors lying in the x y plane it will make 90 degree with the z axis so that means, somewhere your this the x axis is lying in the third quadrant. Let us say this is 214 degrees wanted at is here and to that if you add another 34 degrees. So, it comes here. So, this is vector is located something like this. So, this is your the first axis located the same way we need to locate the other axes. So, nu 2 corresponding to 0 0 1 and this is corresponding to lambda equal to 41. So, that gives you x component is 0, y component is 0 and z component is 1.

So, this axis so your third axis it is a second axis is located on. Second axis is according to this it is it is located here. So, this is your second axis. So, here in this case this corresponding angle is alpha 2, beta 2 and gamma 2 this is your 90 degree 90 degree and this is 1. So, this corresponds to 0. So, with the x and y axis this direction makes 90 degree angle that is very obvious and 0.57, 0.82 and 0. So, this will correspond to alpha three equal to 124.75 and beta 3 equal to 34.9 and gamma 3 equal to 90 degree. And where it is lying, this is lying in the x is negative y is positive and z is 0.

So, it is lying in the second quadrant. So, it is a very obvious once you have got this two directions. So, the third direction must be perpendicular to both of them. So, the third direction somewhere it will be located in the second quadrant like this. So, so 1, 2 and this is 3. Now, we have to look into the order also so it should go in a proper order it should not be arbitrary if you are writing, this as the first axis and then the second axis. So, the third axis putting it like this 1, 2 and 3, we should it should come out it should be in other direction. So, we will we can exchange the direction, we can write rather this as the 3 and we can write this as the 2.

And or either we can write this as 1 so just changing the notation here. So, if we write it has 1, this as 2 and this as 3. So, this makes the proper sense that we have this is the right hand 3 here this is the right handed system. So, this part and this part that makes you the right handed system this is 1, this is 2 and 3. So, according to the right hand rule this is complete and it depends on you how do we it, you can also write this axis as the 1, then you can write this as the 2 and then this becomes 3. So, it totally it depends on you however, you want to represent this. So, this completes this particular part then the last part is remaining, calculate the magnitude of the angular momentum.

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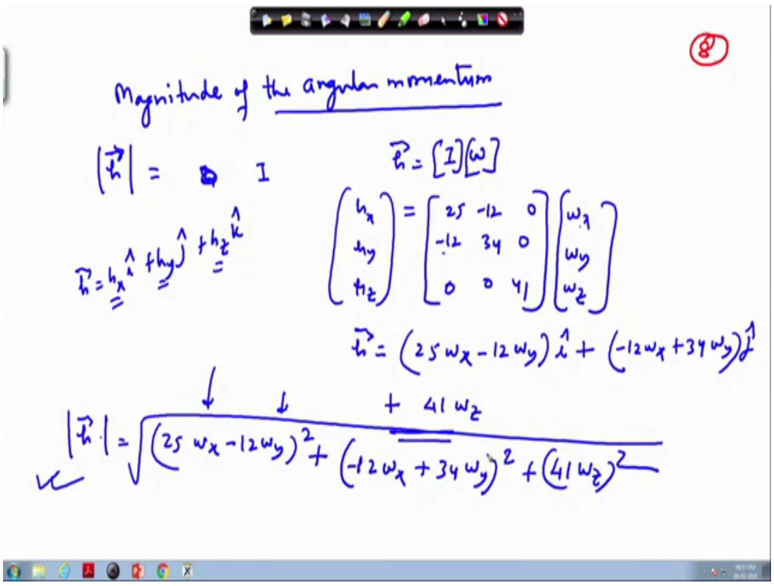

So; obviously, h vector magnitude this will be given by I 1 times or angular momentum, how we have defined this in the h vector we have defined as. I times omega in the matrix notation. So, you can utilize that information because I matrix was already known to you and here omega x, omega y and omega z this was 25 minus 12 0 and minus 12 it is what 34 41 0 0 0. So, according to this scheme then h you can in the vector notation, you can write like this 25 times omega x minus 12 times omega y and this will be along the I cap direction plus the first one, we are taking then the second row.

The hx here this is your h x component, h y component and this is the h z component. So, your h is nothing but h x i cap h y j cap and h z k cap. So, just insert these values here so h y then becomes minus 12 times omega x plus 34 omega y and corresponding to this, this is 0. So, this is j cap and plus 0 0 41 omega z. And therefore h magnitude this will be 25 omega x minus 12 omega y this square plus under root of course, we have to take of this minus 12 omega x plus 34 omega y square plus 41 omega z square. So, this gets you the magnitude, but here omega x, omega y and omega z it is not given if it is given then you can solve this problem. So, we can stop here and we will continue in the next lecture.

Thank you very much.