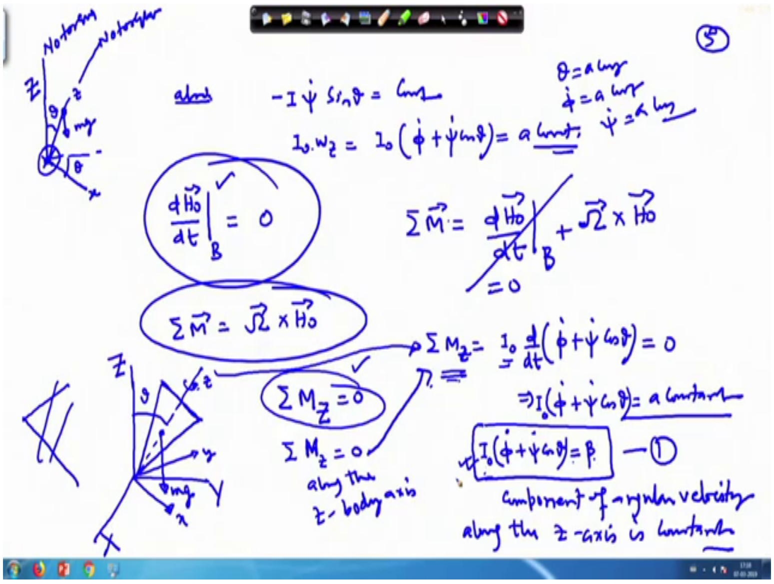
Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

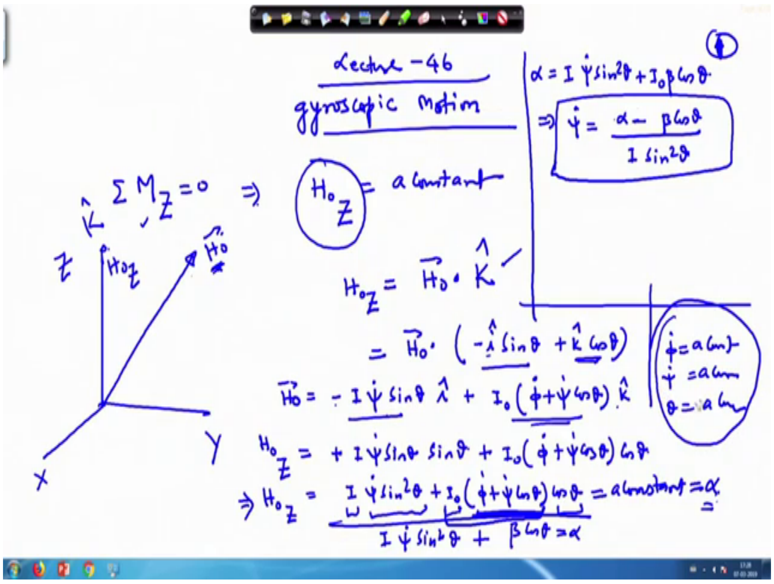
Lecture - 46
Gyroscope / Top Motion (Contd.)

(Refer Slide Time: 00:31)



Welcome to the 46th lecture. So, we have been discussing about the Gyroscope. So, we will continue with discussing with that ok. So, we have been working that along the Z body axis along this axis ok. While we are using this is small z here, we are not using the capital Z ok. Moreover, along the capital Z one, this also this quantity is 0 ok.

(Refer Slide Time: 00:49)



So, M capital Z, this quantity is also 0. So, already so what does this imply that the corresponding H 0 Z, this will be a constant ok. And how to find out this quantity? This capital Z, and say somewhere your H 0 is directed along this line. And you are looking for this component, this is H 0 capital Z. So, for this component as already I have stated that this can be written as capital K, where capital K is the unit vector along this direction, and the small k cap is the unit vector along a small z direction or the Z body direction ok. So, this is not along the Z body direction remember, this has got three components.

And we need to put here the H 0 and all these things. And thereafter, we will get this ok. The other way, there is other way of doing also, like in first we work for this. So, your this is H 0 dot K dot is minus i sin theta plus small k cos theta ok, this we have already done ok. So, look back in the previous lecture. And H 0 previously, we have calculated, so H 0 we need to put here in this place. So, your H 0 (Refer Time: 02:52) is minus I times psi dot sin theta times i cap plus I 0 times phi dot plus psi dot cos theta times k cap. And this is under the condition that phi dot this is a constant, psi dot this is a constant, and theta this is a constant.

So, if we take the dot product here in this place H 0 Z, then results ok, so as of that we get here minus I psi dot sin theta, this is i cap here, this part and this part together, so sin theta and this minus minus that makes it plus and thereafter, we have the second term which is with this and this, so we get here I 0 phi dot plus psi dot cos theta times cos theta ok.

So, this is I times psi dot sin square theta plus I 0 times phi dot plus psi dot cos theta times cos theta. And as you can see here in this place, this quantity is a constant, this is a constant, this is a constant, this whole thing is constant, this is also constant, so that means, this is a constant. So, here we can write this as a constant, and let us write this has beta.

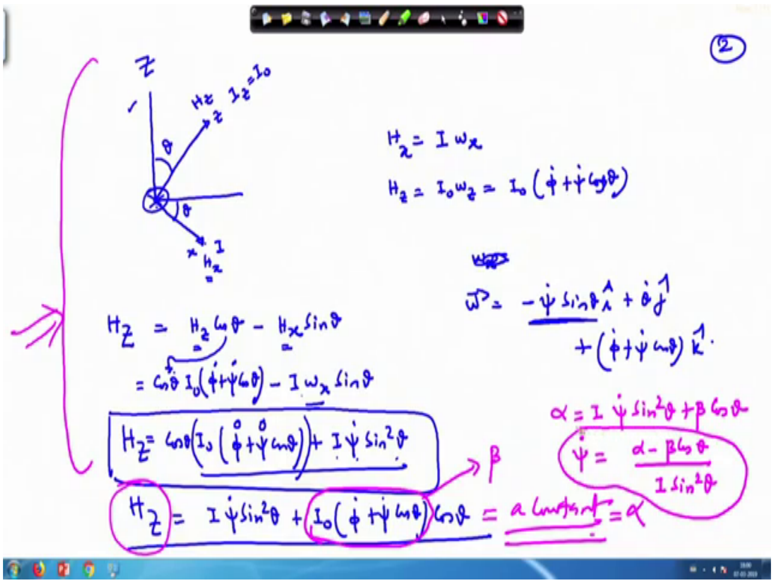
So, we have got two constants here. One constant is ok, here we have already used beta. So, here we will write as alpha, so we will write this constant as alpha. Now, we utilize these two equations for solving further. So, we can write here in this place itself. So, alpha this equal to now if we go back and look here phi dot plus psi dot cos theta, this we have written as beta, so that means, this quantity we can write as beta.

So, then this alpha can be written as I times psi dot sin square theta plus I 0 beta times cos theta. And this implies psi dot equal to I 0 alpha minus here alpha minus I 0 beta cos theta divided by I sin square theta. Beta we have beta we have ok, here this part is missing this is I 0, this is I 0 here.

So, let us say we can multiply it by I 0 itself, and write that quantity to be a constant or either both way it is possible, we can just write for the phi dot psi dot cos theta to be equal to beta or multiply it by I 0 and write equal to beta. So, only thing what we are looking for we are looking for some simplification of the whole process that we do not have to complicate the whole thing.

So, as per your convenience, you can work in this matter ok. So, for that convenience, I will put here by removing this I can put here I 0 and write it like this ok, so that in the next step, this I 0 does not feature here in this place. So, this will go if we include this in the beta term ok, so this whole term we write as beta, and this is cos theta, on the right hand side if this is alpha, and this is I times psi dot sin square theta plus ok. So, thereby we do little simplification here. Otherwise, if even if you put beta equal to just phi dot plus psi dot cos theta, it does not matter, ultimately your result will be the same. Only the simplification results by assuming like this ok.

(Refer Slide Time: 08:45)



So, next we have ok, so this is one way of doing. Before, I progress further, I will take the same issue, but from a different perspective again say this is the horizontal here, this is capital Z, and this is your small z axis. So, moment of inertia about this axis is I z equal to I 0. And your x-axis is inclined downward, this angle is theta ok.

So, along this you have H z, the component of the angular velocity along this you will have H x, and the moment of inertia along this axis is I, this is your small x-axis, this is theta, and y component is y-axis is perpendicular to this. So, this is going into the say into the page. So, from there you will not have contribution toward the H Z, so H capital Z also we can write as this is small H z here small H z here cos theta, and from this place minus H x, here cos theta sin theta, because this is in the opposite direction. And then just insert the value for H z and H x ok.

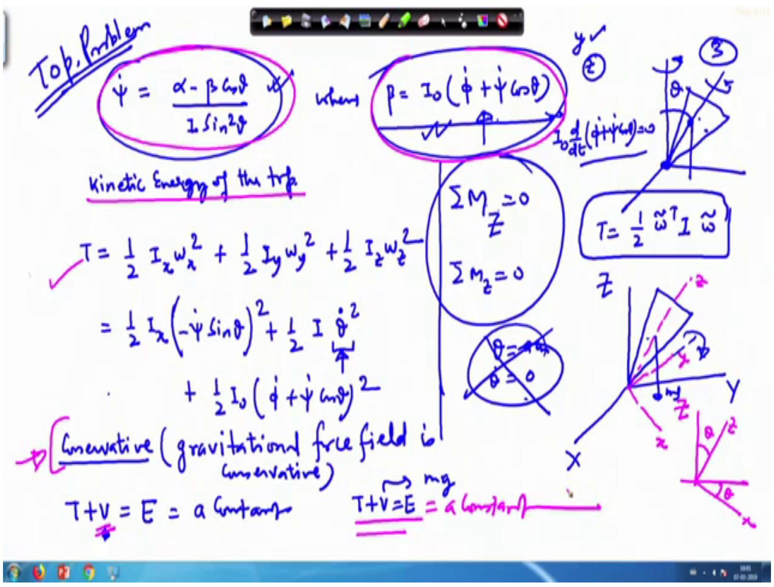
So, we have here H x equal to I times omega x, we need to insert this. And similarly, H z equal to I 0 times omega z, and this quantity is I 0 times phi dot plus psi dot cos theta, we can go back and check. This is your vector here, this quantity, and here this part ok. So, this is I times psi dot sin theta.

So, H x is nothing but I x which is equal to I times omega x, omega x will need to insert here, this comes with a minus sign, and this one comes with a plus sign here to insert the value of the omega x. Omega x is how much? So, your omega vector this is i cap plus theta dot j cap plus phi dot plus psi dot cos theta k cap ok, so this is your omega x. So, if we insert this I 0, so this becomes phi dot plus psi dot cos theta, and this minus sign if we pick up, so that becomes plus I times psi dot sin square theta. So, this is your capital H capital Z.

So, go back and match this I times psi dot sin square theta the first term, here I times psi dot sin square theta. Here it is written as a second term, then I 0 phi dot psi dot cos theta times cos theta so ok. Here this part we are missing out, so we need to multiply it by cos theta ok, one more part is here. So, we will resolve it out and write here, this equal to cos theta.

This cos theta is appearing here, this we forgot to write here. So, multiply it by cos theta here in this place cos theta times this quantity, so that means H capital Z that gets reduced to I times psi dot sin square theta plus I 0 times phi dot plus psi dot cos theta times cos theta, and this is what we have exactly got by the dot product method. So, the various problems we can solve from different perspectives. Keep your mind open, and try out with different methods, you will be able to solve the same problem in many ways, if this is a beauty of the rotational dynamics ok.

(Refer Slide Time: 14:01)



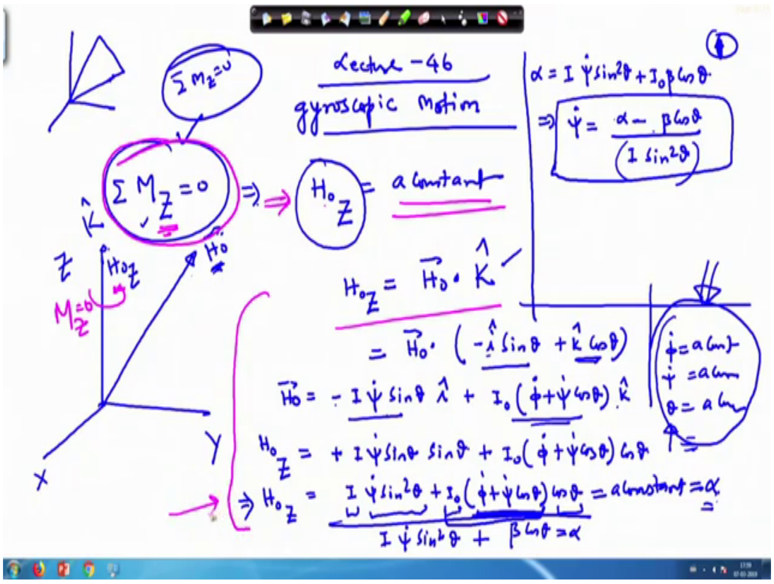
Now, we can progress further, and look into what we were looking for ok. So, we have got psi dot equal to alpha minus beta cos theta divided by I times sin square theta, where beta equal to I 0 times phi dot plus psi dot cos theta. Now, the kinetic energy of the top ok, we are looking into the motion of the top. So, this is spinning about this axis, then this has motion along this direction also. So, already from your attitude dynamics, you know that the kinetic energy T can be written as omega tilde transpose I times omega tilde. This we have derived ok, this is for the simple case.

So, the axis we have chosen the body axis, so this is your capital X, capital Y direction, this is capital Z direction. This is small z axis, and this is a small x axis, and this is the y axis. So, z and y are perpendicular to each other, while z and x they are inclined. So, how they are inclined? This is capital Z, and this is your x, this is a small z, already on the previous page, I have shown it.

So, the kinetic energy of the top, we can write as T equal to 1 by 2 times the moment of inertia. So, once you are doing this, so we need to because it is rotating about this point. So, we need to take moment of inertia about this point. So, moment of inertia about this point, we will have this access this is the axis of symmetry x and y and z ok. So, the moment of inertia we can write as I x times omega x square plus 1 by 2 I y times omega y square plus 1 by 2 I z times omega z square.

The insert the values for all this omega x, and omega z, omega y. So, omega x is your psi dot sin theta with minus sign plus 1 by 2 I times theta dot. So, now you are discussing about the top ok, so while we come to this portion ok. So, if you are theta is not constant if theta is varying, then there is a different issue. If theta is a constant, then there is a different tissue ok. So, from here do not get confused from this place, while we have written here in this place, till this place we have discussed one issue.

(Refer Slide Time: 18:09)



Now, we are entering into the domain of your this top motion, earlier we have discussed under the condition that if this condition is satisfied, then what will happen? Next we are going to do that there is a top, and it is a torque is acting. So, under that under what condition, what result you are going to get, this is what we are looking for ok.

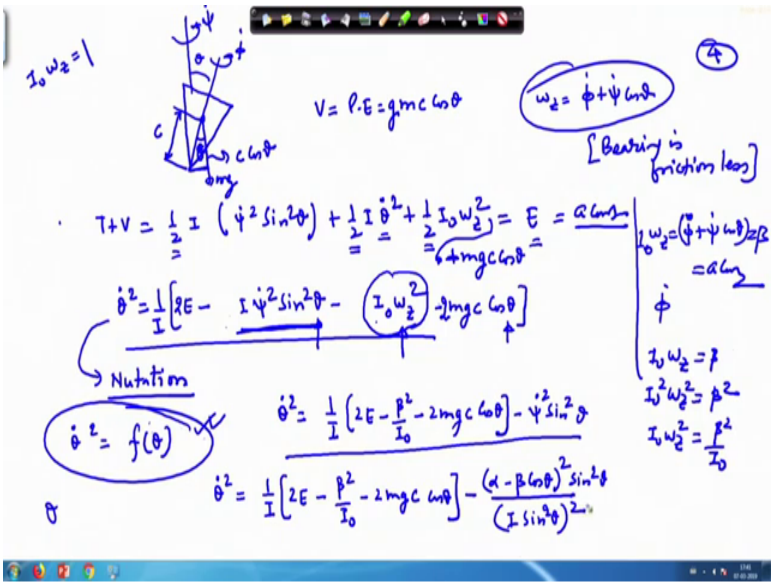
So, if you look for this m g is acting here downward, so there is only one torque which is acting, this torque is about the, because you are applying a torque. So, this torque is acting just along the y-axis ok. And there is no other torque involved here in this case. So, if we discuss for this particular case, so on the previous page what we have written here E M Z equal to 0. So, this condition is still applies, and small E m z equal to 0, this condition is still applies.

So, here this is the top problem, we are discussing. So, E M capital Z, this is also 0. And E m small z, this quantity is also 0 ok. And under this condition, we are going to derive this particular one. And what we have got here ok, this also applies here in this case for the top, because along the y-axis only the torque is acting along the small z-axis, there is no torque.

And therefore, this condition is satisfied ok. This we have got from the result I 0 times d y d t times phi dot plus psi dot cos theta, this equal to 0, so this condition is also applied. So, this is valid, and this is also valid here ok. So, under this situation, but here in this case for this particular top case, the theta will not be theta dot will not be equal to 0. As we earlier we have assumed, this theta to be a constant, and therefore theta dot equal to 0. So, this condition is not applied here, this condition is not valid.

So, omega x, omega y is simply theta dot. So, this we put here in this place, and plus omega z 1 by 2 I 0 times phi dot plus psi dot cos theta, this is your omega z, so this square. So, this is your total the kinetic energy of the system, here this is a conservative system, because the gravitational force field is conservative gravitational force field is conservative. And therefore, T plus V the total energy, so this will be remaining constant, so this will be a constant. And here T plus V equal to E, so your V will be equal to mg, so here in this place. If this is the mass centre ok, and this distance is c will take it on the next page, maybe 4.

(Refer Slide Time: 21:55)



So, this is a mass centre. So, mass centre to the distance to this place, this angle is theta. This distance we have taken as, once we have stated this problem, this we took as c. So, this vertical distance this is c cos theta ok, this angle is theta. So, this is c cos theta, therefore the potential energy V is the potential energy, this becomes equal to m times c cos theta times g.

So, your total system energy T plus V, this will be equal to 1 by 2 I times psi square sin square theta plus 1 by 2 I times theta dot square plus 1 by 2 I 0 times omega z square, where omega z is phi dot plus psi dot cos theta, this will be equal to E, which is a constant. Because, the potential energy and kinetic energy together, it makes the total energy which here in this case it is not getting dissipated means, the bearing the ball socket the ball socket joint or the bearing is frictionless. And therefore, no energy will get dissipated, and this can be solved ok.

So, your we can solve here for this theta dot, so theta dot square this can be written as E the total energy minus 1 by 2 I times psi dot square 1 by 2 I 0 times omega z square divided by I, so this divided by 1 by I. And there is a factor of two here, so this 1 by 2 times I theta dot is there, so we will make here 2 E. So, instead of writing two here in this place, let us make this as we can take this 2, this 2, and this 2 on the right hand side, so that becomes 2 E. And therefore, we can remove this 2 from all this places. And thereafter, we can write here just in terms of I. So, this is your nutation rate. Remember we are not assuming here that theta dot equal to 0.

And we need to solve this in order to get the solution, so this theta dot square this is 1 by I times 2 E minus psi dot, we have I 0 times omega z, this we have written as beta ok. So, this quantity we can reformulate ok. Here the potential energy term, we are also missing out. This is m g times c cos theta plus m g times cos theta, we add to this part here. So, we need to account for this here in this equation also minus m g c times cos theta. And there will be a factor of 2 ok.

So, we rewrite this equation ok, basically the idea here is to that this equation is just a function of theta here, and omega z is a constant ok. You can look into this omega z, this quantity is already we have solved this omega z equal to phi dot plus psi dot cos theta, and this multiplied by I 0, this we have written as beta, and which is nothing but a constant ok. So, I 0 is a constant, so therefore this quantity is a constant, so phi dot is not manifesting anywhere.

So, omega z is a constant, it enters with along with this. So, this phi dot this theta dot is not a function of phi dot or any other term, it is a just a function of theta. And therefore, this can be integrated ok. And if you integrate, you will get the solution in term for the theta that means, the question was that we were looking for the motion of this torque.

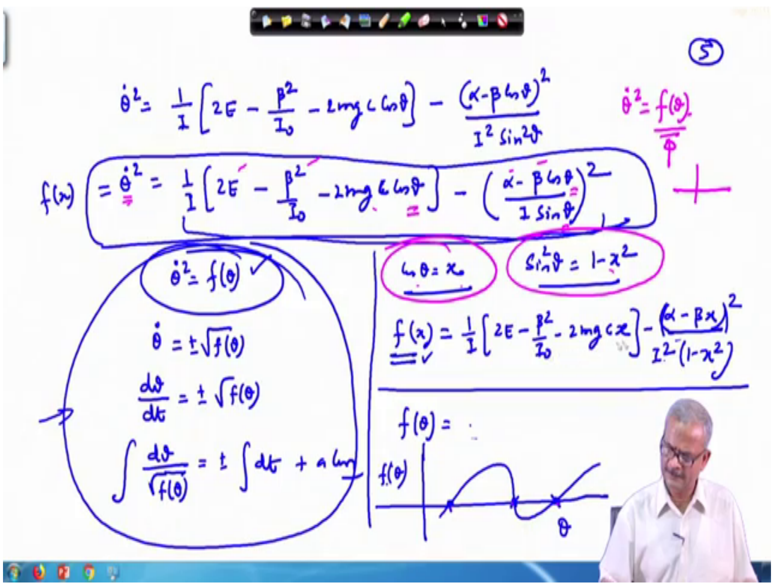
So, if we have a torque, which is a spinning about this by phi dot, and then it is a precessing at the rate of psi dot ok, and this angle is theta. And we leave this torque, while m g is acting downward here in this place. So, what will be the resulting motion? Ok, theta will grow up to what values the theta will be going, what will be the limit in which limit the theta will lie. So, all these are the questions which needs to be answered ok.

So, for this part we can rearrange it, and we can write this as 1 by I times 2 E minus I 0 times omega I 0 times omega z is beta. So, we write it little in a better way, beta times omega z or we can also write it like this I 0 times omega z, this equal to beta; so, I 0 square plus omega z square this equal to beta, so this quantity will be beta square.

And therefore, I 0 times omega z square, this quantity you can be written as beta square divided by I 0. So, we can write using this notation also ok. I am following a particular notation, you can use the other notation also, there is no problem. B a square divided by beta square divided by I 0 ok, this term we are going to take it outside the bracket minus 2 m g c cos theta. And here then I, I cancels out psi dot square sin square theta.

Psi dot already we know what that quantity is so theta dot square, so 1 by I times 2 E minus minus 2 m g c cos theta minus psi dot, we have written as alpha minus beta cos theta whole square and sin square theta, and this is to be divided by let us go back and look into what be the beta, we have written beta, this psi dot we have defined like this. So, alpha minus cos beta divided by I sin square theta. So, I sin square theta whole square. And this last term it can be simplified sin square theta is appearing here, so this will be sin to the power 4 sin theta to the power 4. So, we can cancel it, and rewrite this term.

(Refer Slide Time: 31:09)



So, your theta dot square, then gets reduced to 1 by I times 2 E minus beta square divided by I 0 minus 2 m g c cos theta minus alpha minus beta cos square theta cos theta, so this whole square, and then I square sin square theta ok. So, we get here, I square sin square theta, so that means, this can be written together 1 by I 2 E minus beta square by I 0 minus 2 m g times c cos theta alpha minus beta cos theta divided by I sin theta whole square.

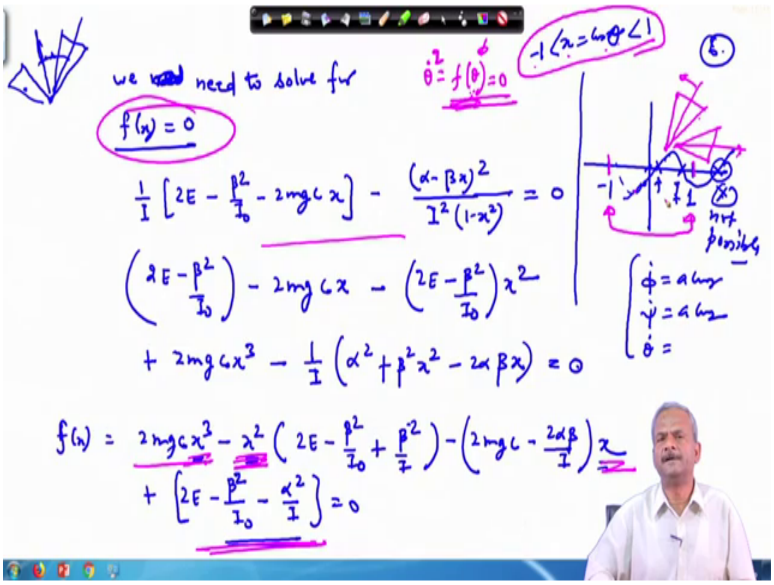
So, look here in this, this theta dot square this is a function of f theta. So, integrate it, and solve it. So, theta dot this will be under root f theta plus minus or d theta by d t f theta under root. And therefore, this is the normal thing we can write, but this equation is little more complicated plus a constant ok.

So, to solve this problem, we need to apply some more trick ok. So, put here cos theta equal to x, and so therefore sin theta square will be 1 minus x square ok. So, the above expression if we write left hand side as f x, so this expression can be written as f x equal to 1 by I times 2 E minus beta square I 0 minus 2 m g c times x minus alpha minus beta x by I sin theta, so I times 1 minus what we can write it separately. This is I square, and sin square theta, so sin square theta will be 1 minus x square. So, this is the equation, and this equation we need to solve ok.

So, if you look into the right hand side, this particular part, this is only a function of theta ok, so your right hand side which is a function of theta, we can write here the write as expression f theta equal to this quantity, which we have written right now in terms of x. Now, if we plot this if we plot here say the f x or f theta, and theta we plot on this axis ok. So, you will get certain kind of curve. So, this curve it crosses here, the x-axis in different places. So, in this places f theta will be equal to 0 ok.

So, in all the places f theta is 0. So, wherever the right hand side only is only a function of theta, so this gives you once you solve the right hand side, so what will be the maximum possible value of the theta, you can get from this place. Here this is a maximum possible, and this is the minimum possible value. So, we need not we do not need to integrate this part to solve it, rather we just take the this part and solve it ok.

(Refer Slide Time: 36:35)



So, from this place we have so f x, we need to solve for; we need to solve for f x equal to 0. So, your f x write that equation 1 by I times 2 E minus beta square divided by I 0 minus 2 m g c x minus alpha minus beta times x square divided by I square times 1 minus x square, this quantity is set to 0 to solve for the values of x ok.

So, I will take a shortcut here, you can check yourself. So, this can be rearranged as 2 E minus beta square divided by I 0 c times x minus 2 E minus beta square divided by I 0 times x square, which can be written as f x equal to 2 m g c x cube minus x square beta square divided by I ok, you can expand and check it. Beta square divided by I 0 minus alpha square divided by I ok.

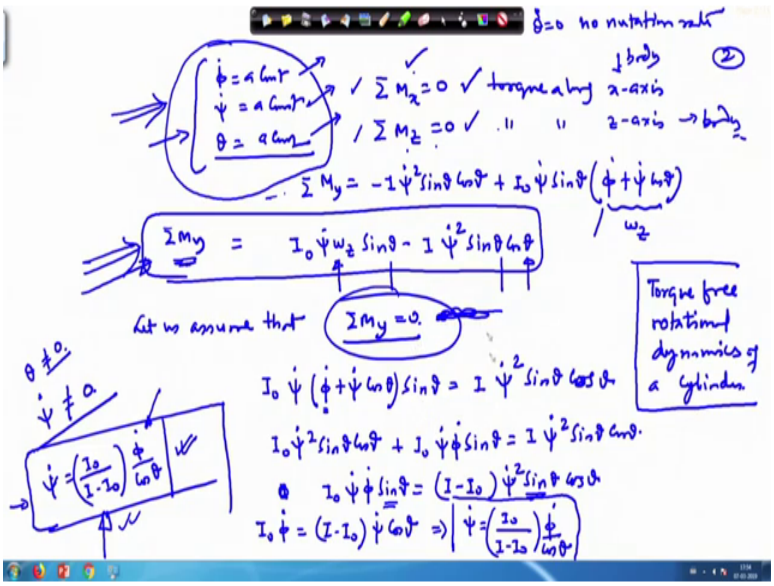
So, with this you can see that this equation is of third degree ok, this is a third order polynomial. This is x cube, x square, this is x, and this is a constant, so it is a third order polynomial ok. And once you try to solve this, so because x we have written as cos theta. So, this is bound to lie between 1 and minus 1 ok, x equal to cos theta. So, it is necessary that the solution lies between 1 and minus 1 cos theta cannot be greater than 1, and it cannot be less than minus 1, here this is a minus 1. So, this is 0, on 1 on this side, and minus 1 on this side, so solve this equation.

If your equation looks like this, suppose assuming that this equation looks like this, so you get the solution 1, 2. And this will be rejected, because this is not possible not possible, it is a not the value of cos theta, beyond 1 it is a not possible. This is plus 1, and here it is a minus 1. So, these are the two solutions. So, if this happens, then this is the maximum value, and this is the minimum value.

Here it is a going on this axis, suppose that it comes here on this side, so this will be on the negative side. So, either on this is on the negative side, your top is on this side or either this side, it does not matter. How much from the vertical the displacement is that is important. So, these two values will be than important to (Refer Time: 40:55). So, based on this we will solve the problem which we have stated earlier, we will assign certain values, and we will try to look into that problem.

Now, overall what we have done that we have started with the case, where phi dot equal to a constant, and psi dot this was a constant ok, and theta dot this was 0, but later on because we are referring to this case, where theta is also a variable. So, for this case we relaxed it. So, again we go back just once, so we have completed this part. We go back and just review the whole thing, so that and thereafter, we wind up this particular topic ok.

(Refer Slide Time: 42:05)

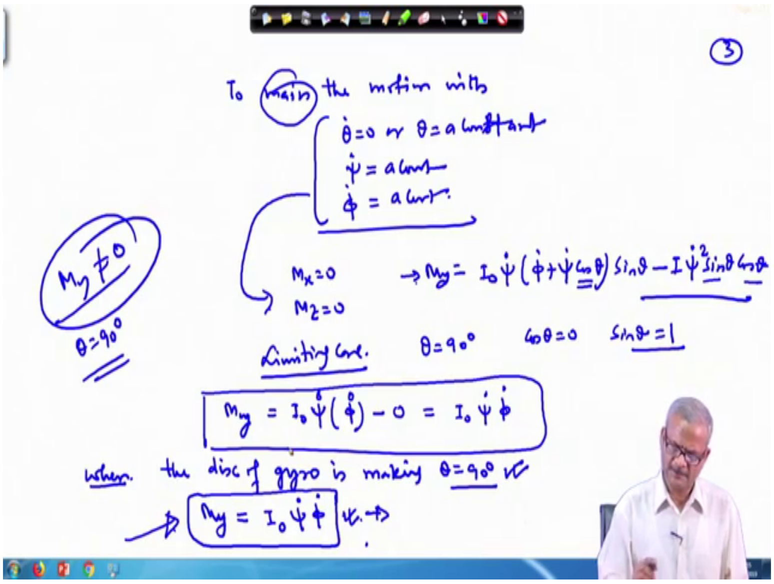


So, we have started with in the 45th lecture by taking a gyroscope solve this the we wrote the equation of the gyroscope already, then we assumed that phi dot phi dot is constant, psi dot is constant, and theta is a constant, so that give me the theta dot turned out to be a constant, and that imply that ok.

So, this imply that M x will be 0, and M z will be 0, along the body axis x body axis, and the z body axis ok. So, only the torque will be acting along the y-axis ok. So, the torque along the y-axis is given by this equation ok. Thereafter, what we have assumed, let us assume that the M y is also 0. So, this gets reduced to a problem, where this is torque free situation ok, which we started with in the rigid body dynamics.

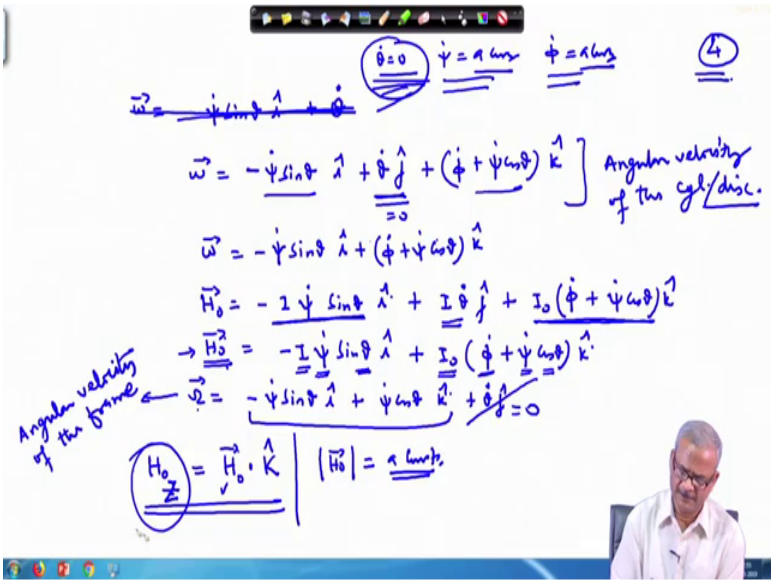
And then we solved for this, so from there we got the rigid body dynamics this precession rate, how it is a connected with the spin rate. And this we have done already, so there is no need of long back we have done this. So, no need to discuss this part ok. So, we recovered the torque free rigid body dynamics from this place.

(Refer Slide Time: 43:25)



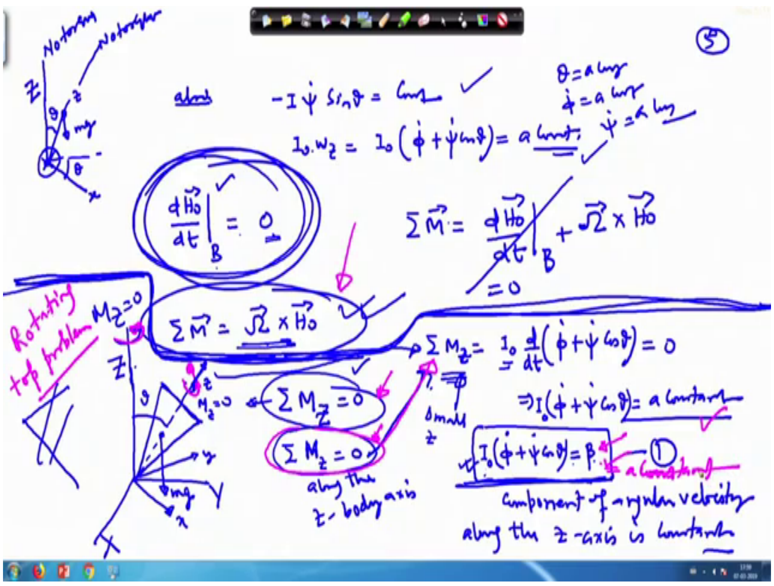
Thereafter, so this what we have written here, I cannot read myself ok. Then we will looked into the limiting condition that when theta equal to 0, how your the torque will be described then, so it gets a lot simplified in that case ok. So, M y is present M y is non-zero, and in that case we are taking theta equal to 90 degree. So, this is the equation that we got ok. So, this topic got over in that place ok.

(Refer Slide Time: 44:17)



Thereafter, we were also interested in some more things, we have we workout and written that context that theta equal to 0, psi dot equal to constant, and phi dot this is a constant. So, how your omega is given, and capital omega this gets reduced from there we have got H 0 z. So, these are few of the derivations, you can always work out. It is a always good to do in various ways the same problem, so that it exposes it to the different techniques ok.

(Refer Slide Time: 44:45)



So, from there we have got this conditions ok. So, what we looked at this is the d H 0 by d t with respect to the body axis that gets reduced to 0 in this case. In that case, where your theta is a constant phi dot and psi dot, they are constant. So, only this part was remaining ok.

So, from there we derived this condition ok, but this condition is also applicable to the case of the torque why, because if we look here in the case of the torque M z means, the torque about this axis this is 0. Your M capital z, this is 0 about this axis ok, we can change the colour here. So, this torque is 0, and also the torque about this one this is 0 ok. So, this is going from the top.

So, this part also M small z, this also turns out to be here in this case 0 ok. So, for that case we are trying to solve this problem ok. So, we are started you can demark it here that we can separate out from this place, it may appear to be the things have got mixed in, but this is a separate problem, which was start from this place. So, here e z this is a small z, this is a small z small z ok. So, M z this quantity, which we have shown here, this particular part this is shown here in this place.

So, we are setting this to 0, because there is a no torque ok. So, the torque equation along the z direction from there, we are getting a constant ok. So, this is set to then this is said to beta. So, you can see that we are not taking off the upper part, this is a fresh part, we have written here as beta which this is a new problem. This the top problem basically, this is the rotating top problem ok. So, all together this is a different issue from this place.

And this things just I tried to show you that how the things will look like under the same thing can be looked from the different perspective ok. So, here under these two conditions, we are trying to work out the things. So, this condition gives me this equation from where we are getting I 0, this equal to beta, this we are writing as a constant ok.

Then we again here this part M Z, this equal to 0 means, about this axis M Z, this equal to 0. So, this implies that H 0 capital Z, this quantity will be a constant ok. And therefore, we have worked out all this things, it is a easy to do using one method, and another method also we have to done the same problem by another method. So, we get H capital Z ok. And this quantity is also a constant, because there is no torque along that axis ok.

So, thereafter, we defined so from our earlier equation, so this quantity is you we are writing as this as beta ok. And therefore, your and this quantity you are writing as alpha. So, alpha we have written as alpha equal to I times psi dot sin square theta plus beta times cos theta. And this we rearranged to get psi dot equal to alpha minus beta cos theta divided by I times sin square theta ok.

So, once we got this equation, so on the next page it is shown here in this place ok, where beta is this quantity. And then we went into the kinetic energy of the system. So, we have written the kinetic energy equation here ok. And thereafter, we added the we considered the conservative case, because the here the total energy of the system is a conserved. So, we added the potential part here. So, this is a constant ok.

Then we got the potential term which is m g times c cos theta ok. So, here we have added that part, this is a potential part added here T plus V, this is the V part here ok. And we rearrange that equation, in terms of theta dot. And we tried to by putting the psi dot part, and omega z part here ok. In terms of the beta here using this informations we tried to put this equation in terms of theta dot, we tried to put in terms of theta only.

Everywhere, this theta is manifesting, alpha and beta they are constant. So, there is no other term, beta is a constant, E is a constant. So, this just become a function of theta dot, so theta dot square this is a function of theta. So, the right hand side, because it is a polynomial in theta, so it will have certain values as shown here in this place ok.

So, we sorted this problem more elegantly by putting cos theta equal to x sin square theta equal to, therefore 1 minus x square. We inserted here in this equation, put it here in this format. Thereafter, we rearranged it and we solved f x equal to 0 ok, because we are looking for on the right hand side theta dot equal to square equal to f theta ok.

So, what will be the maximum value of theta, because this is a polynomial. So, and theta already you know that, it will be cos theta will be limited in this range, but what that range will be? So, how do we get to know that? So, if we solve for the say f theta equal to 0 at the extreme position, these are the extreme positions, where your either the torque is here. So, no longer the theta dot will be there, if you go on the further down. So, here also this is extreme position, no longer theta dot will be there.

So, we solved for that and once we solved for that, we get the equation in this format which simplify it little bit, and put it in a proper format, so that it appears as a cubic equation x cube x square that is the x term, and this is a constant term. So, this cubic equation thereafter, we argued that because the cos theta if x equal to cos theta, this will always lie between minus 1 and 1. So, we put here minus 1, 1 here in this place. And only the solution lying here in this range from here to here, they will be acceptable, other solution will not be acceptable. So, if this curve cuts in other places, so those solution need to be rejected, only these two solution to be accepted ok.

So, we will continue in the next lecture, we will stop here. So, we will do one numerical problem on this. And thereafter, this top problem will be over, then we can go into the reaction wheel and control moment gyros, how they are used to actuate the satellite, how the satellite equation changes.

If the wheel is rotating inside, and wheel is speeding of means its accelerating or the gyroscope itself the control moment gyros already you have looked into, but the gyroscope you have already looked into, but that the frame weights, we assumed it to be the frame mass we assumed it to be 0, and therefore, it is a moments of inertia were 0 ok, so that was a simplified condition. No longer we will assume that ok, because that is never true in reality, and it is impossible ok, you cannot make it mass less.

So, once we put that kind of system where the frames are also having weight ok, then there will be motor actuating the frames ok. Those things need to be accounted, then it is a house inside the satellite, then the satellite is itself rotating. So, the whole equation gets very complex and that equation then you need to solve it, so in many cases we cannot solve it, but numerically you can work out and stability of the system you can show, by if you use the properly a (Refer Time: 54:45) no function. So, it will be difficult for us to go into quite details for the (Refer Time: 54:50) function, but as much possible I will try to give you the present here in this class ok.

Thank you very much.