Satellite Attitude Dynamics and Control

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Lecture – 04
Kinematics of Rotation (Contd.)

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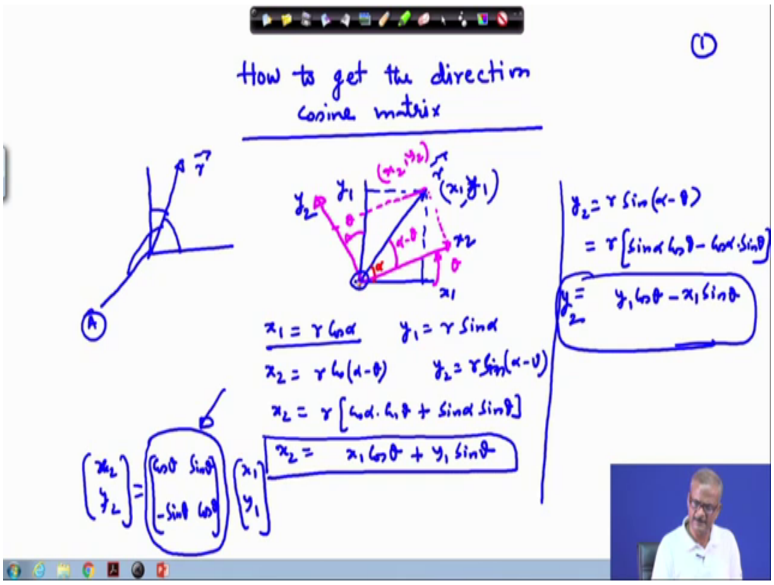

So if what we have observed that if we have this A friend and here this is any vector r. So, if this direction cosines are not directly available. In general, what happens that if we look into say if this is the mobile and if we indicate x axis along this direction, y axis along this direction and z axis along this direction so, taking a right hand triad.

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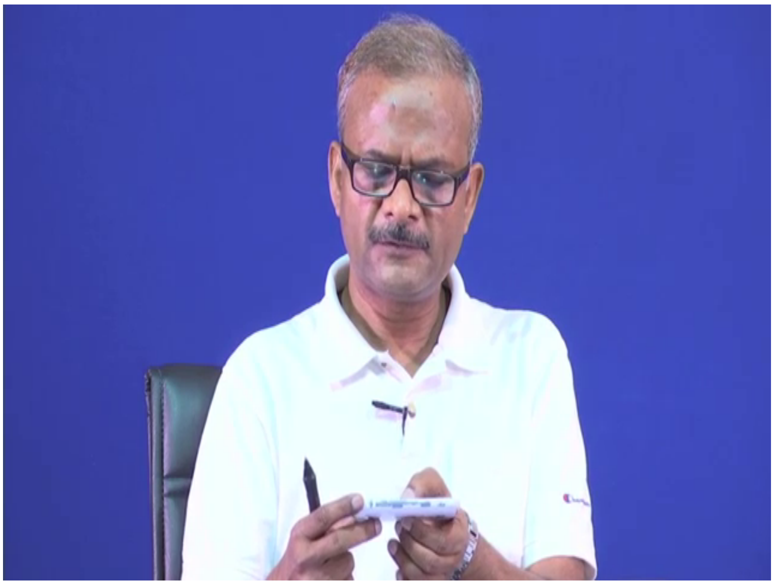

So, for this kind of system generally, the orientation of a body it can be represented in terms of 3 angular rotation. That is one along this direction along the x axis, one along the y axis another along the z axis.

So, this direction cosines in many of the cases will not be available to you, but the it will be known to you along which axis the body has been given rotation. And from there then you can derive the direction cosines which forms the in terms of which you form the rotation matrix. So, let us go into this from the various scratch ok. say this one is x 1, x 2 and there is a vector r whose coordinates are x 1 let us make this as y 1 x 1 y 1 ok.

So, in this coordinates frame x 1 y 1 except coordinates are x 1 and y 1. Now, if we rotate this frame by say some angle. So, let us make this angle as this is alpha. And if we rotate this by theta angle, and this goes from this place to this place. So, this is your theta angle this is theta angle ok. So, we consider this angle will be alpha minus theta. So, in the new frame your coordinates of this vector will be indicated by x 2 y 2.

So, x 2, y 2 are the coordinates of the same vector if we rotate the reference frame. And this we can write as x 1 equal to r sin alpha and a cos alpha. So, we have x 1 equal to this is the r vector. So, it is a component will be r cos alpha. And similarly y 1 will be r sin alpha. In the new system if you look for the coordinates, then in the new system x 2 here this component will be r which is the magnitude of this vector times cos alpha minus theta. And similarly y 2 will be r cos alpha sin r sin alpha minus theta ok expand this so, it will be expanded.

So, this will be r cos alpha times cos theta plus sin alpha times sin theta. This is your x 2 and r cos alpha; obviously, from this place this is x 1. So, we write as x 1 cos theta plus r sin alpha which is y 1. So, y 1 sin theta so, this is your x 2.

Similarly, if you look into the y 2, this is r sin alpha minus theta r sin alpha into cos theta minus cos alpha into sin theta. And r sin alpha is y 1. So, here you have y 1 cos theta minus r cos alpha. So, r cos alpha is x 1 so, this becomes from sin theta. So, this is your y 2 ok.

Now, if you combine this together. So, x 2 y 2 in the matrix format it can be written as x 1 y 1. So, just say from this place x 1 cos theta. So, put here cos theta and y 1 sin theta so, this is sin theta. And from this place you have y 2 equal to x 1 sin theta with minus sign. So, this is minus sin theta and the y 1 cos theta. So, cos theta comes here in this place so, basically this represents a rotation matrix which indicates the rotation of this reference frame x 1, y 1 to x 2, y 2 positions.

So, this is your x 2 and y 2 position. So; obviously, if the question is where we have rotated? So, we have rotated about a vertical axis here axis coming out of the page. Which is the xyz taking the right hand rule. So, xy this is a right hand triad. So, z is coming out of the page. So, therefore, this rotation has been given about the z axis and you can see that the z axis is remaining intact.

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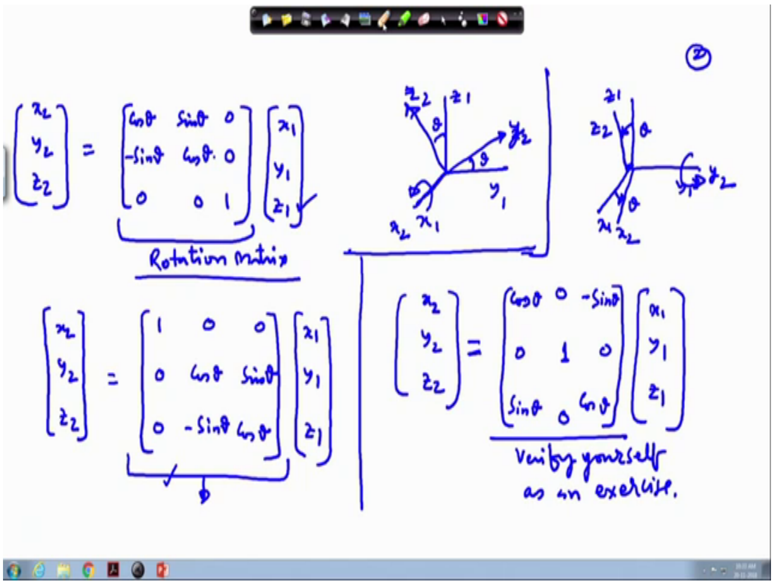

So, now if we want to write the same thing, in terms of x y and z because z axis has not been changed. So, we can write this as z 1 y 1 x 1. So, this axis is not changed it remains intact. So, you considered z 2 equal to z 1 there is no change in that because you are giving rotation about that axis. And rest other just you copy it from the previous page. This is a cos theta sin theta minus sin theta and here this is cos theta. So, this is a rotation matrix which gives you rotation about the z axis by theta angle in anticlockwise direction ok. Here we have taken the right hand triad ok.

In the same way you can give rotation about the say you choose to give rotation about the z axis. So, we have already given the rotation about the z axis. Now, we give rotation about the x axis. So, if we give rotation about the x axis means this is a x y and z. So, this is the y you are rotating about the x axis by theta angle. So, what will happen this y will go to this position and this will go to this position.

So, this is x 1 y 1 z 1 so, this will be x 2 this is y 2 this will be z 2 and here x 2 will be along the same because you are you are giving rotation about this. So, this same logic you have to apply and if you do so, here because in the x axis which remains unaltered.

So, this will be put here in this format and here you have x 1 y 1 z 1 ok. So, if a 1 0 0 and rest you need to copy this cos theta cos theta sin theta and minus sin theta. So, and here we get x 2 y 2 and z 2. So, if this is the rotation matrix, which rotates about the x axis. And this you can verify it by writing the same expression what equation we have written out here you can arrive at result. Then the next question is if we give rotation about the y axis ok.

So, now this is x 1 y 1 z 1 and we rotate about the y axis. So, we rotate like this. So, we are rotating like this. So, if we rotate this will go from this position to this position and this will go down ok. So, here this angle is theta this angle is theta if we are rotating by theta angle, we consider y 2 remains along the same direction this become z 2 and here becomes x 2. Now, here little change you can observe as you use the equations developed earlier.

And what you will see that x 2 y 2 z 2 x 1 y 1 z 1 this will be equal to here we are rotated about the y axis. So, y axis remains unaltered. You can see here y 2 equal to y 1 and the rest other things cos theta cos theta. Here one changes here this is minus sin theta will come here in this place and sin theta will go here in this place ok.

And this you verify yourself as an exercise so, this rotation matrix. So, x and z rotation about the x axis and the z axis, they will look similar to each other. But rotation about the y axis it is a little different because the sign here you can say that plus sign is here minus sign is here plus sign here minus sign here in this case, it gets flipped. Minus sign comes here and plus sign comes here in this case.

So, this is about the rotation of the coordinate system. What happens if we rotate the vector? What happens if we rotate the vector?

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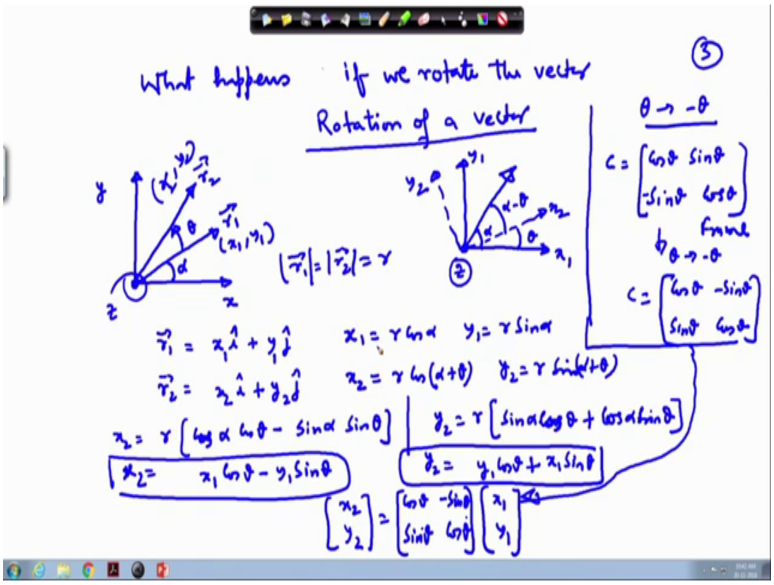

So, this will come under rotation of vector. So, this is our x y plane and let us say this is vector r 1, this is vector r 2. So, this vector is rotated from this position to this position. Instead of rotating the this frame x y frame we are rotating the vector from one position to another position.

One thing is very obvious, that if you look into the previous work we have done. So, if you are rotating the frame in the anticlockwise direction like this ok. So, you are taking the frame from this position this is x y x 1 y 1 to x 2 y 2. So, we can see that with respect to the frame the this angle is alpha is large now this angle if you are rotated by theta. So, this has become alpha minus theta.

So, rotation of the reference frame x 1 y 1 in anticlockwise direction by theta is equivalent to rotation of the vector in the clockwise direction. This is very obvious from here. Let us draw these conclusion, if we look into this figure. So, r 1 vector has rotated to r 2 position has r 2 vector by theta angle. So, magnitude remains same only the direction changes. And r 1 it is a coordinate this r 1 vector coordinate.

Obviously a (Refer Time: 14:42) if you like you can write it like this in terms of x 1 y 1, r 2 similarly this will be x 2 because, now here in this case coordinate system you are not changing. You can also write in this way, but if you so, if a here as if we work in the previous way. So, x 1 can be written as now r 1 magnitude this equal to r 2. So, this we will write as r. So, this becomes x 1 equal to r cos alpha and y 1 equal to r sin alpha as usual. and. Similarly, x 2 this coordinates of this. So, here you have the coordinates x 1 y 1 and it is coordinate is x 2 y 2, but in the same frame we are integrating. So, x 2 becomes r cos alpha plus theta. And y 2 is now r sin alpha plus theta.

So, if we expand this x 2 becomes r cos alpha times cos theta minus sin alpha times sin theta. And similarly the y 2 will be r sin alpha times cos theta, plus cos alpha times sin theta. Now replace with the corresponding value r cos alpha is x 1. So, x 1 cos theta minus r sin alpha is y 1. So, y 1 sin theta so, this becomes your x 2.

And y 2 then results in r sin alpha, which is y 1 cos theta plus r cos alpha which is x 1 sin theta. So, this is your x 2 and y 2 in terms of x 1 and y 1. Now, if we write the same thing in matrix notation. So, this can be written as. So, from this place this is cos theta x 2 equal to x 1 cos theta and y 1 here comes with minus sign. So, this is minus sin theta and this will this sin theta you can look from this place x 1 sin theta and here cos theta.

So, what we observe that here in this case rotation we have given about the z axis. So, z axis is here coming out of the figure. You are rotating about this one z axis it is a coming out, but the sign of this has got flipped. In the case once we rotated the frame if we rotate the frame. So, in that case what we have got that this sign was plus and this sign was minus. Here what we are getting if we are rotating the vector that this sign is minus and this sign is turning out of the plus 1.

So; that means, the sign has flipped and we can observe this as I told you that noting that that if we are rotating the x 1 y 1 frame here, in the anticlockwise direction. So, that is equivalent to rotation of the vector in the clockwise direction that is in the negative direction so; that means, if you want to write in terms of the vector notation. So, theta you have to replace in terms of minus theta in the frame rotation equations. So, for the frame rotation equation your c matrix was the rotation matrix 1st cos theta this is sin theta minus sin theta and here cos theta. So, this is for frame rotation.

Now, if you replace this in by theta by minus theta. So, what we will get, we will get c equal to cos theta will remain unaltered ok. Sin theta will get affected. So, sin theta this will become minus sin theta and this will become plus sin theta. So, we can see that this and this matrix they are the same. So, it implies that the rotation of a matrix here this frame in counter clockwise direction is equivalent to rotation of a vector in the clockwise direction.

So, for this you can think another way that if you are sitting on the frame ok, and if there is a vector which is remaining fixed in the direction. So, if a say if this vector is stick sticking to the page and the, but the frame is rotating and you are sitting on the frame. So, what you will see that the vector is rotating in the clockwise direction as the frame rotates in the anticlockwise direction. Or the other way if you are sitting on this vector which is fix to the page and the frame is rotating.

So, what you will see that the frame is rotating in the anticlockwise direction. So, it is a just a matter of perspective that which way you look and both are. In fact, equivalent to each other, but once you use any one of them one of the representation you have to be fix to that. And so, you have to carry it throughout your computation otherwise it will create problem.

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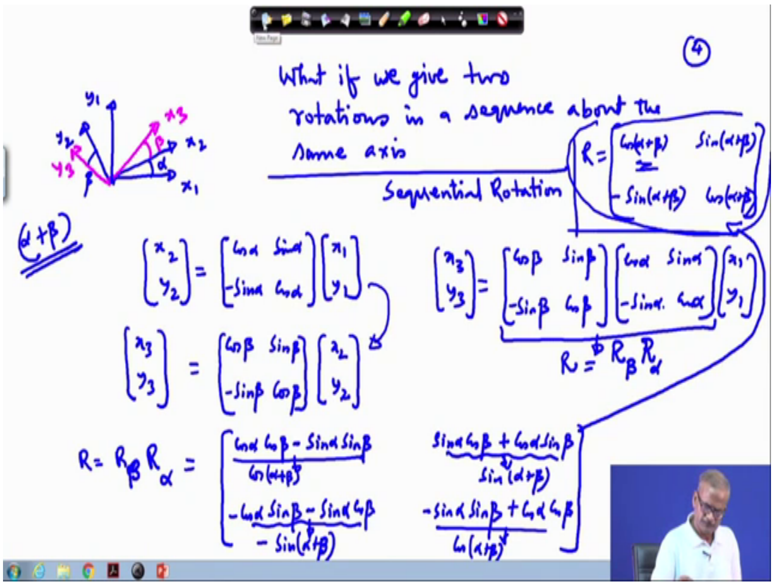

Next we look into what if we give 2 rotations in a sequence about the same axis. So, this is called basically the sequential rotation. So, say we have this frame this is x 1 and this is y 1. First you are rotating this frame from x 1 to x 2 and this gets y 2 and thereafter to give another rotation to x 3. And then this is y 3.

So, as we know the x 2 y 2 the first rotation let us say that we are given it by alpha. And the next rotation we are giving it by beta. So, x 2 y 2 this will be equal to cos alpha cos alpha sin alpha minus sin alpha x 1 y 1 ok. Now, this coordinate then you are forward rotating by beta angle. So, this is beta so; that means, the x 2, y 2 this will be rotated to x

3 y 3 and that you are doing by beta angle. So, here you will have sin beta because we are rotating the frame. Now, if we write x 3 y 3 in terms of x 1 y 1. So, here x 2 y 2 just replace using the upward one just insert this into this one. So, this will be cos beta sin beta cos beta and times cos alpha sin alpha minus cos alpha.

Now, simplify this matrix. So, let us say this matrix is r. Beta first rotation we are giving by alpha then write beta. So, first alpha comes and then beta comes. So, these are the rotation matrices which operate on this vector x 1 y 1 ok. And if we represent this as r so, your r equal to r beta times r alpha multiply this. So, take the do the matrix operation. So, this becomes cos alpha cos beta, sin beta minus sin alpha times sin beta. Then sin alpha cos beta, then plus cos alpha sin beta taking this one.

So, cos alpha this is sin beta, with minus sign. And this is minus sin alpha and then cos beta. And this one then gets reduced to minus sin alpha times sin beta. Minus sin alpha times sin beta and cos alpha cos alpha plus cos beta cos alpha times sin beta and then sin alpha times cos beta.

So, this matrix I will write here in this place. What is this quantity? This quantity is nothing but your cos alpha cos beta minus sin alpha sin beta. So, this is cos alpha minus beta alpha plus beta. So, this is the 1st one is cos alpha plus beta this one is sin alpha cos beta cos alpha sin beta this is sin alpha plus beta. Look into this minus sign we can take outside. So, this is cos alpha sin beta sin alpha cos beta. So, this is with minus sin alpha plus beta and here this one this is cos alpha cos beta minus sin alpha sin beta so, this is cos alpha plus beta.

So, this one is your cos alpha plus beta. And this one is your minus sin alpha plus beta and this one is sin alpha plus beta. So, what we can see that this rotation matrix if you look into this. So, this is nothing but a rotation about the z axis. So, from this rotation indicates that if we give 2 sequential rotation alpha and beta in 2 sequences. So, that is equivalent to one single rotation of angle alpha plus beta.

So, basically this 2 this 2 rotations in sequence by alpha and beta is equivalent to 1 single rotation of magnitude alpha plus beta. This is for a here from these operations.

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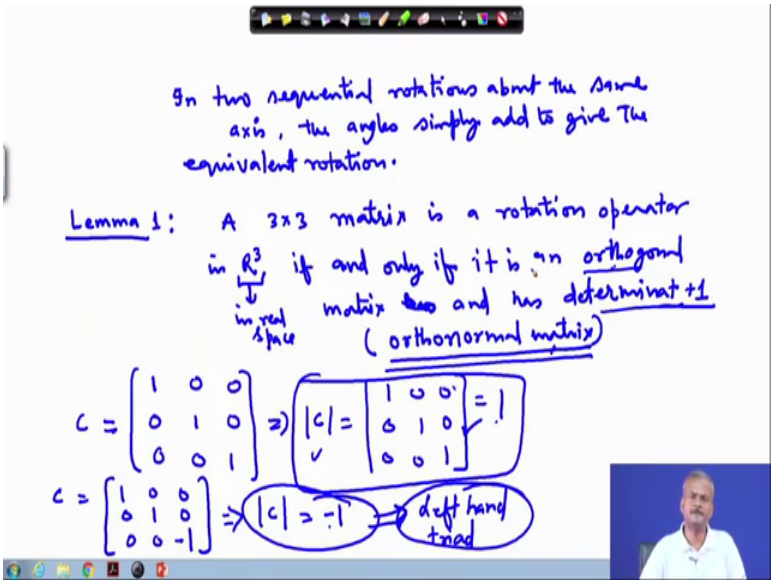

So, this implies that into sequential rotations about the same axis. The angle simply add to give the equivalent rotation. So, one can we can a state a lemma with 3 into 3 matrix is a rotation operator in r 3 so, this implies in real space ok. Within the real space, if and only if an orthogonal matrix and matrix and has determinant plus 1.

So, this basically implies that you are dealing with the this orthonormal if you have an orthonormal matrix say 1 1 1 this is your rotation matrix let us assume. So, what will be it is determinant 1 0 0 0 1 0 0 0 1 so, this is equal to 1 ok. So, determinant of this matrix is 1. On the other hand, if we take a matrix like this 1 0 0 0 1 0 0 0 minus 1. So, we consider this determinant of this will be minus 1.

So, this determinant here minus 1 is not acceptable. This belongs to left hand triad. So, it does not belong to the system which we are working with ok. So, we have taken a right hand triad and this belongs to the right hand triad. So, here orthonormal matrix with it so, it is a orthonormal matrix and has determinant 1 this ensure that you get this orthonormal matrix of this form.

So, if a with this we will conclude this chapter. And next time we will look into the proof of this lemma it is a small proof and we will we will continue further, once we finish the kinematics portion then we will go into the satellite attitude dynamics so.

Thank you for your time. We will start with the next lecture.