Satellite Attitude Dynamics and Control
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Lecture – 37
Gravity Gradient Satellite (Contd.)

Welcome to the lecture number 37. Last time we have been discussing about the satellite pitching motion in elliptic orbit.

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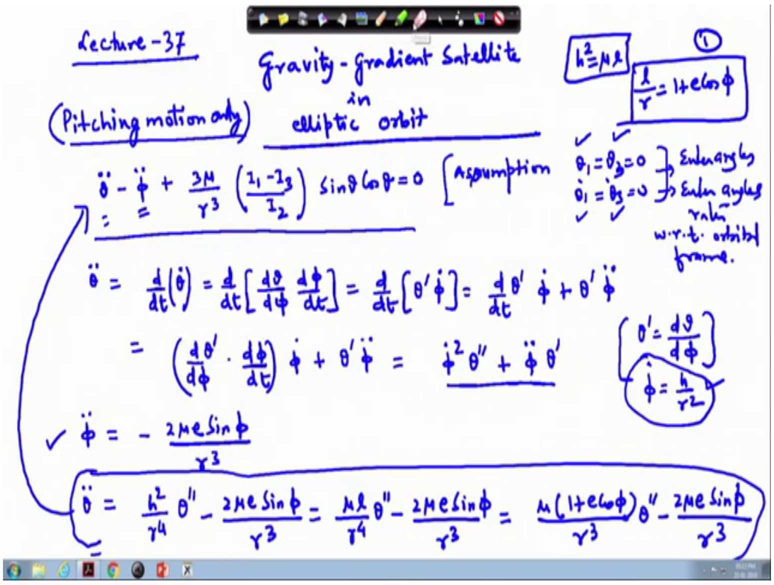

So, in that context we derived the equation which is shown here. So, this is only pitching motion equation and the assumption was that theta 1 and theta 2, these are 0 and theta 1 dot and theta 3 dot. These are also 0. So, these are the Euler angles and Euler angles rates with respect to the orbital frame. And then so, in this context there after we derive that, how much this theta double dot will be? So, theta double dot we have derived using the assumption that this, we can write as d by dt theta dot.

And then we have described this with respect to the d by dt equal to sorry d by dt operates on theta dot. So, this theta dot we are writing as d by d phi and d phi by dt ok. So, d by dt and d phi by dt we have written as d theta by d phi we written as theta prime and this part we have written as phi dot and there after we differentiated this again. So, this was d by dt theta prime times phi dot plus theta prime times phi double dot and this we have written as d by d phi d theta prime times d phi by dt and then this is a phi dot and plus theta prime times phi double dot. And of course, this quantity is phi dot square times theta double prime plus phi double dot times theta prime where we have use the notation theta prime equal to d theta by d phi and here phi dot is the quantity which we have written as h by r square.

So, under this then we derived other things. So, I will not go again into the same details. So, phi double dot we have derived as minus 2 mu e sin phi by r cube and theta double prime thereafter, we have worked out using now is a phi double dot we have to insert and phi dot from this equations. So, phi double dot is available phi dot is available. So, your theta double dot then we have so, theta double dot we derived it to be h square divided by or say here phi dot is there. So, we can write here from this place using this equation we can write h square by r to the power 4.

And theta double prime plus phi double dot from this place. So, this is plus so, here this will come with a minus sign. So, this is minus 2 mu e sin phi divided by r cube and then we rewrote this quantity. So, h square is nothing, but mu times l as we have discussed last time h square this equal to mu times l. So, we use this information divided by r to the power 4 theta double prime minus 2 mu e sin phi divided by r cube. Then we have rewritten this part.

So, we are use the information that l by r equal to 1 plus e cos phi these are information also we have used. So, using this information this becomes mu times 1 plus e cos phi divided by r cube times theta double prime minus 2 mu e sin phi divided by r cube. And now there after we are using this information, this particular information inserting this in this equation along with the values of.

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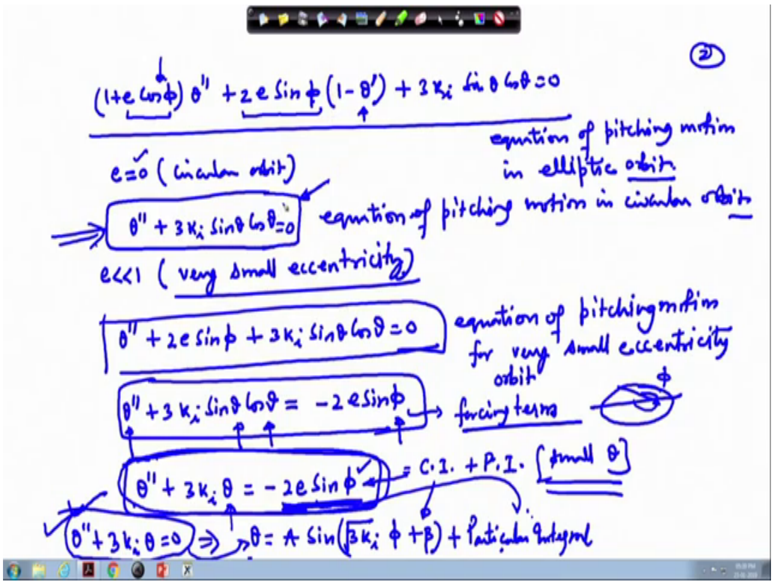

So, we reduce the equation. So, I am writing you can again insert it and check it. So, r q from the denominator it gets eliminated and we get the equation 1 plus e cos phi times theta double prime plus 2 e sin phi times 1 minus theta prime plus 3 k i sin theta time cos theta this equal to 0. So, that this is the equation of pitching motion in elliptic orbit and let say solution we can obtained for certain assumption. So, if e equal to 0 or almost let say equal to 0 means I it is a circular orbit basically. So, for circular orbit this will get reduced to theta double prime plus 3 k i sin theta times cos theta.

So, and this is the equation of motion equation of pitching, motion in circular orbit and there is no assumption this equal to 0; mostly this quantity equal to this equal to 0. While we take a small if e is less than quite less than 1, but not near nearly equal to 0. So, it is a very small value this is small eccentricity of the orbit. So, in that case with respect to 1, this quantity e cos phi it can be neglected because cos phi can never be greater than 1. So, we can neglect that quantity and this gets that the first term gets say these 2 theta double prime.

Similarly the term which is present here theta prime this quantity is already is small e sin phi this is already a small, but once you multiply it by theta prime. So, theta prime which is small quantity. So, we can re write this term as 2 e sin phi and this can be ignored as compared to 1 and the other term we can write as 3 k i sin theta times cos theta and this equal to 0. So, this is the equation of motion pitching motion for very small eccentricity orbit and this we can re-write as. So, if you look on left hand side, all this theta terms are appearing and this is theta double prime on the right hand side this is the phi. So, minus 2 is sin phi term. So, this appears as a forcing term.

You must be aware of the first motion. General solution to this can be written if the sin theta cos theta is present so, it will be difficult to solve it. But if this quantity we approximate 3 k i and cos theta equal to 1 and theta equal to we write it theta means you are taking a small oscillation. So, in that case you can write it like this; remember that phi can vary between 0 and 2 phi means you have the orbit here and phi is being measured from this position. So, this is your phi. So, orbit can it can go along all along from this place to this place and thereafter it can multiple of 2; multiple of 2 phi it can keep changing. So, this is a quantity where sin phi we cannot just approximate as phi but if we assume theta to be a small.

So, this is this is the way we can approximate it and in this case the equation becomes easy to solve ok. So, for this part we know the if we have only theta double prime plus 3 k i theta equal to 0. So, we know that this is the harmonic motion equation and for this the solution we can write as theta equal to A sin 3 k i under root times phi plus beta. So, this is the solution to this equation, but here on the right hand side we have this extra term. So, for this we will have a part which we call as the force or we call this as the particular integral. So, here I will write this in terms of mathematical notation is called the particular integral.

So, that arises for this part ok. So, we have for this one only this solution is there and for this part, this solution appears. So, the whole thing the complete solution to this one we write this as the complementary integration plus particular integral complementary integral plus particular integral. So, this is your part which goes as the complementary integral which is the solution of this equation while particular integral because of the presence of the term this particular term this particular integral we get. So, we write it on the next page.

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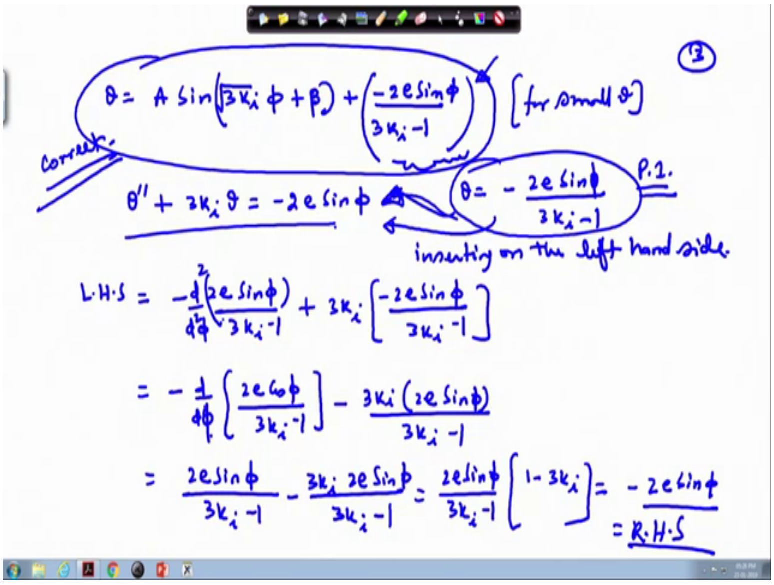

So, we have theta equal to A sin 3 k i under root then we have of course, phi and plus beta. So, this is our complementary part and then for the integral part in the denominator we will have 3 k i minus 1 and in the numerator this will be 2 e sin phi. So, here instead of plus sign we the minus sign if you look on this side so, here the minus sign appears. So, in the solution also you will have minus sign here in this place.

So, this is the solution I am writing here plus you can remove this bracket and put here minus sign. So, let us check that whether this solution the particular integral we have written here in this place is correct or not. So, our equation of motion is theta double prime plus 3 k i so; this is for a small theta. So, our equation is theta double prime plus 3 k i theta this equal to minus 2 e sin phi this is minus 2 e sin phi ok. If we put suppose this is the particular integral so, it must satisfy this equation.

So, we if we insert it here in this place so, putting theta equal to minus 2 e sin phi divided by 3 k i minus 1 in this equation; so, on the left hand side ok. So, we will insert it inserting on the left hand side inserting on the left hand side. So, what we can see the LHS is then equal to minus 2 e sin phi will be cos phi so, d by d phi. So, the first differentiation or either let us write it in proper way 2 e sin phi divided by 3 k i minus 1 plus 3 k i and theta from this place is minus 2 e sin phi divided by 3 k i minus 1 d by d phi.

And, then this gets into the form d 2 e cos phi divided 3 k i minus 1 plus 3 k i and then we can put the bring the minus sign here. So, minus here 2 e sin phi 3 k i minus 1, again differentiate this quantity. So, cos phi becomes minus sin phi so, we get a plus sign here. So, this is 2 e sin phi 3 k i minus 1 plus 3 k i 2 e sin phi by 3 k i minus 1 ok. So, we can take 2 e sin phi outside and also 3 k i minus 1. So, what we get inside is here this is minus sign. So, we get here 1 minus 3 k i and this is nothing, but equal to minus 2 e sin phi means and this is your right hand side.

So, its shows that theta equal to this as the particular integral satisfies your original equation ok. It satisfies this equation and therefore, my solution is whatever the solution we have written here is correct. So, this is correct.

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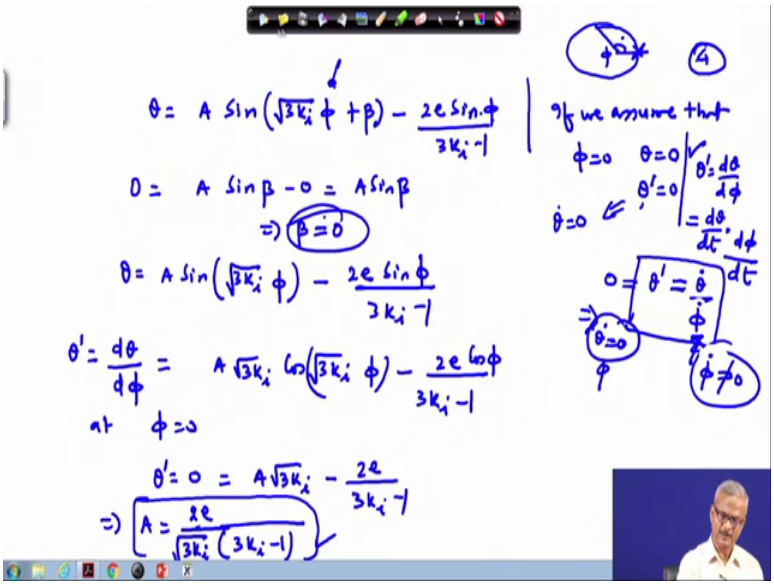

Now, from so, this equation we need to work it little bit more we will consider other cases, but before this let us take this particular equation and work it out. So, we have theta equal to a times sin 3 k i under root times phi phi plus beta and minus 2 e sin phi the minus 2 e sin phi divided by 3 k i minus 1.

So, this is the equation 3 k i under root phi plus beta minus 2 e sin phi divided by 3 k i minus 1. Now if we assume that phi equal to 0 theta equal to 0 and also theta prime equal to 0 means right in the beginning, when your see the phi is your true anomaly phi you are measuring from this place this is your phi. So, when phi equal to 0 means the satellite is here in this point ok. So, if it is here so, at that time you are assuming the theta equal to 0 means the pitch angle is 0 and also theta prime. Now what is your theta prime? Theta prime you have written as d theta by d phi. So, this equal to d theta by d t times d phi by d t. So, this is nothing, but theta dot by phi dot.

So, your theta prime is this quantity. So, it says that if theta prime is 0 means this quantity phi dot is non 0; this quantity is phi dot this is not 0 because a satellite is moving in the orbit. So, phi dot can never be 0 this phi dot can never be 0. So, this implies theta prime this equal to zero. So, this what it implies that theta dot equal to 0 means the pitching motion has not started yet ok; so, with this assumption so, this implies that theta dot this equal to 0. So, writing in terms of getting independent of the time variable, we have expressed the things in terms of phi which is the true anomaly and true anomaly is a function of t. So, indirectly we are working in terms of t only.

So, if we put theta equal to 0 here when phi equal to 0 so, this is A sin beta and this quantity gets reduced to 0. So, this is A sin beta and this implies a is not 0. So, b equal to 0 beta equal to 0. So, therefore, our solution gets reduced to A sin 3 k i under root phi minus 2 e sin phi by 3 k minus 1. Now if we differentiate this with respect to phi so, theta prime then d theta by d phi. So, this gets reduced to a times 3 k i under root sin becomes cos and here of course, we will have 2 e divided by 3 k i minus 1 and sin phi becomes cos phi.

Again at phi equal to 0. So, at phi equal to 0, this gets reduced to theta prime we have written as 0. So, this is a times root 3 k i and phi equal to 0 this becomes equal to 1 and this quantity also can be written as like this. So, therefore, this implies A equal to 3 k i sorry this is A equal to 2 e divided by 3 k i under root times 3 k i minus 1. So, with this amplitude we can rewrite this equation.

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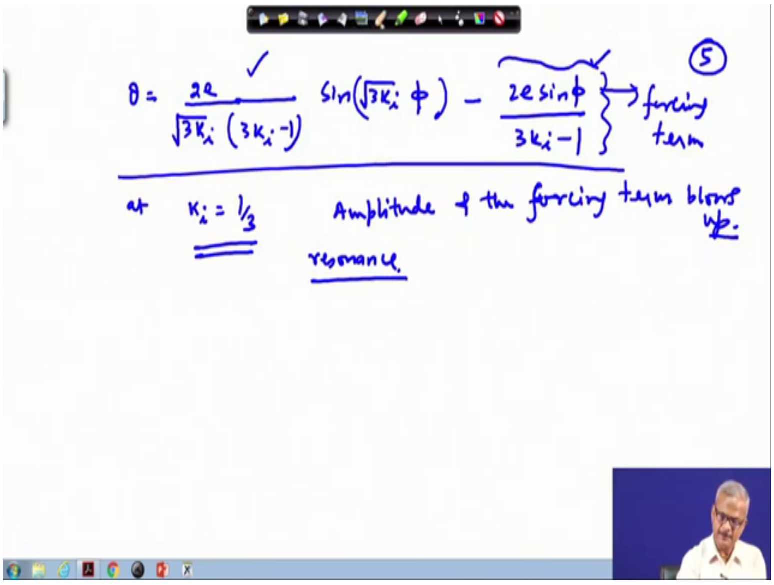

So, theta becomes equal to amplitude we pick up from this equation 2 e by 3 k i under root 2 e by 3 k i under root times 3 k i minus 1 times sin 3 k i under root phi and then the other term minus 2 e by sorry the other term 2 e sin phi by 2 e sin phi divided by 3 k i minus 1. So, if you look here in this equation. So, at k i equal to 1 by 3 these 2 terms, they blow up. So, the amplitude and the forcing term this is the forcing term this particular part and the forcing term amplitude and the forcing term blows up and this condition particular is equal the resonance.

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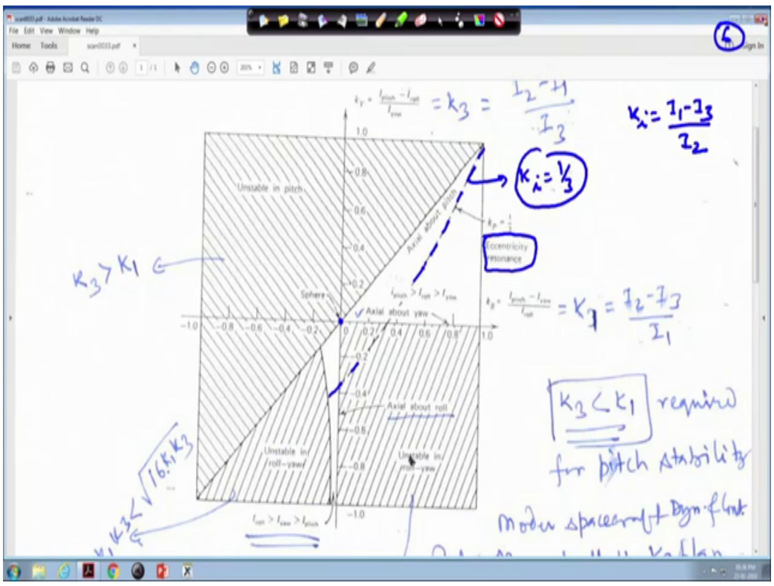

Now we can take up so what we have discussed earlier that this k i equal to 1 by 3. This refers to this curve so; that means, if your the k i equal to I 1 minus I 3 divided by I 2. Iit is such that it is lying on this dotted curve which we have not discussed earlier which is shown here this lines, then the resonance will take place ok. And this is written here the eccentric eccentricity resonance, it occurs because of the eccentricity of the orbit and rest other in this figure already we have discussed.

So, there is nothing to discuss any more and this point is referring to this figure all these things we have discussed in detail. Now, we discuss different cases; so, the equation that we have written earlier; this particular equation. So, here we approximated theta if we do not approximate so, that becomes the case where the orbit is circular because here we are taking a equal to 0, but theta is varying over a large value ok.

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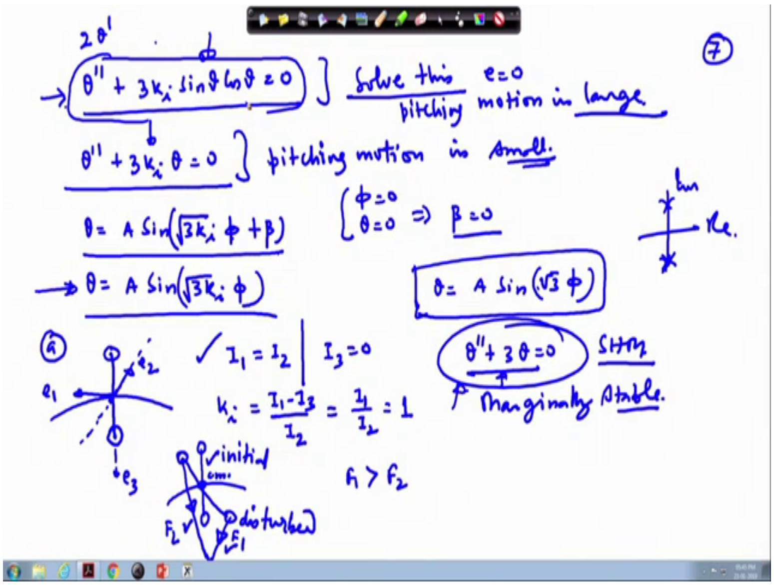

So, this equation if we pickup so, theta double prime plus 3 k i sin theta times cos theta this equal to 0 and solve this equation. Solve this ok. This is for e equal to 0 and this is motion in large pitching motion. So, for this particular part already we have discussed that if theta is small. So, we can solve it very easily means theta double prime equal to 3 k i theta this equal to 0. So, this is for pitching motion pitching motion in small. So, for solving this, we will need to multiply the left hand side by 2 theta prime and there after we can integrate it ok. So, we start with this particular one already we have looked into the solution which is theta equal to A sin 3 k i under root phi and plus beta and under the assumption that while theta equal to 0 phi equal to 0, then theta equal to 0 this implies beta equal to 0. This already we have looked into.

So, for solving this equation we will take up little later; let us first go through the various cases. So, now, here the cases are case a it is of a dumble ok. So, here in this direction you are taking e 1 and as per our reference this is e 2 and in this direction we have taken as e 3 ok. So, the body axis it is coinciding with our orbital axis. So, in this situation if your motion starts the pitching motion so, pitching motion will take place about this axis. So, for pitching motion now look here in this case I 1, this will be equal to I 2 and I 3 will be equal to 0. So, then k i which is defined as I 1 minus I 3 divided by I 2, this gets reduced to I 1 by I 2 and this equal to because this we have taken as equal to I 1 equal to I 2.

So, this gets reduced to 1 and therefore, theta here becomes A sin under root 3 phi ok. And from this place itself we can observe that if k i equal to k i equal to 1. So, the your equation remains as 3 theta theta equal to 0 and this is the equation of a SHM Simple Harmonic Motion and this is marginally a stable system. Stable system means your poles, there are pair of poles lying on the imaginary axis this is imaginary axis and real axis. So, pair of conjugate poles are lying on the imaginary axis.

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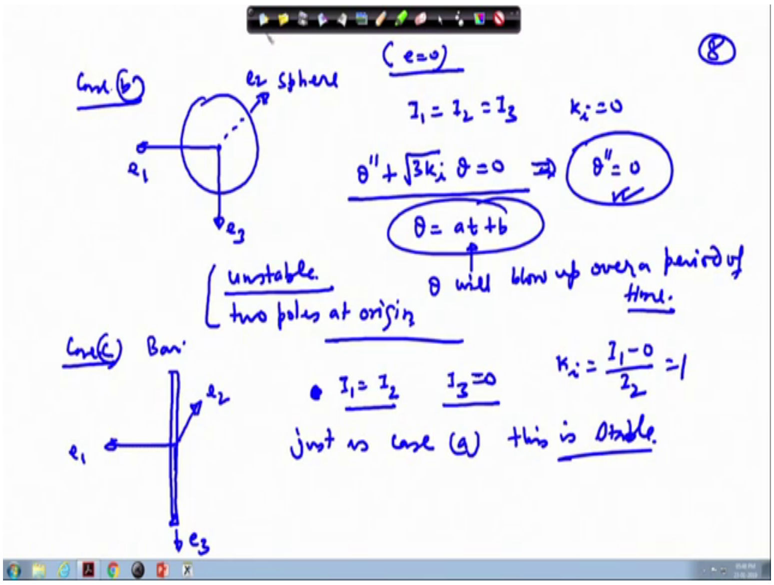

We discuss little bit more about the previous one. So, this already we have observed that if my dumbbell is here and this is the orbit, here is the centre of the earth. So, from the vertical position from this position, this is the initial position so, if you are disturbing like this is the disturb position. So, the restoring force this force is more than this force. If we write this as the F 1 and this as the F 2 in magnitude so, F 1 is magnitude y is greater than F 2 ok. And therefore, this will produce a restoring torque about the centre of mass this is the centre of mass.

And in this case we can also see from the equation that this is the stable case; however, in this case the oscillation will persist because the equation gets reduced in this format. So, it basically it shows that the in the gravitational field because it is a conservative field. Therefore, the pitching moment in a small that is turns out to be the simple harmonic motion of this format and it will keep oscillating forever ok. But in reality you have this term present here and we should take into account if the data is large we should take into account that value.

Case b, this is the case of a sphere. So, e 3 is here e 1 is here and e 2 is going into the page. So, this is your e 2. So, here in this case I 1 equal to I 2 equal to I 3 and therefore, k i this say gets reduced to 0 and your equation which was 3 k i under root times theta equal to 0. This gets reduced to and this we have remembered that we are discussing it for e equal to 0 this is for circular orbit. So, in the circular orbit what happens? It results because the right hand term of the small eccentricity orbit that has dropped out.

So, this gets reduced to theta double prime equal to 0 and this is nothing, but 2 poles at the origin and you can see that this theta can be written as 80 plus b and therefore, this is a function of t. So, theta will blow up blow up over a period of time means it is a unstable 2 poles at origin. So, in this case the system is unstable ok. Case c: so, case c is that of a bar. Again we are here e 1 e 2 is into the page and e 3 is vertically in this direction this is e 3. So, I 1 equal to I 2 here in this case and I 3 is 0 because for a bar is very thin. So, this are all almost it will be negligible and therefore, k i this becomes I 1 minus 0 divided by I 2 so, this equal to 1. So, this is again just like the dumbbell case. So, just as case a this is stable.

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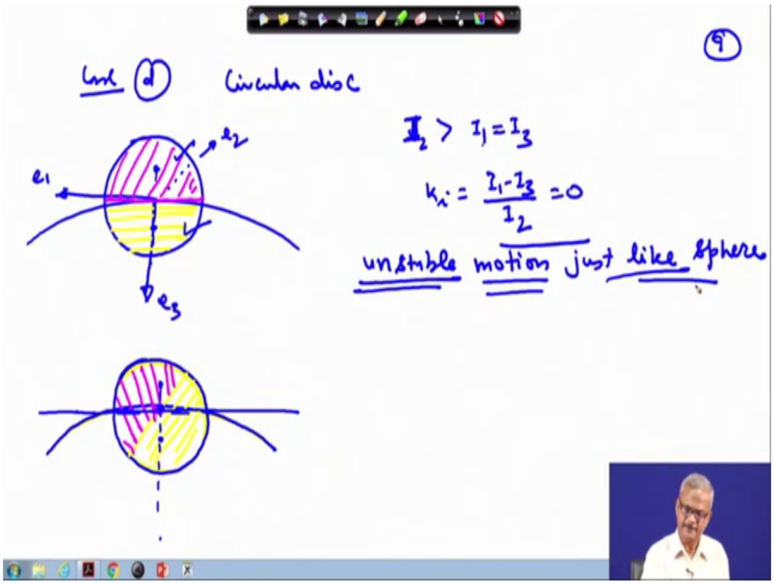

Case d: we can take the case of a circular disc. So, this is going into the orbit along this direction. We have even going inside the page this is your e 2 direction and e 3 is toward the centre of the earth ok. So, here in this case your e 2 I 2 is greater than I 1 and I 3 where this is the situation I 1 equal to I 2 I 1 equal to I 3 and I 2 is greater than I 1 and I 3. Therefore, your k i which is I 1 minus I 3 divided by I 2 this equal to 0 and this keeps you an unstable case unstable motion just like sphere. So, this we could have also analyzed physically that if this is my this is the case here. So, let us break into 2 parts, the upper part i show it like this and the lower part is by some other color; let us say this is my lower part.

If the disc I rotate the disc by certain amount so in this orbit once the disc rotates so, the yellow part. So, yellow part will say it comes here in this place, it comes like; this is your yellow part and the magenta part it goes here in this place. And this is your centre of mass here; the centre of mass is located here in this place ok. So, you can observe that by this change, no restoring torque can be generated because if we look here if I draw a line like this so, still the upper part is here and the lower part is also present.

So, here also the upper part and the lower part and they are symmetrical to each other and you can see that the force acting on the centre of mass of the other part and centre of mass of the lower part here, centre of mass of the lower part and upper part here; they are all passing through this line ok. So, it is a passing like this. So, there will not be any restoring torque. So, therefore, in the absence of the restoring force torque this motion turns out to be a unstable just like the case of the sphere.

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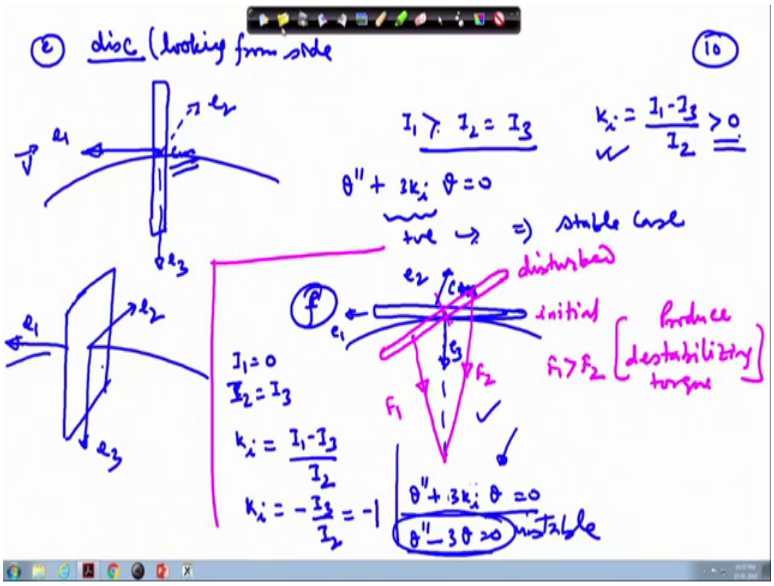

On the other hand if we take the disk like this so, this is a disc you are looking from side. This is the case of a disc looking from side means this is the orbit velocity is here in this direction. So, this is e 1 axis and here this is e 2 vertically down this is e 3 and your velocity vector is here in this direction this is the centre of mass. So, what we can observe that I 1 is greater than I 2 equal to i 3. So, our k i which is equal to I 1 minus I 3 divided by I 2 ok.

So, this quantity is greater than 0 and therefore, in this equation 3 k i under root theta equal to 0 this quantity is positive and so, this is nothing, but your the spring mass system with behaving like a spring mass system and therefore, this implies a stable case ok. So, it is a looking from this side it is a something it is a behaving like a your instead of this you can also have this kind of a square I and here in this direction your velocity is there. So, this is your e 1 axis along this the e 2 and in this direction this is the e 3 axis and you are looking from the side. So, such cases will turn out to be a stable provided in this condition is satisfied.

The case of a rod we have already discussed. So, we can have the case of a circular plate also we have discussed, then another case we can discuss is d e and then this f this is the orbit and your rod is lying like this. So, physically we understand that this case is unstable why? Because once I disturb this from this position to say this orientation this is the initial and this is the disturbed. So, what we see that on the centre of mass which is lying here centre of mass which is lying here for the 2 portion the force here F 1 and the force acting here this is F 2. So, F 1 will be greater than f 2 and therefore, this will result in destabilizing torque. It will produce destabilizing torque and therefore, physically we see that this case is unstable and mathematically also we can look here in this place where k i will be equal to along this direction you have e 1 going inside the page this is e 2 and vertically down, this is the e 3 direction.

So, your I 1 is almost 0 and I 2 equal to I 3. So, k i becomes I 1 minus I 3 divided by I 2. So, I 1 is 0. So, this is minus I 3 divided by I 2 and I 2 equal to I 3. Therefore, this gets reduced to 1. So, k i equal to minus 1. So, therefore, your equation of motion which is theta theta double prime plus 3 k i under root times theta this equal to 0 this is your equation. So, here the k i equal to minus 1, we go back this is under root not under root 3 k i. So, for and here we have written, this is not under root this part we need to correct 3 k i and here we have done the mistake. Inside the in the solution, this is ok.

This is 3 is 3 k i under root because omega square your omega square is nothing, but 3 k i and from that is the omega that enters into this equation. So, this part is ok. So, here this party is also and this is fine. So, here your motion now we have to go to this part, there we have to correct this is only 3 k i ok. So, here k i equal to minus 1 so, theta double prime minus 3 theta equal to 0 and know that this is unstable. It is a very simple to integrate you can integrate and check it ok. So, this way whatever the physically we have concluded this turns out to be correct also mathematically and this is the verification that the way we are thinking it is a right.

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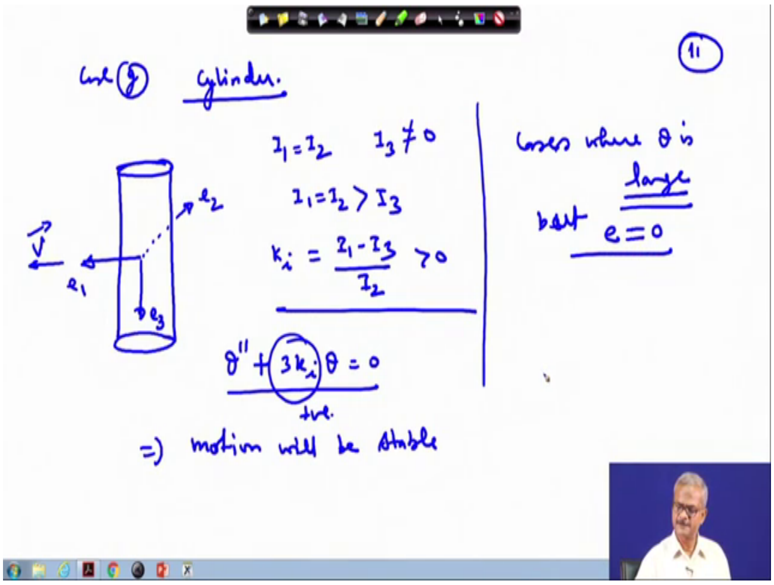

Lastly the case of a cylinder: the cylinder it will look like this. Here in this case this direction your e 1 in this direction, V is there e 2 is directly going inside and here this is e 3. So, I 1 equal to I 2 here in this case and I 3 is not equal to 0, but i equal to I 2 this is greater than I 3 and therefore, k i which we have defined as I 1 minus I 3 divided by I 2, this will be greater than 0.

And therefore, in your equation theta double prime equal to 3 k i times theta equal to 0 this quantity is positive and this implies the motion will be stable. So, therefore, we have discussed all this possible cases and whatever the cases we have discussed, these are easy to verify also physically without any difficulty. Now we go into the cases where the theta is large cases where theta is large. So, that we need to find it out, but still the case of, but e equals to 0 means still we are discussing this in the circular orbit.

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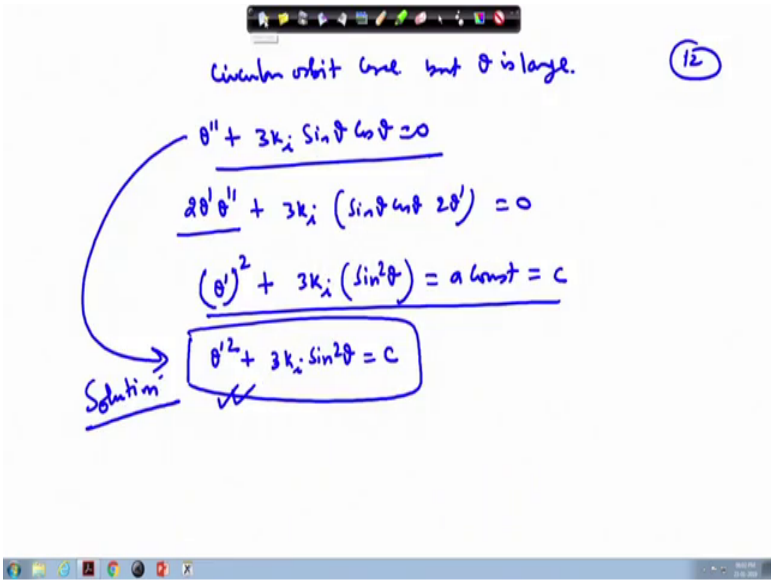

So circular orbit case, but theta is large. So, in that case we have theta double prime plus 3 k i sin theta times cos theta this equals to 0 ok. Multiply both side by 2 theta prime sin theta times cos theta times 2 theta prime this will be equal to 0 and this can be written as theta prime square. If we integrate it 3 k i sin square theta this equal to a constant let us say this quantity is c. So, what we get here? Theta prime square plus 3 k i sin square theta this equal to c, this is the solution to this equation.

If you are if our system is complex so, in that case a proper Lyapunov stability analysis will be required, but we should also remember that Lyapunov stability analysis it gives us a conservative result. The system may be stable in a wider range, then we obtain the result using the Lyapunov stability analysis. So, therefore, we should be careful while dealing with the Lyapunov stability analysis; however, one thing is for sure that Lyapunov stability analysis, it gives you the stability in a range which you can apply to your system without any problem. But it is not it is not that your system will be stable only in that range, it can also be the stable beyond that range.

Because the Lyapunov stability analysis its always conservative in nature in most of the time ok. So, let us move to the analysis of this particular equation.

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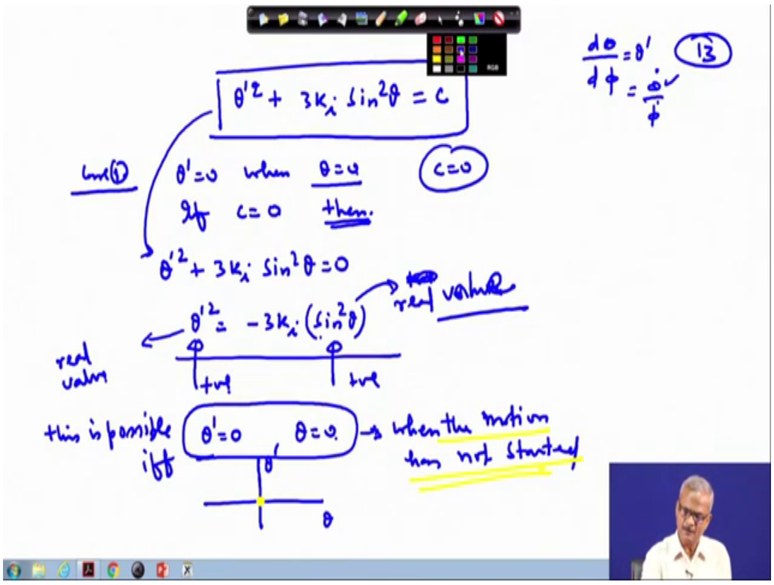

So, theta prime square plus 3 k i sin square theta this equal to c and this can we do the analysis. So, case 1: say the theta prime equal to 0 then theta equal to 0. So, if this is a situation means c equal to 0 or either way we can say that if c is equal to 0 this constant c equal to 0, then what do we get from this place? Let us look into this equation. So, theta prime square plus 3 k i sin square theta this is equal to 0 or theta prime square equal to minus 3 k i sin square theta.

Now, right hand side this quantity is a positive quantity, this quantity also is positive quantity because these are square term. And also these are real quantities theta is a real quantity real value theta is a real value also sin theta is also a real value. So, what does this imply that this is possible if and only if theta prime equal to 0 so, this is possible. If theta prime equal to 0 and theta equal to 0 so, both sides will be satisfied in this case and this is for the case when the motion has not started and on the phase plot the same thing can be shown like this here.

If I plot theta and I plot here theta prime so, these are the 2 states means the angle and angular velocity which we have converted in terms of the using this d theta by d phi, we are writing as theta prime. So, basically this quantity is related to theta dot divided by phi dot. So, angular rate we have converted in this form. So, for theta and theta prime they are basically the states and in this is called the pages phase. So, in the pages phase, the motion has not started. So, this point can be shown here in this place. So, this is just at the origin. So, here the motion has when the motion has not yet started.

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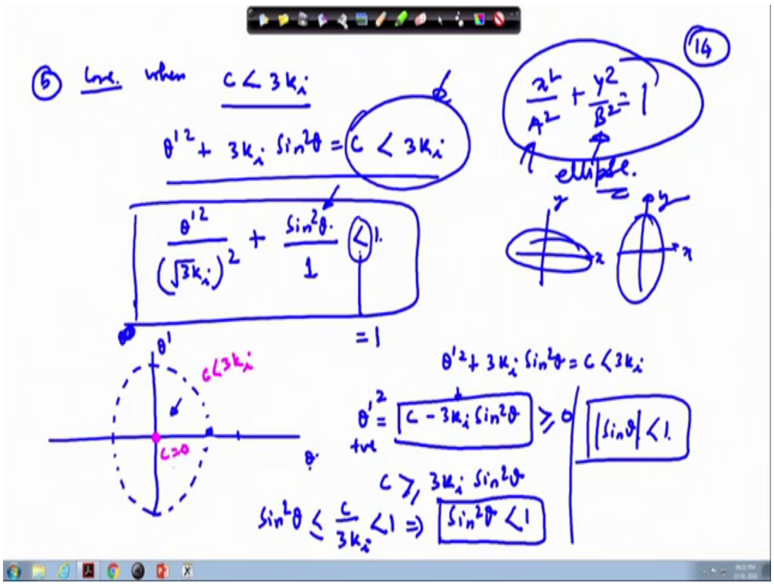

Case when case b when c is less than so, here we are discussing the case where c is less than 1 or let us say that c is less than 3 k i less than 1 also there is a case, but we have to normalize. So, I am not going into that case. So, c is less than 3 k i. So, what we are going to get here? Theta prime square plus 3 k i times this is sin square theta this equal to c and this is less than 3 k i. So, what do we get from this place? Theta prime square we can write like this 3 k i under rooted square plus sin square theta divided by 1, this is less that 1, this is the way we can express it ok.

So, this case once we plot it is a simple to look into this particular page because you know that x square by A square plus y square by B square, this is the equation of an ellipse. And depending on the A and B its magnitude either your ellipse may look like or your ellipse may look like this which one is greater it depends on that. And here we are writing x and t along this axis. So, for this for this particular case if we plot it on the phase plot. So, here we are plotting say for the theta and theta prime we plot here on this axis. So, your plot will look something like this; here sin square theta term is there, but here we are plotting in terms of theta and theta prime.

So, it goes like this. Now here in this case, we will have ok the quantities lying inside. This is the case once this quantity equal to 1 in that case we get here if this quantity say here instead of this equal to sign if we have less than sign if we have equal sign equal to 1 so, we will have the this ellipse sort of figure. But if it is not so, inside part will show and will show this ellipse has the dotted line means all the points lying inside are applicable,

So, if we take the previous case this part here, this it was the point was lying at the origin and here in this case it looks like this. So, combining them together so, this point is lying here and this is for your c less than k i 3 k i less than 3 k i while this case is k i equal to 0 sorry this is for c equal to 0. This case was for c equal to c equal to 0. Now we can analyze few more things here. What we can see from this equation? Theta prime this equal to c minus 3 k i sin square theta.

So, the quantity this quantity is positive. So, this implies this quantity will be greater than equal to 0 or c this will be greater than equal to 3 k i sin square theta or sin square theta this will be always less than c by 3 k i. So, here we have assumed that c is less than 3 k i ok. So, cases we can write it like this. So, this implies that sin square theta this is less than strictly less than 1 and then of course, you know that this will yield sin theta magnitude, this is less than 1 ok. And what will be the maximum value of theta prime?

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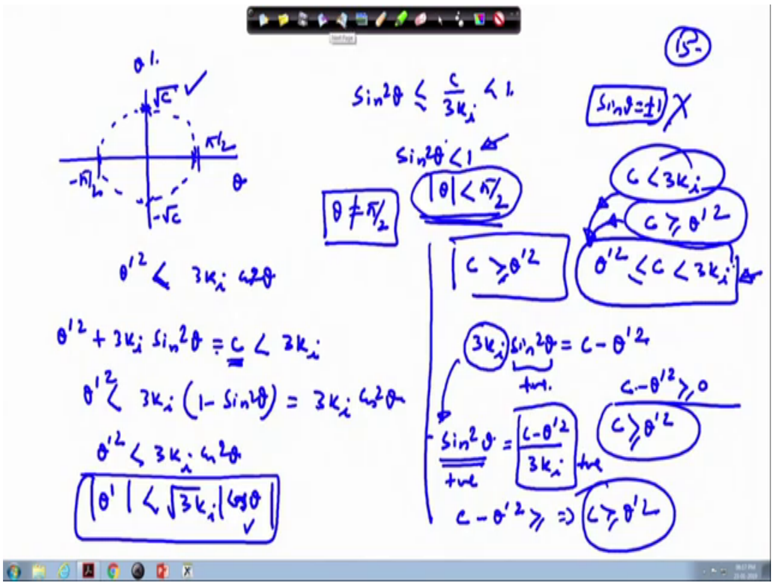

So, writing from the previous page and this is sin square theta is less than 1. So, theta magnitude this will be less than pi by 2. So, in that case it implies that; obviously, here in this case if we are taking it to up to the pi by 2, but theta can go to greater than this value, but not at such value where sin theta will be equal to 1 or either plus minus 1 those values are not acceptable. So, these values are not acceptable. So, if we remain in this range, we can plot it like this theta and this is your theta prime and he the maximum value will be under root c and this we can check using the fact that theta prime square this is less than 3 k i cos square theta.

So, what we are doing actually this is theta prime square plus 3 k i sin square theta this equal to c is less than 3 k i and we can write this as theta prime is less than 3 k i we as take as common 1 minus sin square theta this equal to 3 k i and then cos square theta. So, theta prime square. So, theta prime square. So, this will be less than 3 three ki cos square theta or theta prime magnitude this will be less than 3 k i under root cos theta magnitude ok. So, from this place this cos theta is always less than 1. So, theta prime we can see that this will be always theta prime magnitude will be always less than 3 k i. Now c is the quantity which is always greater than theta prime greater than equal to theta prime square.

So, c is the quantity here c equal to theta prime. So, we can look here look this in this way say 3 k i sin square theta is equal to c minus theta prime square. Now this quantity this is always a positive quantity and this k i if this is positive, then c can be written as c minus theta prime square this can be written to be greater than equal to 0 and c will be greater than equal to theta prime 0. Other ways of expressing the same thing is sin square theta, we can also write it like this equal to c minus theta prime square divided by 3 k i and we know that the left hand side this is positive.

And here therefore, this quantity must be positive. So, if k i is positive so, this implies that c minus theta prime square this will be greater than equal to. So, from where we have got this equation? This we have just written from this page. So, there are various ways you can express it. So, this implies c is greater than equal to theta prime square. So, we use this result and the result we have and the assumption that we have made that c is less than 3 k i. So, less than 3 k i and from here c is greater than equal to theta prime square. So, this says that c lies between theta prime square and this is 3 k i for this particular case. So, using these 2 results, we get this. So, we can conclude it on the next page.

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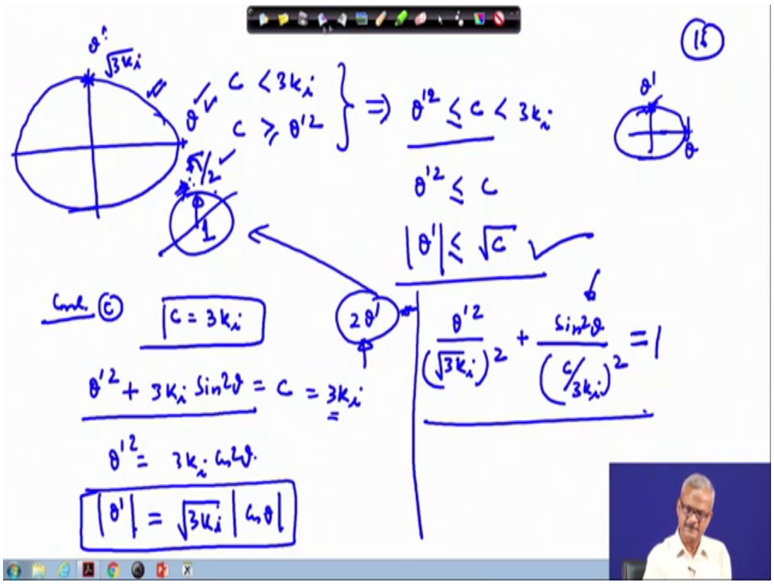

C is less than 3 k i and also we have got the result c greater than theta prime square c is greater than equal to theta prime square. So, these 2 together they imply that c lies between theta prime square and 3 k. So, from this place what we get that theta prime square this is less than c or theta prime this will be theta prime magnitude this will be less than equal to root c. So, going back in this picture so, here in this case this is the maximum value under root c which is shown here in this place.

So, theta prime maximum value can be under root c. So, and this place this is minus under root c and here on this site this is pi by 2 the limit and here this minus pi by 2. Of course, this will go like this if we take the higher values, but writing here the way we have done theta lies between pi by 2 means theta should not be equal to theta is not equal to pi by 2. This is not applicable because in that case this will be violated ok.

The case b: so, this way it can be analyzed case b. This will case c case c where c equal to 3 k i. So, we will have theta prime square plus theta prime square plus this equation we are using 3 k i sin square theta this equal to c and the technique we are used by multiplying by 2 prime theta 2 theta prime and solve the differential equation. So, very this is a very standard technique and it is used also quite often in your orbital mechanics ok. So, here in this case, we can write here this as 3 k i and theta prime square this is 3 k i then cos square theta and theta prime this magnitude, then this becomes equal to 3 k i under root times cos theta magnitude ok. So, from this place this also we can rewrite as we have done earlier theta square divided by 3 k i under root whole square plus sin square theta and because on this side you are writing here as 3 k i.

So, and if we divide it by say in terms of if I write in terms of c so, this will be 3 k i square under root this is equal to 1. And obviously, here c equal to 3 k i so, this quantity here it becomes equal to 1 and this is again the case of the ellipse the earlier case. But, here plotted the difference is that we are plotting here theta prime and here we are plotting theta not sin square theta. So, if I plot sin square theta here in sin theta here in this place and theta prime here in this place. So, then this is representing the equation of an ellipse.

If I am not doing this so, this is little variation, but it is easy to show it like this is. This is a much convenient way of doing it. Now here in this case, you know that I will show it here on the graph this the upper part theta prime we are plotting here. So, this will be because on the right hand side this equal to sign is there. So, this point will be 3 k under root and here the point on the x axis will be 1, but that will be 1 for the sin square theta.

So, if we write in terms of sin theta so, if we write in terms of theta so, this will be pi by 2 if we plot here sin theta. So, this will be equal to here we can write as we can replace this by 1 ok. So, this we are not doing we are writing in terms of theta. Therefore, this is pi by 2 and then your figure will look something like this. And then this is shown by the solid line because here on the right hand side; you have the equality sign.

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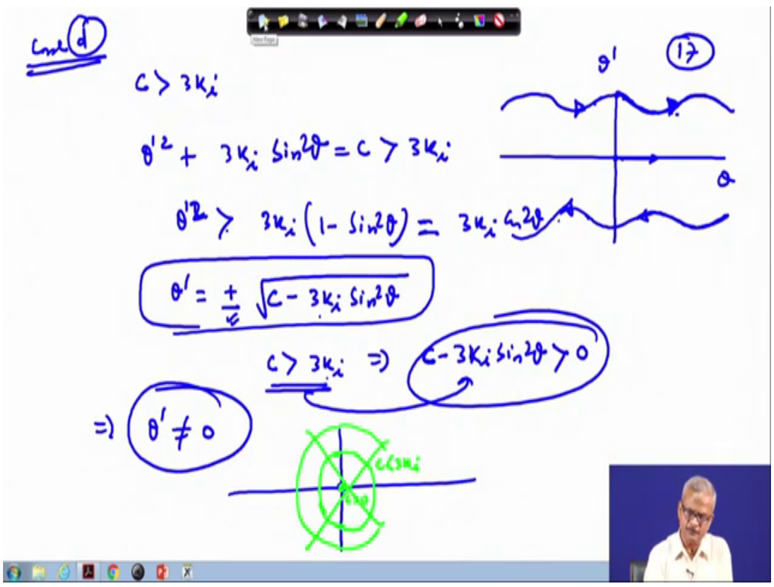

Now next we go for the case where c is greater than 3 k i. So, our equation then theta prime square plus 3 k i sin square theta this equal to c and this c is greater than 3 k i. So, theta prime square, then is greater than 3 k i times 1 minus sin square theta equal to 3 k i cos square theta and theta prime. So, let us first write it in this way c minus 3 k i sin square theta under root c is greater than 3 k i ok.

So, if c is greater than 3 k i so, this implies that c minus 3 k i sin square theta this will be always greater than 0 because of this quantity this is c. And therefore, this implies that theta prime is not equal to 0 at any instant of time and for this case your then the figure for theta prime and theta this phase plot it looks something like this. This goes here in this direction and this is here in this direction. So, as the value of theta increases so, your theta prime it decreases where c is greater than 3 k i. So, this will continuously decrease.

So, as the theta is increasing, the theta prime is increasing and you get a flow here in this direction. On the other hand if we take this minus sign so, you get the lower curve now the 3 cases. We have discussed this was our case c and the finally, we have the case d case. Now if we combine all these cases so, what do we get? For the first case the motion has not started. So therefore, there we get this point thereafter we get the case. So, this was the case when the motion has not started and this is referring to the case where c equal to 0 is pertaining to c equal to 0. Then we have the case where this c is less than 3 k i ok.

Then we have and here this lines are dotted lines or here we will show it using we show it using the solid line. And for c equal to I will do this figure on the next page, that will be better we do not have enough space here.

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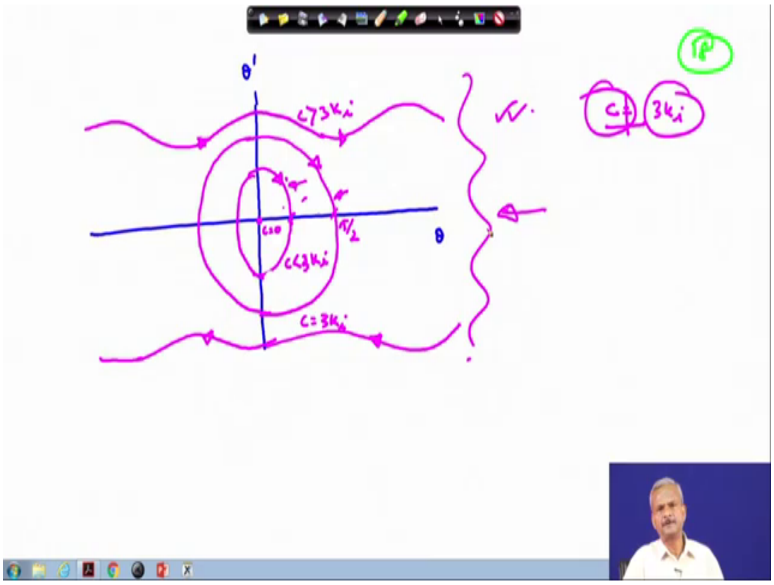

So, this is your theta and theta prime we are plotting on this axis. So, once the c equal to 0 so, this point is referring to c equal to 0 then the next one we are getting as this is for c less than 3 k i then we have the case where c equal to 3 k i and finally, we have the case where c is greater than 3 k i.

So this part here this is this part will write as pi by 2 while here this is not pi by 2. So, there is no need of showing this by dotted line ok. We can show it by solid line itself because we are in the earlier figure I was showing here at pi by 2 as the boundary so, therefore, I have put here in this place the dotted line. So, anything inside this I can show it by the solid line. There is no problem in this. So, your phase plot it looks like this. So, the motion is here in this direction the flow takes place and while here the phase flow takes place in this direction and here the motion has not started.

So, this is how your phase plot looks like. So, this phase plots they give you the idea about how the if I choose the value of the k i different value of k i and; obviously, here in this k c, we are relating it with 3 k i. So, how your motion is going to evolve ok; directly you can look from this picture. So, this is the benefit of getting the phase plot. So, rest other things we are going to do through the tutorial problems because it is a very exhaustive. We can keep discussing for the whole semester only on gravity gradient and it will not get finished. Therefore, we will stop it here today and we will continue in the next lecture.

Thank you very much.