**Satellite Attitude Dynamics and Control**
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**Lecture - 32**
**Gravity Gradient Satellite (Contd.)**

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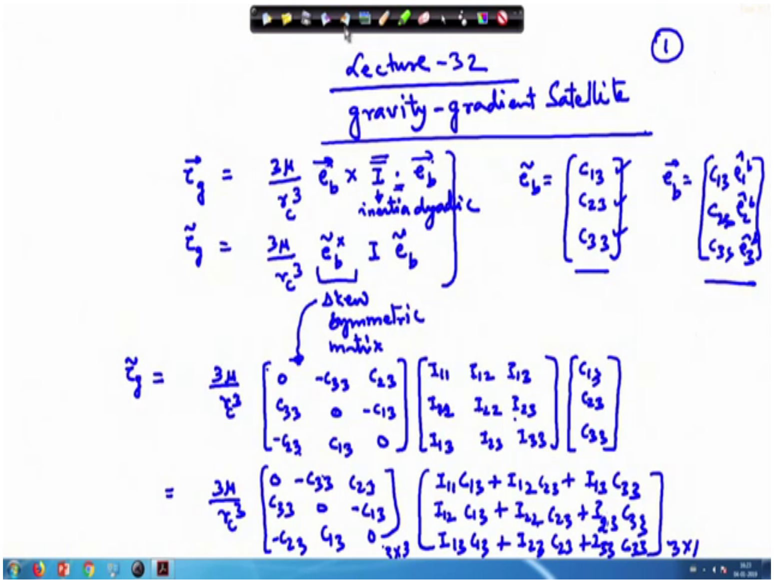

Welcome to lecture number-32. So, today we will continue with the Gravity Gradient Satellite. Now, if you remember so last time we have derived the gravity gradient movement, it is given by 3 mu y r c whole cube times, here we have written e tilde of it can be written in terms of e cap also, we will I will return back to this notation cross I double bar dot e tilde b.

So, if we are using this cross here, so if we are using the cross here in this place, so rather than using this tilde, we should write it this way ok. And the same thing if you want to write in terms of tilde, so you can change it to r c whole cube e tilde b cross, and this cross instead of this cross product, now we will put it here cross upside which indicates a skew symmetric matrix. So, this part is your skew symmetric matrix.

And then here we instead of using I double bar which is the inertia dyadic, we are just using the inertia matrix, and this will get converted into e tilde b. So, these are the notations last time we have worked out. And remember that e tilde b is the last column of the direction cosine matrix which is c 1 3 c 2 3 and c 3 3 ok, this already have derived.

And the moreover, if you want to write the same thing in terms of e b so rather than writing like this, you should write it as c 1 3 e 1 cap c 2 3 e 2 cap and c 3 3 e 3 cap, where e 1, e 2, e 3, these are along the body axis direction.

So, it is a just a matter of whether you work with the this part I will mark it here, this is your inertia dyadic, and this is your dot product, this part is your dot product, while here we are working in terms of the matrix notation. So, either of these two you can choose, and you can work. But, this will be more convenient, once you have written this equation, so it is a much more convenient to work with this.

So, we develop the whole thing right now. So, tau tilde g, then it can be written as 3 mu by r c whole cube, and e tilde b, so these components are here these are the three components. So, 0 in the diagonal term, then here this minus c 3 3 c 2 3, and thereafter if this is c 3 3, 0, and then c 1 3 with minus sign, and here again minus c 2 3, this place c 1 3, and here 0. So, this is your skew symmetric matrix which is here. Thereafter, put the inertia matrix I 2 1 which is nothing but equal to I 1 2, so let us write it as I 1 2 only. And I 1 3, because I 3 1 equal to I 1 3 ok, and this operates on this vector. So, this part is c 1 3 c 2 3 and c 3 3. This can be expanded and written in a proper format.

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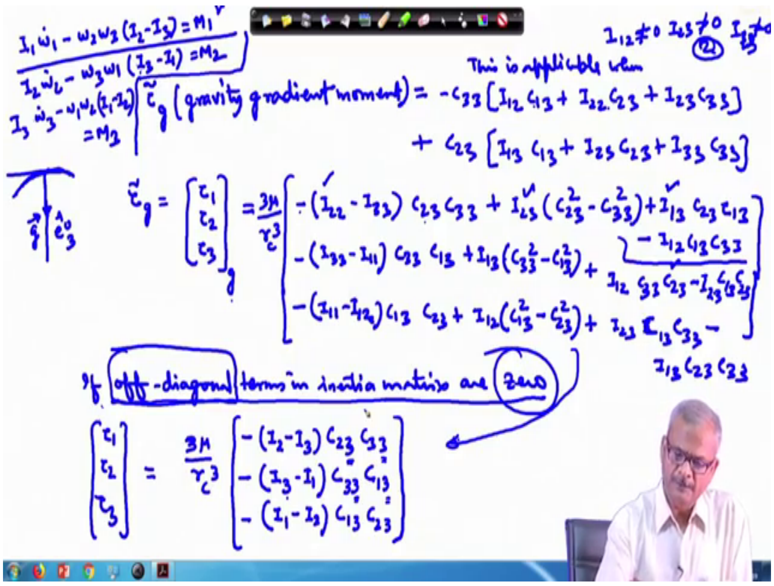


So, this gravity gradient moment; so, this final format, I will write it here or either if one line at least, maybe I will do it a single line. So, this is the first line, this will remain there. And then first multiply this matrix ok, so we need to copy this here in this place.

And this part, then we will have I 1 1 c 1 3 plus I 1 2 times c 2 3, and similarly the second term will be I 1 2 c 1 3 plus I 2 2 I 1 3 times c 3 3 I 1 2 I 2 2 and I 2 3, this is I 2 3 this one. Here this one should go here in this place, so this is I 2 3. And the last one will be I 1 3 c 1 3 plus I 2 3 times c 2 3 plus I 3 3 into c 3 3. And operate on this matrix, we have to multiply this two matrices ok.

So, if we multiply we see that this matrix of size 3 into 3, and this is matrix of size 3 into 1. So, the result will be we get a column vector ok. If we multiply these two matrices, so we get a column vector. And that column vector the first term then once we multiply, so here this is 0; so, if we multiply with this term, this will be 0, this term will count. So, minus c 3 3, so this is minus c 3 3 times I 1 2 c 1 3 and so on. I 1 2 c 1 3 plus I 2 2 into c 2 3 plus I 2 3 into c 3 3, this is what we have written on here I 2 2 I 2 3 c 3 3 c 2 3. So, this is multiplied by this, and then plus c 2 3 times, this last row. So, c 2 3, and the last row then we have to copy which is I 1 3 times c 1 3 I 1 3 times c 1 3 plus I 2 3 times c 2 3 plus I 3 3 times c 3 3. So, the middle term is here I 2 3 c 2 3. This is I 2 3 c 2 3 ok.

And this can be organised, so the delta tilde g this will consist of if this tau g, this will consist of tau 1, tau 2, tau 2 3. So, these are the three torque or gravity gradient movement or the torque components, and this is due to the gravity gradient. So, we put a g here ok. So, the first component of this only I will I am working out, and rest if you just put it here finally. So, this can be arranged as I 2 2 times I 2 3 I 2 2 times c 2 3 and c 3 3. So, with this there is a minus sign, so I 2 2 c 2 3 c 3 3 with this a minus sign of heres. So, this particular term is inserted here in this place.

Similarly, for the I 3 3 term we have to pick up. So, I 3 3, if we search it is here in this place ok. So, for I 3 3 c 3 3 multiplied, and then c 2 3 is multiplied. So, you can see that c 3 3 and c 2 3, it is multiplied and this minus minus sign that makes it plus. So, these two terms are combined together, and written here. Similarly, the other terms we can fill in, you can just check it. So, this is your first term. So, what we see that once we are writing the Euler's dynamical equations, so while writing that I 1 we have written say there we have written I 1 times omega 1 dot minus omega 2 times omega 3, this equal to I 2 minus I 3. And on the right hand side we have written the torque, so torque we have written by M 1.

So, in this equation if the off-diagonal terms are 0, so at that time we tend to write like this I 1 I 2 and I 3. So, we see that if my off-diagonal terms are 0 here, so that is your I 1 3 and this I 1 2 and I 2 3, these are 0. So, this will simply get reduced to this format means, this simply says that if off-diagonal terms in inertia matrix are 0 ok. Then this tau 1 can be written as, and here of course 3 mu by r c cube is there. So, 3 mu by r c cube, and the this term just we have to copy. So, I 2 2 in that case, it will get reduced to I 2 and I 3 3, we write as I 3. And rest other things which remain same.

In the same way for if we have to write for the tau 2, so you remember that I 2 times omega 2 dot minus omega 3 omega 1 omega I 3 minus this is I 3 minus I 1 equal to M 2. So, on the right hand side, the term that appears it will be exactly the same thing, here I 2 minus I 3 see here I 2 minus I 3 ok. So, right side this moment equation we are writing, so the moment term it appears like this. So, this you need not memorise. If you know the how to write the left side, so the right side you can write provided the off-diagonal terms are 0.

So, here from this place, we say that this is I 3 minus I 1, and similarly you just right c c this is 3. So, write here 3, this is 1, write here 1. And this three belongs to because your while we have work. So, for your if you remember that your gravity is acting along this direction, third axis of the orbital difference frame that is e o 3. So, in this direction your gravity vector is acting. So, therefore in all of them this second subscript, it will be 3. You see here this is 3, here also this is 3, so also here 3, also here 3, the same way the tau 3 this will be so here in this case, this will be I 3 times omega I 3 times omega 3 dot minus omega 1 times omega 2. And then the other terms will come ok.

Then we have I 1 minus I 2, and this equal to M 3. So, here exactly the same thing we have to copy, this is I 1 minus I 2. And similarly this 1 and 2 are there. So, we will put here 1 and here 2, and then just copy here this 3 ok. And this one is with the off-diagonal terms non-zero. So, this is applicable when I 1 2 not equal to 0, I 2 3 not equal to 0, and I 1 3 this is not equal to 0. So, for that case it is a valid. So, the other terms we can write here. This is a I 2 3 times c 1 3, you can do the matrix multiplication yourself, and check these terms ok. So, this case gets reduced to this case. If the off-diagonal terms, they are all 0; so, this is very pretty simple to remember.

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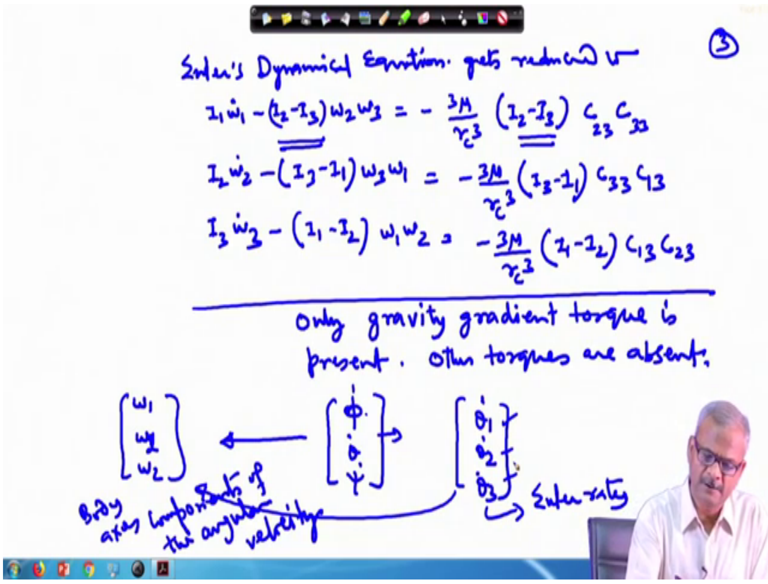


So, our equation of the Euler's dynamical equation gets reduced to a simple format. Here we have inserted 3 mu by r c cube ok, you can see this term is here, so this term is also present here. And rest you just have to put this c which are the components of the transformation matrix, and write here this is 2, so write here 2, write here 3, and then put just 3 3. This completes the Euler's dynamically equation, when the gravity gradient torque is taken other torques are absent. So, here only gravity gradient torque is present, other torques are absent ok.

Now, what this c 2 3, c 3 3 all this things that we need to insert here in this place and moreover, this omega 1, omega 2, omega 3. As if you remember earlier we have discussed that this omega 1, omega 2, omega 3, these are not visualizable. Because, the itself the body orientation is changing, and about the body axis, then the satellite is rotating.

So, you cannot get out of the how the satellite is rotating, so to describe that the Euler angle representation is required. And therefore, if you remember that we have converted omega 1, omega 2, first we wrote here like pi dot theta dot and then psi dot, and this we converted to omega 1, omega 2, omega 3, this omega 2, and omega 3 ok. So, instead of using this notation, we use here theta 1 dot theta 2 dot and theta 3 dot, these are the Euler rates. So, this is convenient in working in terms of theta 1, theta 2, theta 3. And then this is converted into the body axe components of body axis component components of the angular velocity. While these are the Euler rates along the three axis, but those axis are not mutually perpendicular to each other ok. And this whole thing we have derived here, so I will not go into that again ok.

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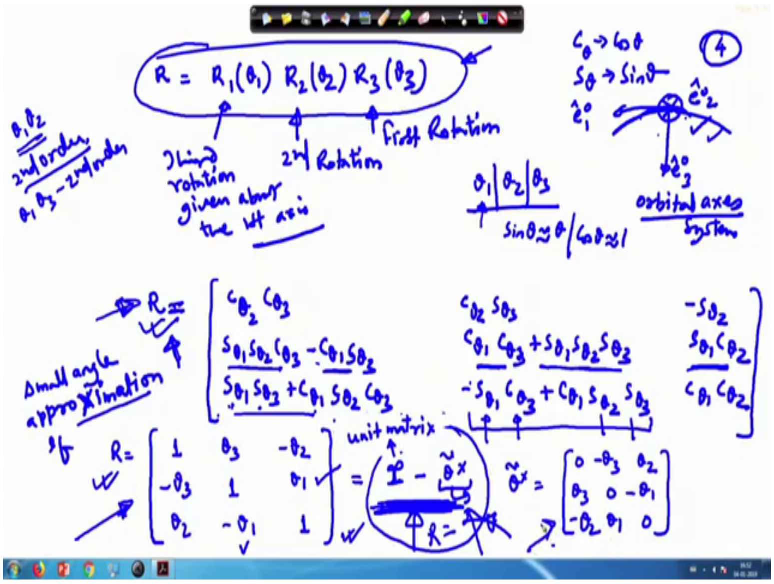


So, we will just recall that if we consider the rotation to be given about the three axis, so this rotation is the first rotation is given about the third axis. This is the first rotation, then this is the second rotation, and this is with third rotation which is given about the first axis. So, this sequence already we have studied. And we have seen that this can be written as c 2 c theta 2 where c theta it implies cos theta, and correspondingly the subscript s theta will imply sin theta just we will verify it ok, it is fine.

If so this theta 1, theta 2, theta 3, these are your Euler angles. So, these Euler angles, if you look in so we have chosen to describe this, this is my e o 3 cap along this direction this is the satellite orbit, along this direction e o 1 cap, unit vector along the velocity vector direction and e cap o 2. So, with respect to this particular orbital axis system orbital with respect to this system, your body is getting oriented ok. So, the body axis will get oriented with respect to this particular axis system. So, how that is getting oriented, it is getting oriented in terms of theta 1, theta 2, theta 3, this three angles.

So, the first rotation you are giving about the third axis, the second rotation you are given about the second axis. The resulting second axis, as we have discussed during the rotation. And similarly, the first rotation is this third rotation is given about the resulting first axis ok. So, these are the three Euler angles. Now, if this Euler angles are small means, the cases where we can approximate sin theta equal to theta, and cos theta this equal to 1.

So, in those cases you can see that this gets simplified c theta 2 c theta 3, because theta 2 theta 3 all these are small. So, a small angle approximation small angle approximation, so this will become 1 c theta 2 theta 3, this becomes theta 3, because cos theta 2 will be 1 here minus theta 2. In this place, s theta 1 s theta 2, this two multiplied together. So, this multiplies like theta 1 theta 2 sin theta 1 and sin theta 2, and this is of second order ok. So, we know such terms.

So, if we ignore, so this becomes 0, this part will become 0, and what remains there is this particular term. So, in this particular time c theta 1 is 1. So, here we get minus theta 3 only. So, here in this place minus theta 3; so, the same way if you look into this last term, so s theta 1 s theta 2, both of them will multiplying together that gives theta 1 theta 3. So, the first order approximation.

So, theta 1 times theta 3, again it is of second order ok, so this gets eliminated. And from this place we get cos theta 1 equal to 1 cos theta 3 is equal to 1, so this gets theta 2 only ok. So, this way if you try to fill, and look into the other parts also. So, this terms have to be a matrix, where this diagonal elements will be 1, and the off-diagonal elements will be opposite in sign. So, this is theta 2 minus theta 2 theta 3, then minus theta 3. Here you are you will get theta 1.

So, in this place you can you will get theta 1, so we can check this term here, let us say this term we want to check. So, here s theta 2, and s theta 3 that becomes 0 multiplied together. So, this c theta 3 is 1, so what we get here, this minus theta 1. So, minus theta 1 is appearing here. Same way here in this place, this is just theta 1 times 1. So, here you get theta 1. And in middle term these are the three terms that makes it theta 1 times theta 2 times theta 3 which is third order. So, we eliminate it and what remains here, this quantity only which is 1 times 1, so that gets 1.

So, this way you can complete this matrix. And earlier if you remember, this we have written as an identity matrix minus theta tilde cross ok, so where theta tilde cross is your skew symmetric matrix which is nothing but 0, theta 3 with here minus sign. So, minus sign we are taking it outside ok. If we will just you can check it, here this becomes theta

2 theta 3 is 0 minus theta 1 0. So, if you put it here, and then you take the minus sign inside. So, you can see that the resulting quantity will be the same as R. So, this R can be written in terms of I minus theta tilde cross.

And this we have earlier proved while doing the rotation part toward the end we have done it. So, what is the benefit that if the angle of rotation is a small ok; if your satellite is here, and deviation from this orbital reference frame or the body reference frame, if it is a small. So, in that case that rotation can be approximated like this ok, you do not have to take the whole this matrix and then work out. So, this is a simplification.

Now, another part that you do not have to memorize this matrix R ok. If you have to take larger angles, only then there is a problem that this matrix needs to be computed anyway this matrix if it is a better not to memorize it, but rather computed from time to time whenever it is required ok. When the angles are small, so it is very easy to remember just use this R equal to this quantity ok. So, this gives you the resultant matrix. And now you can go back, and do the job you want ok.

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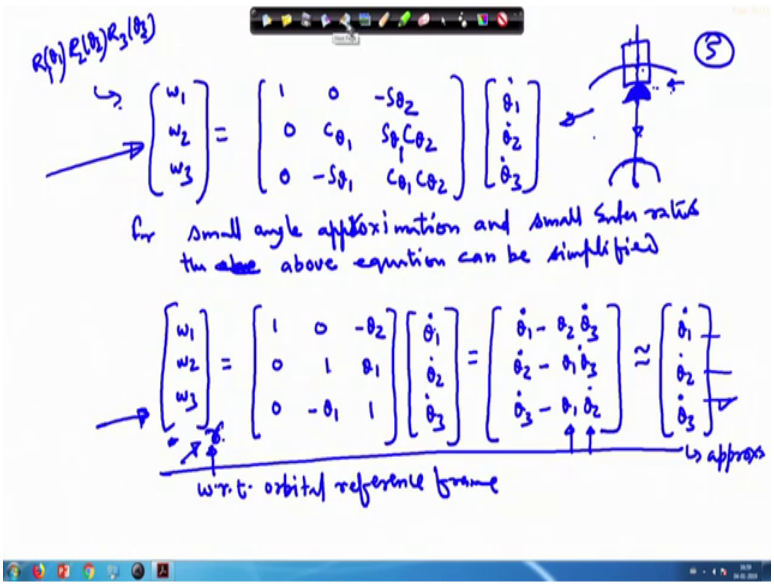


Beside this we have also proved omega 1, omega 2, omega 3, these quantities we have derived earlier. These are for the rotation, where we have given R 3 theta 3 R 2 theta 2 and R 1 theta 1 for these rotations it is a written here. So, these are not the transformation of this is not this transformation, this has come through some other transformation that I have shown you earlier derived it earlier ok. So, this case also, it can be simplified. So, this is 1 0, this is minus theta 2 0 just becomes 1 here, this is s theta 1.

Now, this will become theta 1, and here 0 minus theta 1. And here in this part theta 1, theta 2, so both are cos function multiplying it. So, this is theta 1 dot minus theta 2 times theta 3 dot, then this part of the theta 2 dot times here this is 0, this is theta 2 dot, and then theta 1 times theta 3 dot. Finally, the last one this is theta 3 dot minus theta 1 times this one is theta 2 dot, which comes from here and theta 1 times theta 3 dot. And this is theta 3 dot, and then theta 1 times theta 2 dot ok.

So, if we look into this, we are considering this to be a small, and these dots to be also small. So, if you approximate this can be approximated as theta 1 dot theta 2 dot and theta 3 dot; so, in your Euler's equation omega 1 omega 2 omega 3 was required. So, for a what we see that, they can directly be replaced in terms of theta 1 dot theta 2 dot and theta 3 dot.

The benefit of small angular displacement is that you do not have to worry about the order of the rotation you have taken, because as we have shown earlier in the rotation part that the order of rotation for a small angle of rotation does not matter whichever order we write we take, we can always right in this way which we have shown here in this place. So, I which is this is the identity matrix, we will write it here itself identity matrix or unit matrix ok. This is your unit matrix.

And as usual this is your skew symmetric matrix that we have written here. So, what we are trying to do here, we are trying to develop the Euler's dynamical equation in terms of Euler angles. As I have told you earlier that this Euler's dynamic equation, it is not a physically visualizable. So, to visualize this, we need Euler angle term. But, if you put the Euler angle angles in the equation of motion dynamic Euler's dynamical equation, it becomes very complicated.

So, here if we are approximating it, so in that case it gets simplified. And in most of the cases, it will so happen that for small angle deflection from the orbital axis, you will be working, because your satellite you want to orient the satellite towards the earth. Most of the time, what you will look for that the it is a earth pointing satellite means, this is my orbit there is a satellite on which say that is a transponder or there is a camera. And this camera should always point towards centre of the earth ok, this may be one of the requirements.

So, in such cases, if you have a satellite which can maintain its orientation along this direction, so it is a very good. And what we are looking for that this satellite, once it gets discharged from this particular position ok. It should be not deviating away from this position, so that is what the path is the stability is earlier we have also observed that the rotation about the minor axis it is not a stable, if the internal energy dissipation is taking place.

So, obviously you have a satellite, so there may be some vibration, it is not a perfect rigid body. So, always there will be some vibration involved with the system and energy will start dissipating. So, we have to look also for the whether the satellite is a stable or not, under what condition it will be a stable. So, this is what we are going to work out here ok.

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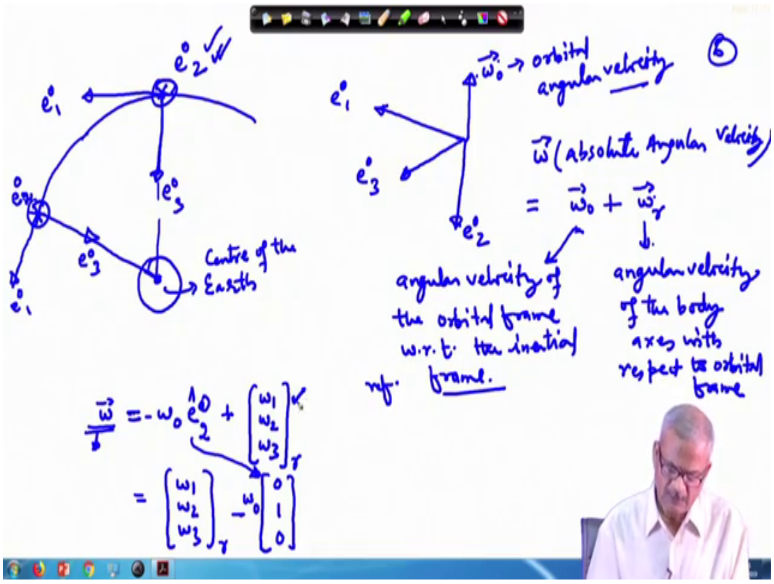


So, finally what is required that in this orbit, your this is e o 3, e o 1, this is the axis we are using, and this is e o 2 ok. So, this is the orbital axis system. And this system after sometime here some there is the centre of the earth. So, after sometime, this axis will come here in this place, so it is a rotating e o 2 ok.

So, as you can see that in the inertial frame itself, this orbital reference frame is having certain angular velocity ok, it is a rotating, right now it is like this. So, in the inertial frame, it is it has rotated to this condition, and where the angular velocity is directed it is a directed perpendicular to the page coming out of this page from this place ok. So, as we can see from this place this is your e o 1, and this is e o 3 direction, and this direction is vertically down e o 2 direction which is shown here ok.

So, your angular velocity is coming out of the phase means, the angular velocity of the this orbital reference frame is along this direction. And that we can write as let us say omega 0, this is the orbital angular velocity. And if we put a arrow over this, so it becomes a velocity vector. The absolute angular velocity which we write as omega this is the absolute angular velocity, this becomes equal to how this frame is rotating that is omega 0. And with respect to this frame with the orbital frame how your body is rotating, so that we write as omega r or either you can write as omega b. So, this is the angular velocity of the body axis with respect to the orbital frame. And this is the angular velocity of the orbital frame with respect to the inertial frame or inertial reference frame ok.

So, to calculate this omega 1 this omega, so what we have calculated here. This is just the relative angular velocity, because we have calculated it with respect to the this is with respect to the orbital reference frame ok. So, we can put a here, I will not tag anything because this is a general equation, while we are reducing it for a small rotation. So, we are then we will tag it here or either finally you can tag it. But, if I put a r here that means, this becomes a relative angular velocity otherwise, omega 1 omega 2 omega 3 that will indicate your absolute angular velocity.

So, coming to this place again, here omega then your this is omega 0. And in which direction this is this is just opposite of the e 2 direction. So, we can write here omega cap along the o 2 direction with negative sign here and plus omega r which is nothing but omega 1 omega 2 omega 3, and we will be writing it like this. This is r minus. So, this is e o 2 means, it is only along the second orbital axis, so that means, we will have 1 here and rest other will be 0, and put omega 0 here in this place. So, gets this can be written like this ok. So, this is a still this whole thing what we have written here. So, what we have done, this is the angular velocity of the satellite with respect to the ok. This term I will remove it.

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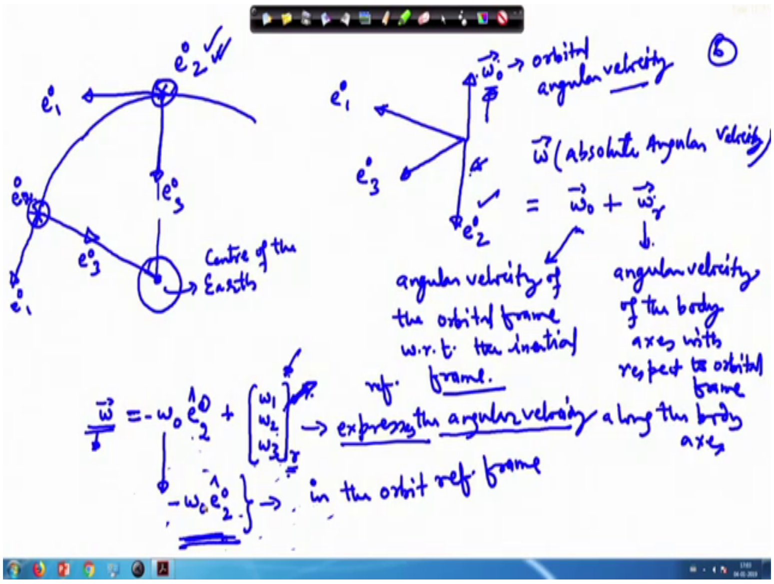


Let us first finish one thing, then we will take up this part ok. First we consider this part, this omega 0 minus e o 2 cap. This is still along the surface this omega 0 was actually oriented, this is taken this is the orbital difference angular velocity or the orbital reference frame, but it is along the negative of the e o 2 ok. So, this is an inertial frame, but along the negative direction of the e o 2.

And therefore, the same thing we have written like this. So, this is along the orbital or in the orbital frame, this is it still in the orbital frame. And we need to convert it, this is in the orbital reference frame. And if what we are doing that we are expressing the angular velocity along the body axis ok, so we have to express the angular velocity express the angular velocity. So, these are the angular velocity components with respect to the orbital axis this particular part ok. This r stands for that again reminding you ok. So, and but these components are taken along the body axis.

So, express the angular velocity expresses the angular velocity along the body axis, but here this is not along the body axis. So, therefore we need to convert it ok. Once we do the conversion to the body axis, then our job is done ok. So, we will continue with this in the next lecture, we wind up it here.

Thank you very much for listening.