Satellite Attitude Dynamics and Control
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Lecture – 31
Gravity Gradient Satellite (Contd.)

Welcome to the 31st lecture on Gravity Gradient Satellite. So, we have been trying to convert the movement due to the gravity gradient from one frame to another frame. And you must be very particular about this understanding, because it can be done in various ways. And generally if you look into the books, they do not explain all these aspects.

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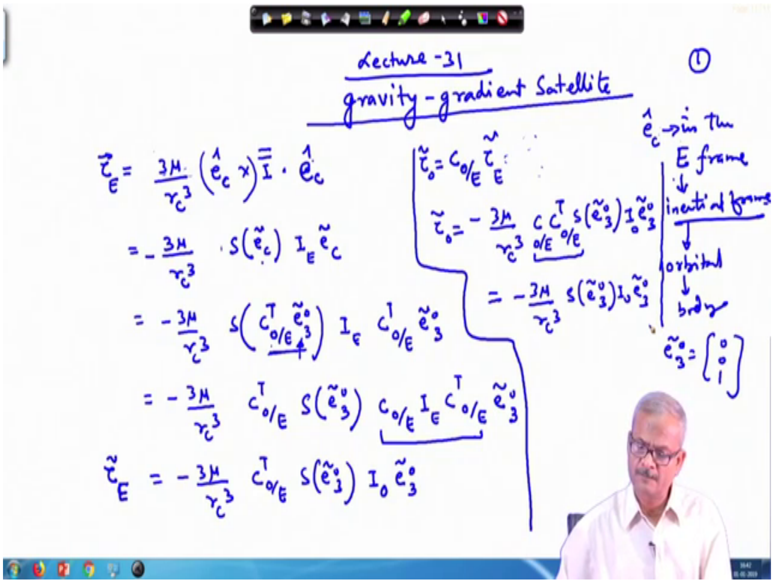

So, I will explain the same thing from others perspective. Let us say that once we started writing this gravity gradient torque which we have written as tau. So, instead of writing this in terms of vector notation without write it as tilde, which is the matrix notation. And the right hand side also, we will convert it accordingly. Just wait for this will wait for some time, before we go into this 3 mu by r c cube, this was our basic equation. And then we have e cap c cross I double bar dot e c cap.

And as I told you that this is a vector e c cap, this is a vector as along the c direction. So, this c direction vector can be converted. Now, we can be express in terms of say this vector is in the e frame, let us say this is in the in the capital E frame which here is the initial frame. And we need to pass from this frame to the orbital frame, from here we will go to the orbital frame, and then we will go to finally to the body frame ok.

So, there earlier once we have done this, so directly we replace this e c by e u. Here I am going systematically from one frame to other frame. So, this e cap as I have told you that this e cap cross, this can be written in terms of with minus sign here. And this I can be replaced in terms of I double bar dot e c cap, this we can replace as e tilde c ok. And this minus sign, we have taken for this e c cross, this part we have written s, so this is minus sign is appearing here.

So, we are converting from E frame to o frame ok, we will write it like this ok. Now, because we are converting along the third direction, we are interested in the vector along the third direction. So, your as usual e o 3, this indicates 0 0 1 ok, the this is what we are interested in. So, we will write it in this way and ok. And we will we will have a transpose here in this place I times c transpose o slash E. So, this is E to o frame ok, and to transpose of this means we are converting from inertial frame to the orbital frame. This is converted to the orbital frame, because here this quantity is in the orbital frame.

And using this skew matrix properties times I c transpose o slash E. Now, if we look into this quantity, this quantity is nothing but c transpose o slash E s e tilde o e, this will be written as this was in E frame. So, maybe we can put a tag here that this is in E frame ok. This is in E frame, now it comes to the o frame means the orbital frame ok.

So, so this is your tau tilde, which is now in still the on the left hand side you can see that once we have manipulated like this, in terms of this transformation matrix. So, on the right hand side we have this in the E frame, and because the whole thing we have written in terms of this e c e c cap which is along the r-direction. And if we are expressing this in the inertial frame, so it will lie along the inertial frame.

So, this is along the still along the E frame, this we need to convert along the orbital frame. So, if we multiply both sides by c o slash E means E frame to o frame, so this quantity will get converted to the orbital frame ok, you can see here. Then this becomes 3 mu by r c cube, and this is c times c transpose I 0, this is equal to here let us write this in this way here tau 0 tilde ok. So, as usual this quantity is identity. And therefore, this gets reduced to r c cube, and this is s e tilde o 3 I 0.

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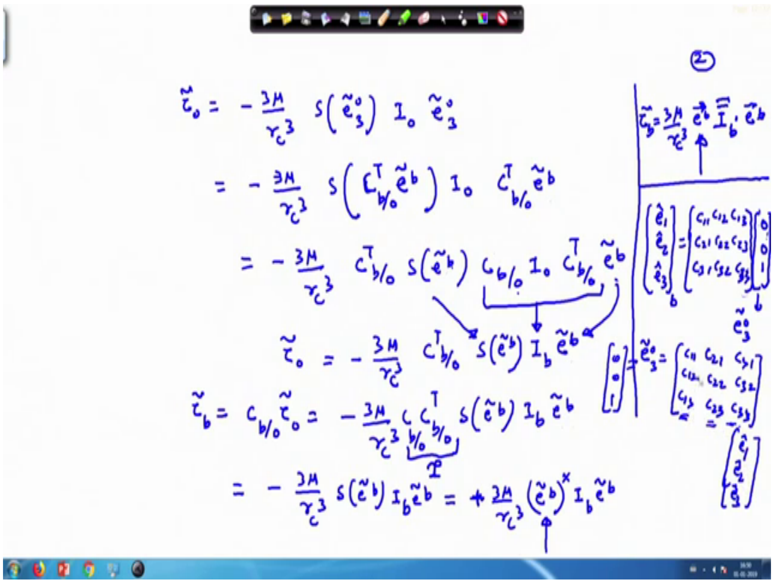

So, we have tau tilde o, this quantity equal to minus 3 mu by r c whole cube, and then s times e tilde o 3 times I o. Then we will start writing this as c transformation from orbital frame to body frame I 0 c transpose b slash o ok. And this can be written as c transpose b slash o, and this quantity is nothing but your I b. This converted into the body frame as I have discussed earlier ok, rest we copy it from this place ok. Here we have not written it correctly.

Once this is converted, so this quantity here. Let me check on the previous page also e c. Here we have done it correctly e c that this is e c has been changed to e o 3 ok, so it is coming in your 3 terms. So, here this will be in terms of e b tilde, similarly here this will be in terms of e b tilde. So, this is converted into this quantity is converted into the body frame ok.

So, this is e b and this part is converted into this place, this we have copied here, this we have copied here, c transpose b slash o 3 mu by r c cube here equal to, but left hand side is still it is in the orbital frame. And to get it into the body frame, what we need to do that we convert this from the orbital reference from to the body frame by multiplying it by c b o, this is the transformation matrix ok. So, this is minus 3 mu by r c cube c times b slash o 3 mu by r c whole cube these quantities and identity matrix. And therefore, only this part will remain here also, here we have also to change c these are the common errors that creeps in.

So, once we have change this, so this should also be in the b notation. While writing it goes in a flow this is b, this is also b, therefore here this would appear as b. And if we write in terms of this minus sign if we want to observe, so this will be plus 3 mu by r c cube, and this can be written as e tilde b cross this minus sign, because that way we have defined this times I b times e tilde b.

I will take it on this side, so this give us tau tilde b this equal to 3 mu by r c cube. And this quantity if you want to write in vector notation, so we should do like this. And what is this quantity we should look into this. So, I will work here itself times this we are converting this is in the body frame ok. And here on the right hand side, this is this is the quantity we are trying to convert ok.

So, e o 3 this is nothing but 0 0 1 here a this space is sorry 0 0 1, this is nothing but your e tilde o 3. You can see from this place that e tilde o 3, this can be described as c 1 1, c 1 2, c 1 3 taking it on the left hand side. And taking the transpose that is by or multiply both side by the transpose. So, this will get eliminated, this will get reduced to identity matrix here in this place.

And therefore, on the this can be described as e o 3 equal to c 1 1 c 1 2 c 1 3 then it c 2 1 c 2 2 c 2 3, and then c 3 1 c 3 2 c 3 3, and this operated by e 1 cap e 2 cap e 3 cap. So, this vector is nothing but your 0 0 1 ok, on the left hand side. So, what we see that this vector is a combination of it is given by combination of these three ok. Let me do it in the phrase or some other space, here we do not have enough space. We do not need to do this part, we will just remove this. This part will be good enough for our work.

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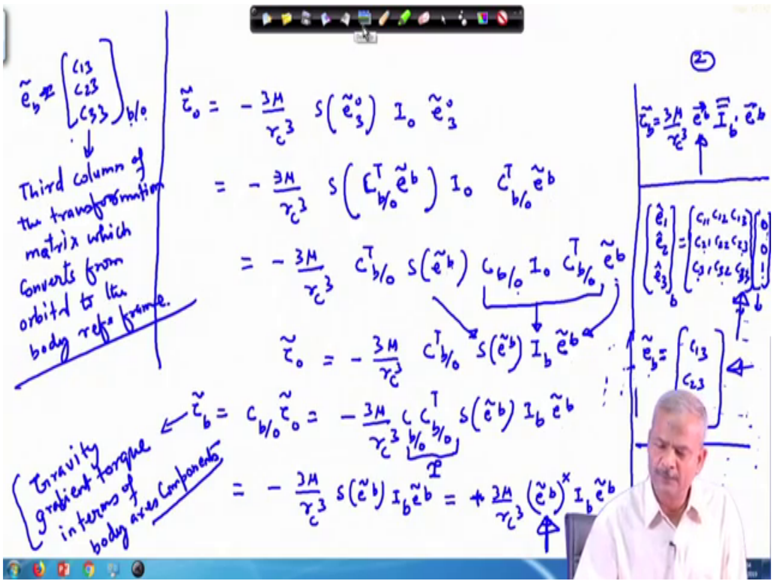

So, from this part we say that this is e tilde b, and e tilde b if we are trying to express, so what this quantity will look like? From here if we multiply it, so this will be 0 0 0 c 1 3 ok, this is 0, this is 0, and only this part. And then if we multiply, so this is 2 1 2 2 here multiplied this will be 0 c 2 3. So, the last only last will count, similarly the c 3 1, c 3 2, this multiplied with this is 0 only the last count so c 3 3. So, the former-1 I was doing it that is not required for our purpose for the time being, so we removed that part.

So, e b tilde which is appearing here in this place, and this place ok. This is the quantity which is mentioned here means, this is the third column of the transformation matrix which takes from orbital reference frame to the body reference frame. So, I will explicitly right here, so e tilde b this is given by this equal to c 1 3 c 2 3 and c 3 3. And this is the third column of transformation matrix which converts or transforms from orbital to the here I will put a tag, this is b slash o means orbital to the body orbital to the body reference frame ok.

So, we need to be very careful. So, this way what you have we have seen that if we go in the systematic way, then we can keep converting one after another. So, the left hand side also, we need to convert ok. While we work in terms of the vector, so at least for this step, this step we simplified the first step ok. Directly we wrote in terms of e c be replaced in terms of e o with the minus sign ok.

So, this is all about the getting the torque, and this is your this is all about getting the torque in the in terms of the body frame components ok. So, this is the gravity gradient torque gravity gradient torque in terms of body axes components. This is a very important ok, because the Euler's dynamical question, we have written in terms of the body x components. So, I will go in the next page, and then we will recall it.

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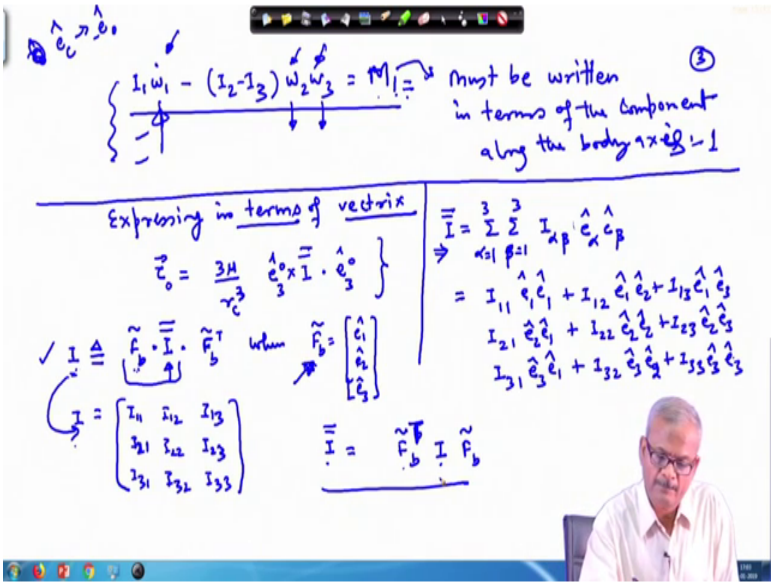

So, if you remember that the Euler's equation, we have written I 1 times omega 1 dot minus minus I 3 times omega 2 time omega 3 equal to the torque on the right hand side which is in M 1. Similarly, the other three two equations ok. So, here omega 1, omega 2 which are appearing so, these are the components of the angular velocity vector along the body axis direction ok.

So, here on the left hand side, this torque is along the body axis direction. So, here you must put the torque along the if your system is described in terms of the omega components, which are along the body axis. And on the right hand side, if you put in the orbital frame or either in the inertial frame system, so that will be a total blunder ok. So, this must be put in terms of the body axis system. So, this is must be written in terms of the component along the body axis component along the body axis, here this case this is the first body axis.

So, you see the way we are proceeding, so this is the difference between you can tend to memorize the whole thing, but without understanding you can memorize and you can write in the exam you can work. But, once the complicated situation arises, in that case the problem starts, because the understanding if there is not understanding how the things are being converted, then that leads to the whole blunder. So, this part I want to make it very clear, though we are going little slow. And we have exceeded the prescribed lecture, but I I want to show you that how the things are working properly ok. So, this part we have finished.

Then next part, I will describe your terms of the vectrix, which I have not done earlier, and this part I want to do it for you ok. So, for that we are start with writing the tau in terms of 3 mu by r c cube, and directly we will start from the orbital frame. So, on the left hand side this is in the orbital frame, you can put here o. As we see that we can just in the from thus inertial frame where the r c vector instead of using this way e c vector, we have converted this into with a minus sign e o cap. So, directly this gets converted one step we can skip here ok. And as usual we have written it is like this, this is e o 3, here in this place it was 3.

So, if utilise the vectrix method, so we can also work this. But, for that we need to develop the already we have developed the vectrix equation for this particular case, we need to write it further. So, by definition this I can be written as F tilde b, I bar double dot where F tilde b this is nothing but e 1 cap e 2 cap e 3 cap. So, this is the matrix notation on the left hand side, do not get confused by the to undersign. So, I am removing it, this is not required. This is in the matrix notation on the right hand side, this is the dyadic. And these are the vectrix this two, this is F b, this is the vectrix, and it is transpose is here.

This I double bar this dyadic is written as say here, I can write is a alpha alpha equal to 1 to 3, and beta equal to 1 to 3; I alpha beta times e cap alpha, and e cap beta. So, this implies that there will be nine terms in the inertia dyadic, this is what the inertia dyadic is ok. So, there are nine times comes in this, and which expands like this.

We put alpha equal to 1, and then beta equal to 1, so this is e 1 cap 1 right ok. Then I 1 2 bête equal to 2, so e 1 cap e 2 cap. And similarly other terms then set alpha equal to 2, beta equal to 1, so this is 2 1. And the third row this the other terms are alpha equal to 3 setting alpha equal to 3, and ranging beta from 1 to 3 e 2 cap. And so these are the total nine terms in this dyadic.

And if you take a dot product of this quantity, if you suppose that we are operating first with this, so if you take the dot product, so and finally you take the dot product with this. So, you will get a matrix ok, which will be this matrix where I equal to this is I 1 1 I 1 2 ok. So, this quantity is nothing but here ok.

On the other hand, the same thing we can also write as this dyadic we can express in terms of F tilde b transpose, so I can be converted into this part where F f tilde b is shown here in this place. So, if you if you operate on this matrix by this vector here transform, you can check it yourself. This part at least you can do as your homework, so this quantity will result.

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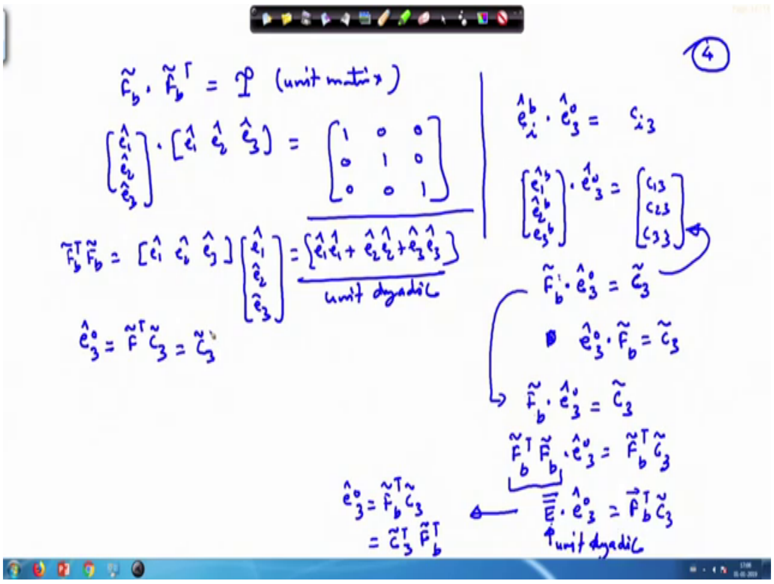

Also we have F b tilde dot F b tilde transpose, this will be a unit matrix. As you can check this is e 1 cap e 2 cap e 3 cap, so if we use this, so e 1 dot e 1 this will be 1, and e 1 dot e 2 this will be 0 0, so 0 1 0 the other things. So, this is a unit matrix ok. Some more expressions are required to work it completely.

So, this is e cap b i b use this notation, this we write as c i 3 that implies that if we have your e 1 cap e 2 cap e 3 cap, and this we can put b here. To indicate this is in the body frame, and this is in the orbital frame ok. So, on this side then we can write I will be replaced with 1, 2, and 3. So, this will be c 1 3 c 2 3 and c 3 3. So, this is basically showing the because both are the unit vector, so this is nothing but the angle between these two vectors.

So, this is showing that how this you have three vector, it makes angles with e 1 b e 2 b and e 3 b ok. And this quantity as usual we have written this as the F b, we have use the notation F b tilde F b we have written in the bottom. Here we are using in the bottom, so we write it here like this. So, this dot e cap o 3, this will be equal to this, and this part, we will write as c 3. To indicate this is along the third direction ok. And we will put a tilde here to indicate this is a vector, so this is your this part.

Similarly, the along the same line, because this is a dot product; so, this can also be written as ok. And if we use here F tilde b transpose F tilde b ok, we have to basically express this in a proper format. So, F b tilde dot e cap o, this equal to c tilde 3 multiply both side by b transpose which we are b tilde b dot F tilde transpose b c tilde 3, this quantity we have to look into.

So, this quantity here F b transpose F b tilde, this is nothing but e 1 cap e 2 cap e 3 cap, and e 1 cap e 2 cap e 3 cap. So, if you take this product, so this comes out to be e 1 e 1 cap plus e 2 e 2 cap plus e 3, so basically this is a unit dyadic ok. So, therefore this we can write as E double bar dot, this we are using as a unit dyadic the unit dyadic. So, this is e cap o 3 equal to F transpose b C tilde right ok.

And as we know that dot product of unit dyadic with any vector leaves the vector intact, and therefore this implies e cap o 3, this is equal to F transpose b c tilde 3, and because this is inner product. So, we can also write as C transpose 3 times F tilde transpose b. So, finally this expression also we write here transpose C tilde 3 equal to 3 transpose F tilde ok.

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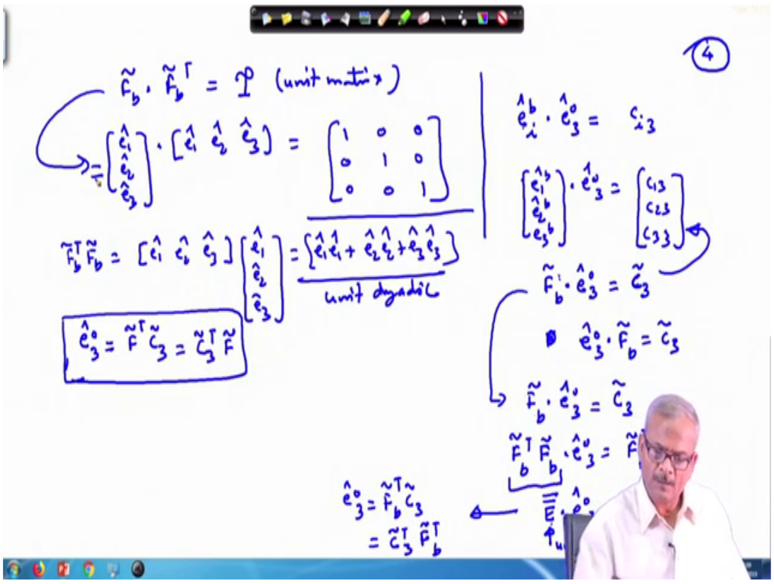

So, these are the results which are required for working with the vectrix method ok.

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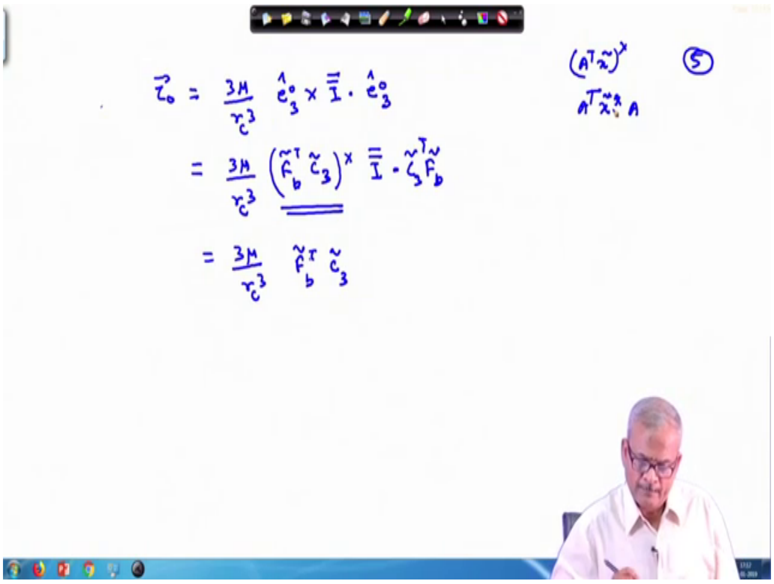

So, finally we are start like this the tau term, we have written, going back this particular part, this we pick up from in this place. Now, this is tau 0 3 mu by r c cube times e cap o 3 cross I double bar dot o 3 right, every thing is in terms of the vectrix notation. So, this part is our given here e o 3. So, we replace in term of this. So, F tilde transpose b times c tilde 3 ok, then cross of this I double bar dot, this we will write as because this is a dot product. So, we will use this one ok. So, here c tilde 3 transpose times F tilde b, we will write it like this.

And we know that this quantity can be written as F tilde transpose b times c tilde 3 ok, there are various ways of expressing this thing. And we are going to say like we [ha/have]- if we have a vector like this A transpose x tilde and this cross, so this can also be written as A tilde times A times x tilde, we have use the notation x tilde cross times A.

So, we will follow the notation here, what we have written earlier ok. This notation will we need to change it little bit, we will put this part here in this place ok. Rather than following this part, we do is just little change here, and make it like this. We will right here this c tilde transpose F tilde b cross, and then in this place we will replace it with F tilde b times c tilde 3 ok.

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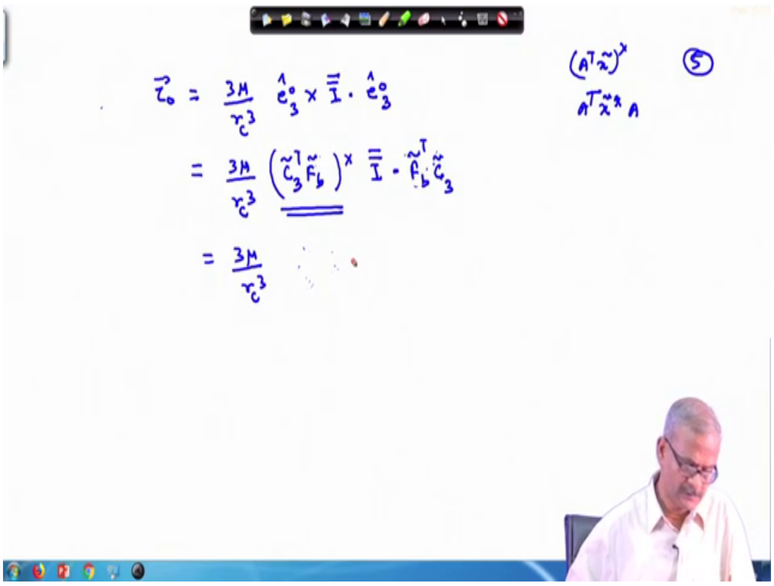

Whole idea is to get the final answer, what we have been looking for.

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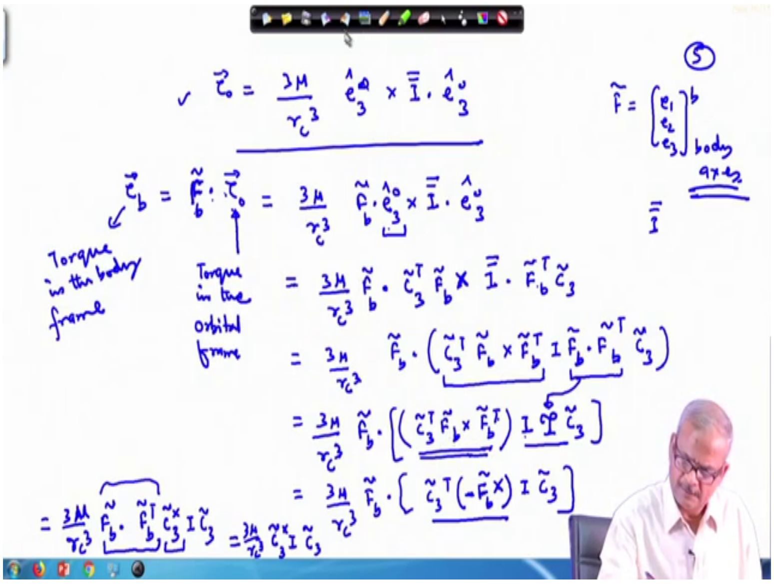

So, we have this equation, now let us multiply on both side by F tilde, so we take the dot product here F tilde dot, and this is tau 0. So, what we are doing, by doing so we are taking the components of this torque along the F tilde directions. F tilde is your vector which is e 1 e 2 e 3, and this is along the body axis ok.

So, if you take the dot product, so we get the directly the components of tau 0, which is the torque in the orbital frame gets converted into the body frame. So, left hand side then we can express in terms of tau b. So, this is the torque in the body frame torque in the body frame, and this is the torque in the orbital frame. So, the right hand side, then therefore will be 3 mu by r c cube F tilde dot.

And we are putting here b. This is and e o 3 we need to convert it, this part we need to convert. So, we need to put this, already we have worked for this. So, we convert this in the that format which is c tilde transpose 3 times F tilde b, and then cross this is a vector ok. And on the right hand side, we have this part I double bar. And this we will write as F tilde transpose b times c tilde 3.

Now, this I double bar we need to express it in terms of the this is this is in the form of a dyadic, so we will convert it in the form of the this kind of format we have used earlier. So, this I double bar, we have written as F tilde transpose b times I times F tilde b. And and there is a cross in mid between, so this cross is put here. And ultimately, we have on this side dot F tilde transpose b times c tilde 3.

So, this part we have merged together, and this part we know from our earlier expression F b times F b dot we have written as F b dot transpose we have written as the unit matrix. So, this I times unit matrix times c tilde 3 ok. This part can be written as c tilde 3 transpose times minus F tilde b cross, and this is your I times this unity matrix this will be I times c tilde 3. So, you may be wondering that what this whole thing is all about. ah

We can we will do it on the next page, this particular part. First let us complete this, so this gets reduced to 3 mu by r c cube. This part can be written as F tilde b transpose times c tilde 3, which I am going to do it on the next page I times c tilde 3. And we know this property r c, now this quantity is 3 mu divided by r c whole cube. And c tilde cross times I times c tilde 3. So, let me work all these things.

So, first let us look into this quantity F b dot F b dot transpose, the this perhaps we have done it earlier. So, F b dot F b dot transpose here itself. So, this is just I the unity the unit matrix, and therefore and this quantity which we will look here in this place, this is a skew symmetric matrix. So, therefore, just we can write this as I, this gets eliminated. And skew symmetric matrix appear, so we need not work on this particular part I will work. So, this is c transpose 3 F b cross F b transpose. So, first we will work on this F b cross F b transpose.

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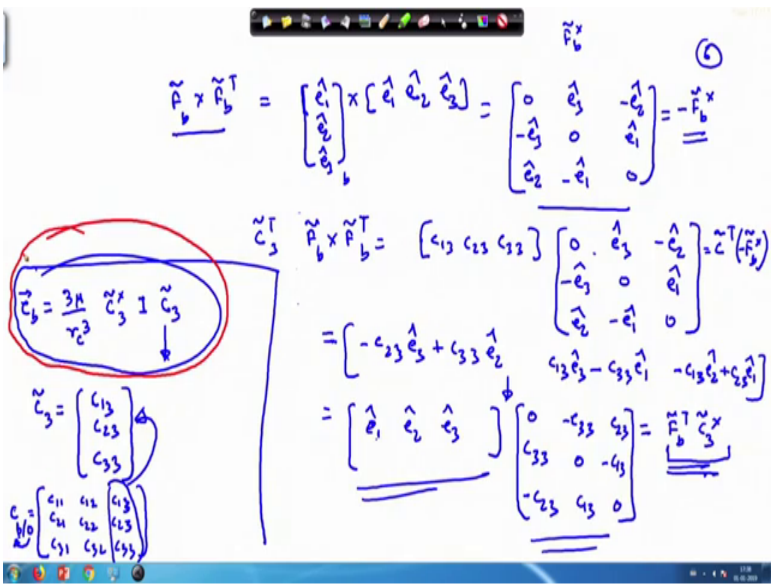

So, F b tilde cross F b tilde transpose, now this is the e 1 cap e 2 cap e 3 cap. This is all along the body or direction which I may not write it, because it is unnecessary to carry this tag e 1 cap e 2 cap e 3 cap ok. So, this is appearing in the outer product format, but there is a cross here. And if you take the cross product, so this will be 0 e 1 cross e 2, this will be e 3 cap e 1 times e 3, this will be minus e 2 cap. Similarly, others can be filled in ok. So, this get reduced to this skew symmetric matrix ok.

Then we have this times c tilde transpose c 3 c tilde transpose 3, and times F tilde F tilde b cross, so this will be your c 1 transpose. So, we pick up this particular part here we have written c 1 3, c 2 3, and this is c 3 3. We have written for this, this notation is very important to us while working with this and c 1 3 c 2 3 and c 3 3. So, this is the transpose here, and then this is multiplied by this part 0 e cap 3 minus e 2 cap 0 e 1 cap ok.

So, this type operation, already we have done if you remember, you just multiply it, and if you try to solve it. So, this can be reduced to let me maybe I will try to do it for you right now quickly. So, this is minus c 2 3 times e 3 cap, this part plus c 3 3 times e 2 cap this is the first term. Then this multiplied by this is c 1 3 times e 3 cap, and then c 2 will be this will be 0, and c 3 three plus c 3 three times e 1 cap here this is minus sign, so we have to put a minus sign here. And then for the last term this is minus c 1 c 3 times c 2 cap and plus c 2 3 times e 1 cap ok, the last term will be 0.

If we take out this e 1 e 2 outside, say we write it like this e 1 cap e 2 cap e 3 cap, and take the c 1 c 2 inside ok, so how the this equation will look like ok? So, first we have reduce is to this format, and then we are reducing again into this format, but only thing that e is outside, and c is inside ok. So, just looking into this the first column it will appear as if we have written like this, so this is e 3, so e 3 will e 3 is combined with this so minus c 2 3 will be here, and then it e 2 is here. So, e 2 is in this place, so c 3 3 will be appearing here in this place there it will be 0.

And similarly, you can look for other terms like the e 3 cap is here, so e 3 cap is here in this place. So, this is c 1 3 correspondingly. And here c 3 3 which is related to e 1 cap, so this goes here in this place minus c 3 3 ok. Here this place it will be 0, and accordingly we look into this place. So, e 1 cap with this it appears the c 2 3. So, here we will put c 2 3, which is multiplied by e 1 cap, and which e 2 cap this is minus c 1 3, and here this quantity 0. So, you can see that this is the skew symmetric matrix. And therefore, we can write this as this is F tilde b transpose, and this is nothing but your c tilde 3 cross ok.

So, now looking back here in this place; so, you can check that what we have done here. So, this quantity we are writing in terms of this one ok. So, this is 3 tilde c 3 tilde transpose, we look into this place. This quantity already it was in c 3 transpose format, so what we have written here 3 mu by r c cube c tilde cube I c 3 transpose.

Now, here in this part; if you look for this quantity so this quantity; so, this quantity is nothing but F tilde b tilde cross with minus sign, because minus sign is appearing here as per our definition minus if we are writing F tile b cross. So, here all the minus sign should be here, this should be plus, this should be plus, and wherever the plus is there that it should be minus. So, this quantity is nothing but this quantity.

So, multiplying together this F b tilde cross F b tilde transpose, this is nothing but your this quantity which is appearing here this particular quantity. And c tilde 3 transpose, which we have written here in this place, and rest of the things we have copied just here in this place ok. Now, c tilde 3 transpose minus F b cross ok, this is this whole quantity ok. So, instead of copying this whole thing here writing in terms of we could have equally written in terms of like here this is c tilde transpose times minus F b tilde cross, I could have written it like this. So, this is exactly the same thing which is written here in this place. So, this part it is getting reduced into this format.

So, finally we are getting this quantities multiplied together, it can be written like this. And this F b tilde transpose c c 3 cross here. So, go back and look here this is the whole thing from here wherein this part ok, this whole thing has been done F b dot is there. So, F b dot we have kept it here, this is F b dot here. So, F b dot here present, there after you have this quantity or either this whole quantity from this place to this place.

So, this quantity is nothing but F b tilde transpose times c tilde 3 cross. So, here you see this is this the same thing that we have written there ok, and rest of the things I times c c tilde 3, so times c tilde 3. So, this gets reduced into this format. And then what is this quantity here, so we will go on the next page itself we will work.

So, finally we get the result as tau b, this is the tau b which is the torque in the body frame this equal to 3 mu by r c cube 3 mu by r c cube. And then we have this c tilde cross 3 c tilde cross 3 I times c tilde 3, what this quantity this quantity is this quantity c tilde 3, this we have earlier denoted as c 1 3 c 2 3 and c 3 3. And if you remember this is the if you take the c matrix, which converts from the orbital frame to the body frame it converts from orbital from to the body frame, so this we have written as c 1 1 c 1 2 c 1 3 c 2 1 c 2 2 c 2 3 c 3 1 c 3 2 and c 3 3,

So, if you look, this is nothing but this is the third column third column ok, third column of this transformation matrix. And if so here in this equation this appears as the third column of the transformation matrix, and this is exactly what we have derived earlier using the matrix method there. So, there is no difference either you work from the vectrix method or either from the matrix method that is exactly the same identical result you will get. And this part especially you have I have worked for you, so that you understand how to work with the vectrix method, and how to work with the matrix method. Any one of can be applied, whichever you feel comfortable with ok.

So, we will end our topic today here in this place. And I will continue with this gravity gradient still some part is left, because till now what we have done, we have just calculated this part ok. This is your the torque acting on the satellite due to gravity gradient, but this is not enough. There after this must be applied for applied to the Euler's dynamical equation. So, once we apply to the Euler's dynamical question, and there after we solve the dynamical equation maybe for small perturbation from the equilibrium position or from the reference position, we are looking into. Then we will know whether this system is stable or not stable.

So, now we are in a stage where we are going into the stability of the system, we will look into the controls of the system. So, we started with the rigid body dynamics, then we have looked into how the gravity is acting on the satellite. So, this was fairly long, so as I told you that we have considered the earth to be a massive particle or in assumption we are made that earth is a perfectly uniform density sphere, so that while we apply the Newton's law of the Newton's gravitational law, so it gets reduced to a particle.

And there after we have got these equations which look complicated, but still it is a very simple. If we start trying with the higher order terms, like which appears in the case of when you are taking the oblique shaped earth. So, you know that the earth is not a perfect sphere, and also it is density is not uniform.

So, if you try all those things, it becomes massively complicated which are not a matter of this is not a matter of the classroom teaching, you can write a paper, you can do some professional work, like you have to work for ISRO. So, you can do those precise calculation, but for the classroom we cannot keep it stretching, it takes lot of time working on a small thing putting in the proper notation.

So, thank you very much. And next time we are going to apply this the torque that we have derived due to the gravity gradient to Euler’s dynamical equation and progress further that. So, it may take next maybe two lectures to wind it up. We are going somewhere 30-lecture, it was scheduled 25 to 26 to 30 lecture means, 5-lectures we are schedule for this, but it is lightly that we two more lectures. So, 2 to 3 more lectures will be required for this particular one.

Thank you again, so we welcome you to the next lecture next time.

Thank you.