Satellite Attitude Dynamics and Control
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Lecture – 29
Gravity Gradient Satellite (Contd.)

Welcome to the lecture number 29. So, we have been discussion about the Gravity Gradient Satellite so, we will continue with that.

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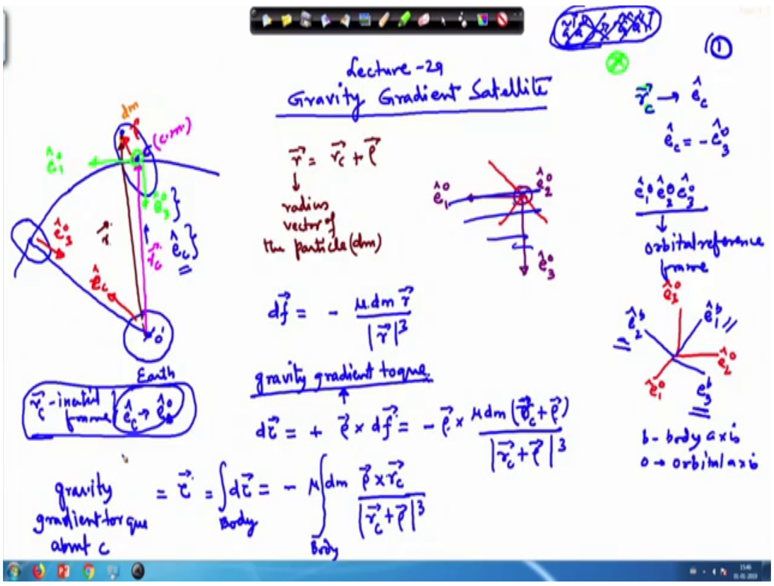

If you can recall that if a satellite is moving in an orbit around the earth. So, this is your earth and there is a satellite here in this place. So, that satellite we showed by may be some oblate shape or whatever it can be. Then we have radius vector to this point which is the center of mass. This I will be indicated by c and this is c this is centre of mass.

In this we denoted by r c and then we took any mass in this body which are the satellite here in this case. So, let us say this is the mass delta m or d m and the radius vector to this point from here to here, this is rho. So, radius vector of this point from the centre of the earth this point ok; from the center of the earth and we have shown by r where r we have written as r c plus rho is the radius vector of the particle d m ok. Then if you remember that we fixed the triad about this axis, I have this point and which we have written as along this direction we took e o 3 cap; we have shown it by e o 3 cap.

Along this direction we have shown by e o 1 and 2 is going into the page which going into the page always, we show it like this means you are looking at the back of the arrow. So, arrow is going inside and therefore, you are looking at the back of the arrow. So, e 1, e 2 and e 3 so, this forms the right handed system. So, as you can see that this radius vector r c, this is the radius vector r c. Let us say units corresponding unit vector, we can write as e cap c and e cap c and this vector e o 3, these are related by e cap c. We can write this as a e cap o; e cap o and here in this place we can write 3. So, this is the e 0 the system e 1 cap, e 2 cap and e 3 cap and these are 0s a small 0 shown here.

This stands for the orbital reference frame which we have discussed in the previous lecture and then the body acts as it body at system. So, it is a oriented with respect to this orbital reference frame. So, I will not show it here, otherwise this figure will get totally destroyed. So, let me show it somewhere else like if you have, this is your e o 3 cap in this direction you have o one and going into the page.

So, it this is your orbital reference frame to with respect to this your body reference frame it may be oriented arbitrarily, but again it will form a right handed triad. So, and this orientation; obviously, there is the set two dimensional figure from the this page, it cannot be shown that correctly, but if you look like if I have this e 1 cap e 2 cap e 3 cap.

So, instead of this place I am showing at here. So, we can show it something like this e let us name this point as e cap b 1, then e cap b 2 and e cap b 3. So, this forms here right handed system means the body axis already body system has already rotated with respect to the orbital axis system ok. So, with this then we wrote the force acting on this small particle which is here in this point d f equal to minus mu times d m, then r which is the radius vector of this point from the centre of the earth divided by r magnitude whole cube.

And therefore, the gravity gradient torque about this point which I am showing here by blue point ok. So, gravity gradient torque it is important to remember this gravity gradient point because it is a because of the variation of the gravity along the radius vector r. So, gravity gradient torque if we show it by say we can show it by some tau or some any other notation we can use it; it is not a problem. So, let us say that we showed it by tau. So, this will be rho cross d f and therefore, this is minus sign will come here rho cross d f equal to mu d m r we can write as this is r c plus rho, so, r c plus rho and divided by r c plus rho magnitude whole cube.

And if you see that this if we break this bracket so, this cross product rho cross rho cross will be 0 and this gets reduced to mu d m rho cross r c divided by r c plus rho whole cube. And if we integrate it over the whole body that gives me tau this is the gravity gradient torque gradient. So, this point is c here; gravity gradient torque about c which is the centre of mass of the satellite. One thing I will advice you sometimes writing inadvertently some error will get introduced because like here the minus sign is there if I go to the next page so, automatically sometimes this error may introduced that instead of minus, there it is plus. This minus sign is left, but later on it most of the time I have corrected it.

So, just wait for in the case you find that there is any error like this sign or say some way of instead of instead of like if I remember some in some place I have written something like a tilde a tilde dot and then instead of this, it should have been a tilde times a tilde dot transpose something like this ok. So, this kind of error can get introduced while working it. So, because of the attention getting from away from this place or going from one place to another place, this kind of problems can appear but I correct it all the time, so, just keep tracking ok.

So, if you find at somewhere some error is there so, go to the next page or whatever you wait for sometimes mostly because if the result is not match so, I correct it I go back and then correct it.

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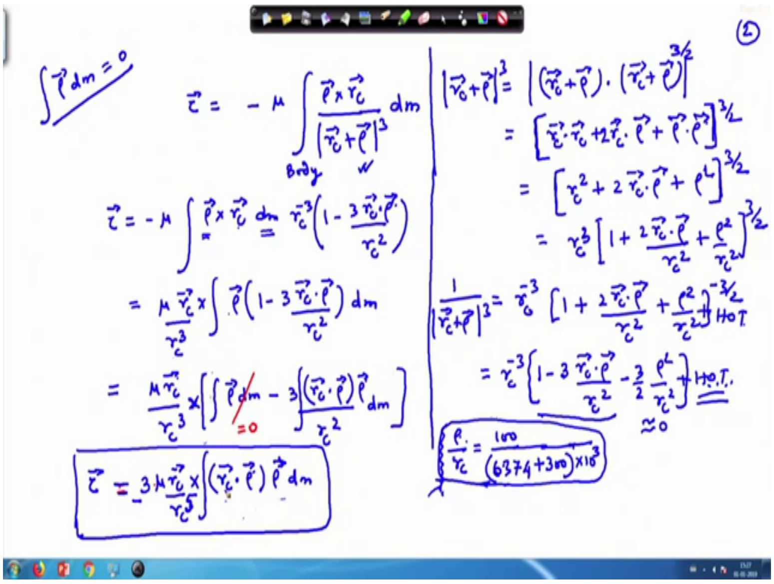

So, we have here tau this equal to minus mu times rho cross r c divided by this the place where most often we do the mistake copying from one place to another place. So, this is mu rho cross r c over the whole body, we have to integrate it. So, first of all we will determine this factor as we can see this r c plus rho whole cube this can be written as this is r c magnitude.

So, this is r c rho dot r c plus rho 3 by 2 because if I am taking a square term here this is the dot product. So, this becomes a square and therefore, I must divide this 3 by 2 and then break this bracket. So, this will be r c dot; r c dot plus 2 times r c dot rho plus rho dot rho this is r c square plus rho plus rho square. And if we take r c outside so, this becomes r c cube and once we have taken this dot product here and we written in this way. So, this mod sign we can remove it this magnitude sign and we can put a bracket here. So, r c cube 1 plus 2 r c dot rho divided by r c square plus. Now if this term here it is a lying here in the denominator.

So, if we take it so, we have to take it upside like here in this place itself I will work and then transfer it here. So, 1 by r c plus rho whole cube this quantity will be then r c minus 3 and then 1 plus 2 and if we do the binomial approximation to the first order so, this will be minus 3 by 2. So, and then multiplied by 2,so, that gets cancelled out. So, this is 3 r c dot rho divided by r c square minus. So, these are the places where the mistake can occur ok. So, here once we are taking the binomial just I wrote it plus, but it should be minus here, so, this is minus 3 by 2 rho square divided by r c square.

Now, looking into this quantity this rho this rho can be of the order of say rho by r c your satellite. In the case of the international space station if we are considering so, in that case rho can be your of the order of 100 meters ok. So, let us say this is of the order of 100 meters and r c; r c is the distance from the centre of the earth to the centre of the centre of mass of the satellite. So, that distance normally if you take the earth radius to be is 6 this is a round 6374 kilometers 6374 kilometers and to that you can add the altitude of the satellite, let us say that it is in the 300 kilometers orbit. So, this becomes here r c and then multiplied by 10 to the power 3 to convert into meters.

So, you can observe from this place that what will be the order of this quantity. So, this quantity and then once you a square it so, by a squaring you are making it further a small. So, therefore, this quantity is negligible. So, we can consider only this quantity here. So, if you look that look here in this place r c and here r c is also there. So, r c r c will magnitude by it cancels out, then we get rho by r c. So, again rho by I a rho by r c it will be given by this ok.

So, once we square this term so, this term becomes very small but that is to the second order. This is we call this term to the first order term and these are the higher order terms in r c and rho ok. So, this time for here we could have written plus higher order terms; plus higher order terms n. So, this kind of terms we can ignore, but if you are looking for a very precise calculation then you should take into the account otherwise for getting into the physics these things should be ignored. Now once we ignore this so, what we get? This tau equal to minus mu rho cross r c d m times this quantity we have to pick up from here. So, this is a r c minus 3 1 minus 3 r c dot rho divided by r c square.

This term from here to here we have picked up and this term we have ignored. Now this term will take this rho on this side and this r c will bring on this side. So, this minus sign can be a result and this and therefore, this can be written as mu times and one thing more that this r c is integration over d m is just related to this rho this is not depending on r cs. Therefore, this r c can be taken outside we can write it like this and then we have the rho minus sign, we are absorbing because we are flipping it the these two quantities this two vectors and then there after write all these terms. So, here we have similarly we can take here from this side the r c cube 1 minus 3 r c dot rho divided by r c square d m.

So, this r c we have taken out side cross we have taken out side this rho is here, this term we have picked up from this place to this place and this d m appears here in this place. So, in the next step ok; we take this rho inside and if we take it to inside so, we can break it like this. This is rho d m minus 3 r c dot rho times rho r c square and this also we can write here this is d m ok. So, these are the 2 integration and we know that rho d m because rho is taken from the centre of mass of the satellite. So, this quantity will be 0 as we have discussed earlier also, so, this quantity vanishes. This quantity will be equal to 0 and therefore, this gets reduced to mu r c by r c cube.

Then cross is here and 3 we can bring from this place to this place and as we see this minus sign is present here. Then r c dot rho by r c square this r c square is there. So, we can take it here in this place, make it 5 and this r c a square we can remove from this place. And finally, we need to process this. So, if you see this is a scalar r c dot rho; this is a scalar and this is operating on. So, this r c is a vector which whose dot product you are taking with rho. Now this quantity we need to work out, so, this is your tau.

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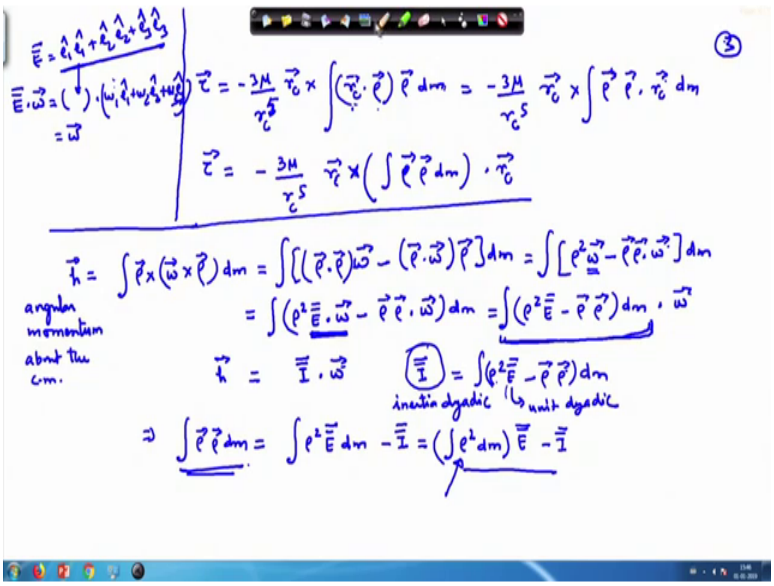

So, this previous ones this equation we have copied here in this place ok. Now, this particular equation, it can be rewritten as 3 mu by r c 5 r c cross and if you look here in this place we can flip this. We can write this rho because this is a scalar this whole thing.

So, this rho and be brought in front and moreover if you look here. So, this is a dot products. So, its either it is in this place or this is in this place it does not matter. So, we can change the order of this. So, if we change we can write it like this is d m and therefore, this can be simplified. So, we write this as 3 mu r c 5 rho rho d m and dot r c because this r c is not dependent on the integration. So, if it is not independent depending on the integration, it can be written outside.

Now, from here we need to go back and look into what this quantity is. So, if you remember that the equation for the angular momentum, this is angular momentum about the centre of mass about the centre of mass. We have written as rho cross these things we have developed earlier while dealing with the rigid body dynamics and then this we wrote as rho dot rho times omega minus. If you remember this thing we have already worked out ok. So, in the in dyadic notation so, this can be written as and we can rewrite this as while working with the rigid body dynamics thus equation we have written. And we know that this quantity ultimately we have packed this as the inertia.

So, if you want to write in the form of matrix, we can write it in the form of matrix if we are write in terms the dyadic we can also do that. So, from this place this appears like here this quantity we have written as inertia dyadic from this place to this place. So, I double dot ok. So, if this is the quantity h we have written. So, what is of interest here in this place this is the I double bar which is nothing, but the inertia dyadic inertia dyadic and which we are writing as rho a square e double bar minus rho. So, if we try to rearrange this. So, this implies rho rho d m rho rho d m this quantity will be equal to rho square E double bar d m minus I double bar and this can be written as rho square d m.

And E double can be taken outside because E double bar it is a vector ok. It maybe a these are related to the its dyadic and where it is the unit dyadic, this is the unit dyadic. Unit dyadic, how we have written? E double bar this quantity will be equal to e 1 cap e 1 cap e 2 cap e 2 cap and e 3 cap e 3 cap. If you operate with this unit dyadic on any vector that vector remain some change as you can check it. So, copy it here in this place and take the dot product with omega where omega is omega and this omega we can write it as let us write in the full form omega 1 e 1 cap plus omega 2 e 2 cap plus omega 3 e 3 cap ok.

So, if we get the dot product you will see that this will result in omega. So, that was the thing that we have done this part this omega we have written in terms of unit dyadic and this omega and therefore, this is just a dyadic which is related to the reference frame and it has nothing to do with the mass and therefore, it can be taken outside. So, here this can be written as this minus this. And this result what we are getting here in this place this quantity equal to this quantity can be inserted here in this point and then the system can be solved.

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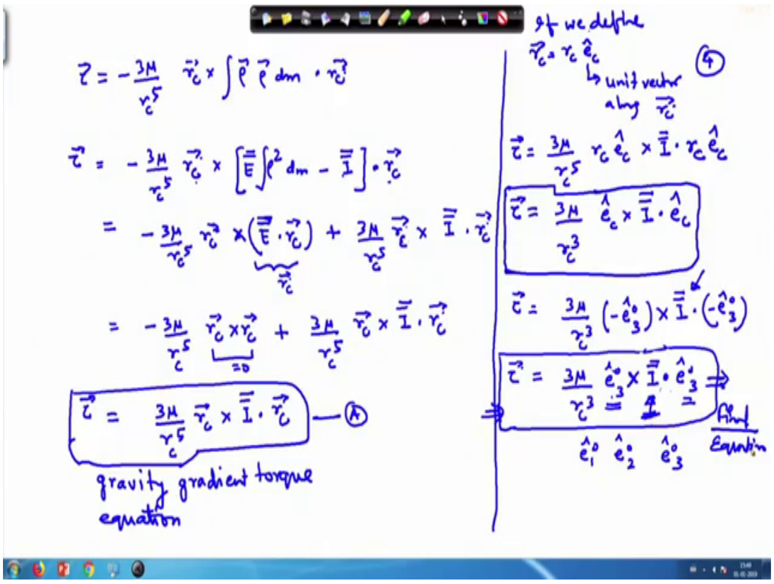

So, we have the gravity gradient torque. We have written as tau; tau, this equal to we need to copy it here minus 3 mu by r c to the power 5 and then r c cross this quantity and then dot r c ok. So, more over we use the result that we have derived on the previous page I mean this result we can use it we need to replace it. So, in that case this gets reduced to r c to the power 5 r c cross; now this part we have to replace. So, this is rho square d m E double bar minus I double bar this is I double bar dot r c e double bar dot r c.

Now the this bracket and this is minus and then other part will be plus. So, this is 3 mu by r c to the power 5 r c cross; this is r c cross I double bar dot r c. So, this is dot product of this r c with the unit dyadic so, ultimately this will be nothing, but r c. So, this quantity is nothing, but r c ok. So, therefore, in the next step we can write this as 3 mu by r c to the power 5 r c cross r c plus and this is nothing, but 0 because it so, the cross product of the same vector. So, this part vanishes and what we get is 3 mu by r c to the power 5 r c cross I double bar dot r c. So, this is the gravity gradient torque equation

Now look into this quantity. If we define r c equal to r c times e c cap ok. So, this can be this equation can be reduced to if we define where e c cap this is the e c cap this is the unit vector unit vector along r c that implies that in this figure here r c vector is going from this point to this point now. I hope its a visible on the board. This is visible through the cursor its a going from this point to this point. So, r c e c cap is the unit vector along this direction itself along this direction ok. So, if we insert this here in this place so, this tau this gets reduced to 3 mu by r c 5 and here r c times e c cap cross I double bar dot r c times e c cap.

So, this is your final equation, but still here if we look into this is expressed in terms of this e c vector this is expressed in terms of this e c vector which is going from this point to this point. But most of the time we work in terms of the body axis coordinate system means we will be working in terms of this e 1 b e 2 b and e 3x b b type is stands for the body here b stands for body axis and then this o stands for orbital axis ok. So, we need to convert this in proper format. So, you can see that e o 3 this victor e o 3 which is shown here by this green and this vector they are directive they are obviously, unit vector, but they are directed opposite to each other ok.

So, instead of using this r c I can equally write in terms of e o 3 because they are the vector all along this two vectors will be opposite to each other. Suppose the satellite, it moves from this place to this place it comes here in this place. So, this is the centre of mass and then at that time your the direction of the e cap o 3 will be like this while the r c vector will be directed like this. So, this will be your e c cap direction ok. So, they are opposite to each other and they are of unit magnitude. So, if you are writing the whole thing here in terms of e c. So, it is a directly it is a convenient to express this as 3 mu r c cube and instead of this we can have here with minus e cap o 3 cross minus sign this cancels out.

So, you will question that what to do for this. This I double bar it is written in terms of if you go and look into the previous page it is written in terms of rho if you look into this I double bar this is written in terms of rho and this rho vector rho square and rho vector and this is a unit dyadic ok. So, rho vector as we have sent it to from the r c notation to this notation. So, it says that we can express everything in terms of this e o cap or otherwise we can go very systematically using the matrix method and change from one reference frame to other reference frame like first assuming that this r c is described in the universal reference frame.

Therefore, this e cap c we need to convert it from this frame to the orbital frame ok, we will need to convert it from e c frame to the orbital frame. So, this will be little long I can do it here. So, that I will do, but I will do it in the next class for the time being let me finish this. So, this can be this is basically here if you are writing in terms of the r c you were using this r c vector for which you have replaced it by the e c cap the unit vector so; that means, from here we have then transition to the orbital reference frame vector which is along the z direction of the orbital frame this part ok. So, therefore, this can be expressed in the this is basically expressed in the orbital reference frame.

And once we go for the this one this particular one so, here this dot product is there. This is dot product this is cross product, we have to be very careful. This system is I am not able to show you properly because this is on the screen computer screen. If it is a page paper system, then I could have easily shown you everything. So, e if you look here in this place this I double bar vector is a this dyadic is there which we are telling it as the inertia dyadic. So, this inertia dyadic is currently we are expressing in terms of orbital reference frame vector here it is expressed in the terms of these vectors e 2 cap e 3 cap. And if you do not want to do this, then come from start from the c c vector and assuming that this c c vector was in the inertial frame.

So, we can go to the orbital frame system and from orbital then we have to finally, go to the body frame system, but here in this case it. So, happened that e c and e 3 this e o 3 cap, they were directed opposite to each other. So, we have been able to replace, but there after going to the body frame we cannot do it. So, conveniently and we need to do certain transformation. So, I will keep this equation for the time being here and I have something more to tell you. So, I feel that time is over for this lecture. So, we will continue here. So, what we have concluded today that this is my final equation, but still not in the format we want; finally, this is the final equation ok. So, I will stop this lecture here.

Thank you very much for listening we will continue here in the next lecture.