Satellite Attitude Dynamics and Control
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Lecture – 26
Stability of Torque Free Rotation

Welcome to the lecture number 26. So, already we have discussed about the rigid body dynamics under the torque free condition especially for the symmetric case. In this lecture we will continue with that, but here we will look into the stability of such a system which is torque free and having the symmetry as discussed earlier, ok.

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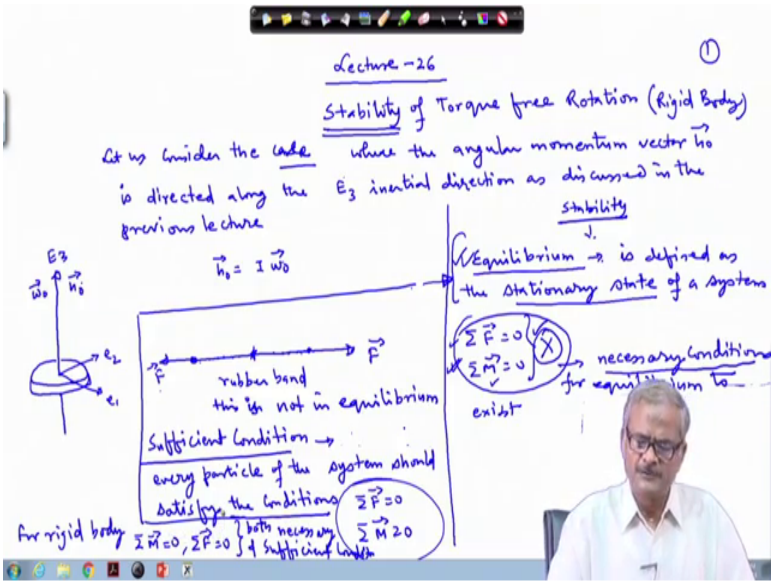

So, if let us consider the case where the angular momentum vector h 0 is directed along the E 3 cap or E 3 inertial direction as discussed in the previous lecture. So, this is a case where your, this is the E 3 direction and h 0 is directed along this direction ok. So, the initial configuration is something like this. So, here it is a disk and which is rotating on this axis with angular velocity omega 0.

So, we can write here h 0, this equal to I times omega 0, both are in the same direction and these are the body axis e 1, e 2. Now, once we talk about the stability of the system ok. So, stability we always we can discuss in terms of like the global stability or the local stability, this kind of notion they arise. So, what I will do that, first I will let you know what the exactly the stability is because without that it will be little abstract to discuss this topic.

So, here we have ok, heat up the stability. This notion first it is a derived from the equilibrium ok. It is called the equilibrium is defined as, the stationary state of a system. And what you have learnt till today that E F equal to 0 and E M equal to 0 that is the vector sum of all the forces and vector sum of all the moment if it is 0, then we call it equilibrium. This is not equilibrium. The definition of the equilibrium is this one ok. It is defined as the stationary of the state of a system.

If the state of the system it is not changing with time. Now these two conditions, these are the necessary condition for equilibrium to exist. But, this does not guarantee that your system will be in equilibrium. Just like take the case of a rubber band, say this is a rubber band and if we apply equal forces in both the directions, so here F and F, they are directly opposite to each other and also the moment if you take moment about this point.

So, this moment will also vanish. So, both these conditions are satisfied. But as you apply these two forces, so, this will stretch ok, this will stretch. So, therefore, this is not in equilibrium, this is not in equilibrium ok. So, why it is not an equilibrium, because we will see that the particles in different places, they are as you apply the force this say this is a rubber band and if I stretch it, so the particles on this they will accelerate on both the sides of this point.

So therefore, if the particles of this band they are accelerating means this simply implies that this is not under equilibrium condition. So, to ensure that this part, ensure a proper definition we need to further estate that. So, this is the necessary condition and the sufficient condition, then we have to also estate and sufficient condition is that, here this is the necessary condition each and every particle should satisfy the conditions E F equal to 0.

And so, if on every particles of the system will remove this, every particle of the system should satisfy the condition. So, you can say that once you have a stretch and the rubber band is a stretch to full extend for the force supplied. So, all the particles will be then at rest. So, the equilibrium condition can be divided in two parts like the dynamic equilibrium or the static equilibrium, something is moving.

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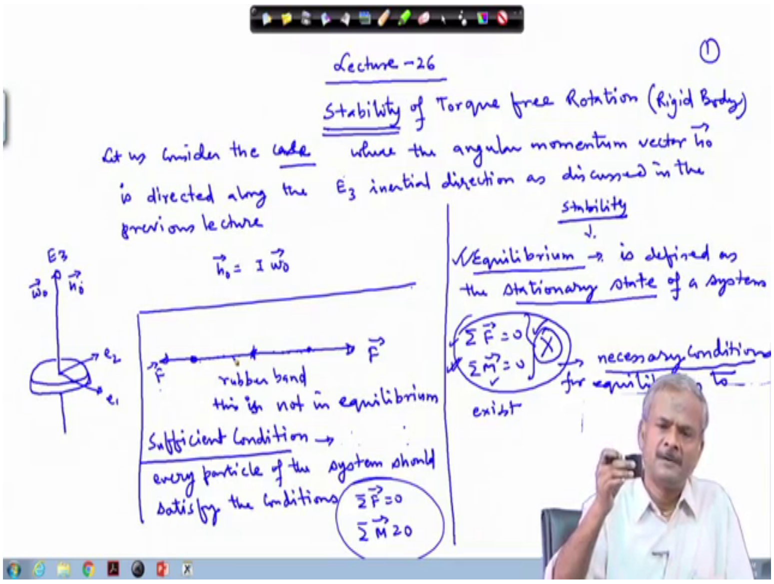

So, if I take this cups out of thing and if this is moving with a constant velocity ok, so you can see that this will be in a condition of equilibrium but it is a dynamic one it still moving ok. And the in the static equilibrium, if the particle is at rest then we will see that this is in static equilibrium condition. So, this definition what we have given this is a very broad definition and it is applicable to many system, it may be thermal system, it may be mechanical system, it may be electrical system, all sorts of system it can apply.

But definitely, whatever you have learned till today that E F equal to 0 and E M equal to 0 this is the equilibrium, this is not the equilibrium this is just necessary condition for equilibrium to exist, but not the equilibrium itself. So, it so happens that for the rigid body; for rigid body E M equal to 0 and E F equal to 0. These are both necessary and sufficient condition because in that case, once the body is rigid. So, there will be no extension ok. So, in that case you are like here in this case this is a pen. So, if I am stretching it, there is no extension in this. Virtually obviously, under certain load, there may be some extension which will be very on the minute scale.

So, which we are not discussing about but anyhow, even if it is stretched say even on the minute scale, so if the rigid body implies, it is a perfectly rigid means whatever the forces you apply it will not get extended. If it is an elastic body only under that condition once the force is applied, it will get extended. So, for the rigid body there is no question of any movement of any particle in the system. And therefore, E M equal to 0 E F equal to 0, that gives you the necessary and sufficient condition ok.

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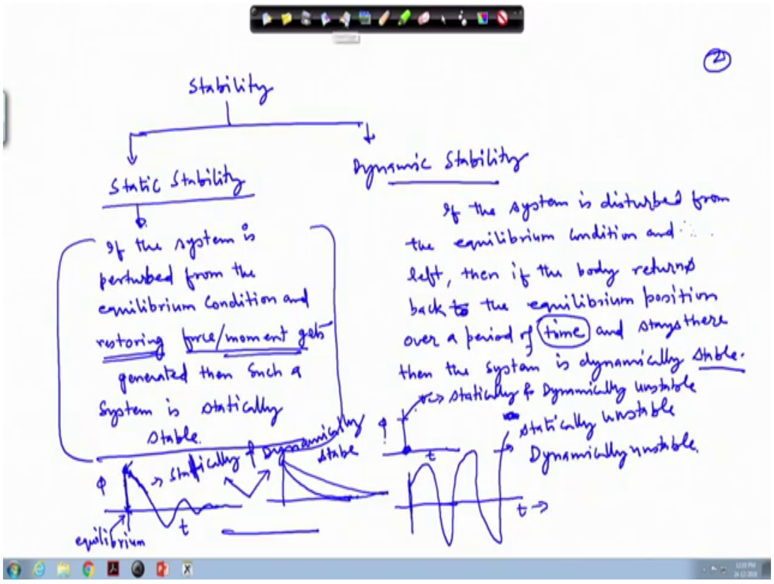

So, if, so that way from here, from the equilibrium we get the notion of stability say if about the equilibrium position, if it disturb the system from the equilibrium position in it is never old and live it ok, then if the system returns to the equilibrium condition over a period of time. So, a term another term we use it is called the dynamic stability. So, we will have the stability notion; it is derived from the equilibrium. So, stability we can divide into two part; the static stability and the dynamic stability.

In the static stability, there is no time involved ok, there is no motion involved. Once you have disturbed the system this is my pen and obviously, know that this is under stable system. As you leave it so, it starts falling means until unless it is a perfectly in the vertical condition. It cannot stand like this. So, little bit of disturbance, even time is speaking, so, the air coming from my mouth can disturb it and it will fall ok. So, if the system is called statically stabled, that if you disturb it from the equilibrium condition and then it returns back to the equilibrium condition like if I take this inverted case.

So, if you see till this plus minus 90 degree or even until unless it goes to 180 degree what we see that if I leave it so, it tends to go back here in this place. So, if I just take it from this place to this place, what is happening? There is a restoring torque which is generated about this hinge. This is this fingers are acting as the hinge here ok. So, as soon as I perturbed it little bit, so immediately the torque gets generated about this hinge because of the gravitational forces which is acting about the centre of gravity of this pen.

So, this generation of this restoring torque if the torque is generated which is restoring in nature, then we say that those kind of system gets a statically stable. So, here there is no motion involved remember. So, if the system is perturbed from and restoring forceless moment gets generated, then such a system ok; accordingly, if the chemical system is there where you are it is an equilibrium concentration of various liquids in that it is an equilibrium condition and if you are disturbing it and again it gets back to the same condition. So, getting back to the same condition will fall on the dynamic stability but if the restoring forces moments or the potential all this things are generated, then we call this as the static stability.

So, restoring force moment slash movement gets generated then such a system is a statically stable. While here in the dynamic stability, if the system is disturbed from the equilibrium condition and then left, then if the body returns back to the position over a period time and stays there, then the system is dynamically stable. See the difference. Here, there is no time involved. Any work, I have not mentioned which is time here, but here in this case; obviously, this time appears ok.

So, if I show it on a graph, so I can show it like this. If this is t and this is the equilibrium condition here, this is the equilibrium position and say I am disturbing this by something like phi. So, this is the initial condition here. So, as you disturb and leave it here in this place, so if there is restoring torque if there is restoring torque. So, system will have tendency to move toward this direction initially ok. So, that we will call as statically stable like here, say if this has a tendency to move toward this direction. So, this is a statically stable.

But, there is no moment I am telling at just at this point this will have tendency to move toward the equilibrium condition or toward the equilibrium. So, there is no movement involved. If you see here, if I am drawing a line like this; so, this is only in time ok. So, time is not involved; just start the initial position, there will be a tendency that tendency will come from if there is the restoring force and moments are present. And if it so happens that it comes and stays over this, then this is dynamically a stable, but here we can categorize as like this is the damped oscillation which we call as the damped oscillation, there can be situation like this is the critically damped or the over damped cases may be there something like this or if we discuss about the next stable situation. So, it can be something like the system may go like this or it may happen like, system may start from here it may oscillate like this and the oscillation can keep growing all the time.

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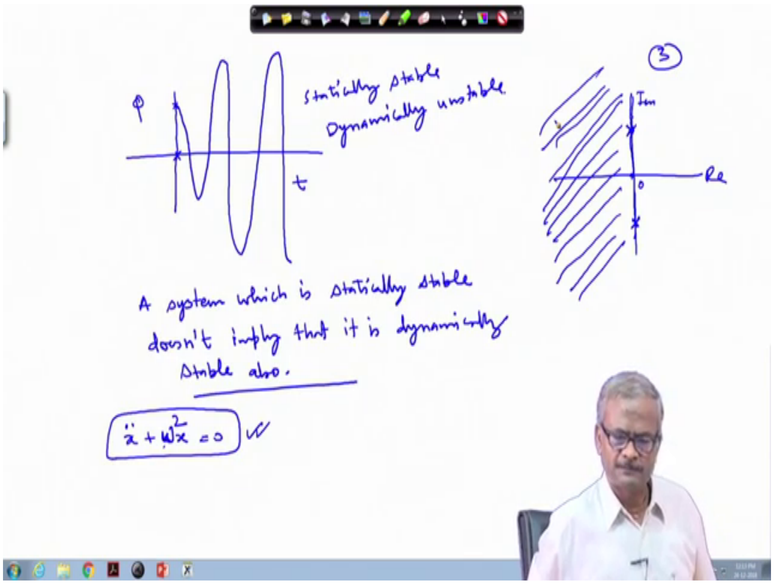

It may also happen that you have perturbed the system from the equilibrium position. So, there is initial tendency to move toward this, but later on what we see that the oscillation grows ok. So, this kind of system it is a statically stable but dynamically unstable. This is also statically unstable; also dynamically unstable. This is statically and dynamically stable. Statically and dynamically stable; both of them ok, both of them are statically and dynamically stable. This is both statically unstable and dynamically unstable.

Why because, their initial tendency to move towards, move away from the equilibrium position. This is the equilibrium position this we are plotting in t, there is some of there is a tendency and thereafter as the time is involved this also keeps moving away from the equilibrium position ok. So, this way you can analyze. So, for a system or system which is statically stable, does not imply it is dynamically stable also.

So, for dynamic stability, it is require that if you consider linear system at least for the linear system you know that if the system is dynamically stable, then the pole should lie in the left half complex plane. And if the poles are lying over this imaginary axis ok, then the system you call this as the marginally stable system because if you have disturbed it. So, the oscillation remains at that level only say if here in this case, just like in the case of this spring mass system, we can write this system like this. And if you disturb it, so you will see that there is no damping present in this dynamical system which is a spring mass system I have removed the same and here I have written, written in this format. You can write here as better as omega square.

So, this will keep oscillating, there is no damping in the system. So, for this kind of system, the poles are lying here over the imaginary axis of sphere of poles. So, it will keep oscillating for all the time. So, this kind of system we call as the marginally stable system. Quite often, this kind of system in practical cases, these are unstable because of because they observe energy from the environment and then the oscillations keeps growing ok. And therefore, this kind of system we do not put in practice. Whenever we are designing a system, we ensure that that the system remains stable when the poles stay lie in the left half complex plane ok. Then only, they are this kind of system will be of practical uses ok.

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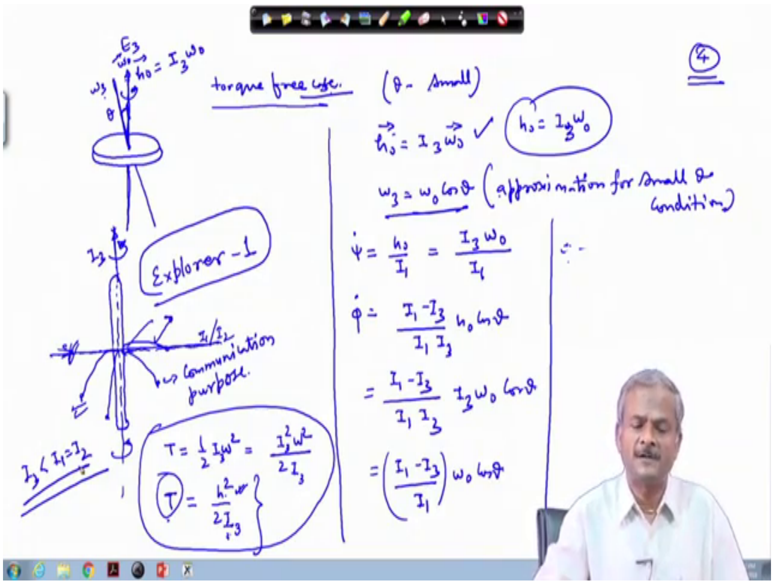

So, in this context what we have been discussing that if I have a disk where which is free from the external torque and this is rotating about it is third axis and this is the initial situation and h 0 is directed toward the E 3 axis. Now, we if we perturb this system and this is a torque free case, torque free case; if we perturb the system, let us say that it is perturbed by some small amount. This axis of rotation, it change from this place to this place by applying certain force or certain torque whatever. Then, how the system will behave and obviously, thereafter left. So, the there is no torque acting on the system. So, what will happen whether the disturbance will grow in over a period of time means it is say you have mutated by some small angle theta ok, you disturb it from this position to this position by angle theta where theta is a small.

So, will this kind of system the, it will remain about the same position or will this disturbance will grow over a period of time? So, this stability is necessary in many cases because if we take the example of explorer 1, this was the first American satellite and there were four trans file antenna ok. These were flexible antenna over this explorer ok. So, explorer was cylindrical in the shape and it was set in rotation over this axis and this was meant for communication purpose. But it so happen that once it was put in the orbit, then it started rotating above the about the major axis means it has started rotating about this axis and the reason we can if we do this torque free because this is a this was considered to be a torque free case.

So, if we look here except for gravity gradient ok if, in the case of the gravity gradient, but those are very minor talks. Here, in this case the things out turn out to be and more over the gravity gradient torque itself the gravity gradient. If you look the motion of the satellites is on the that it most of the time, it will remain, if there is no dissipation of the energy. So, it will remain conservative it is a gravitational force is conservative in nature. So, it is a conservative system. But here in this case, as the satellite was rotating these antennas, they were oscillating and because of the oscillating, the energy kinetic energy of the system it dissipated in the form of it.

And therefore, it is a initial rotation which was along the major axis, this minor axis, this long axis from there it deviated and came to the this major axis and this we can we can check quickly from this place that kinetic energy is written as 1 by 2 times I omega square ok. And this, we can write as I square omega square divided by 2 I equal to h square by 2 I here for this case. So, I is the moment of inertia about this axis for this case, we have written it often as I 3 let us say this you are writing as I 3, I 3 here in this case is the minor axis.

So, as this antenna started oscillating, so energy has started dissipating. So, this t has started decreasing. So, if it decreases. So, what will happen here in this case because it is a torque free case. So, h cannot change, h is a constant. So, what can change? This I can change ok. So, the rotation axis that change. So, this was rotation about the minor axis. So, from here it went to as the energy kept dissipating. So, it went and settled over this major axis which is here in this case I 1 and I 2 axis ok.

Here, we can have one axis like this perpendicular to the page of the paper and one axis in the page of the paper and here one axis like this. So, the minor acts rotation; obviously, this is not stable and we will come to all these things. So, we will look into the dynamics of this system also and also the stability of the system. So, let us quickly work for this in next 5, 10 minutes. So, we have h 0 equal to I 3 times omega 0 and once it is perturbed by a small amount. So, a still we can what we can write that omega 3, this equal to omega 0 cos theta; you can, we can see from this place omega 0 is along this direction initially and omega 3, then comes along this direction.

So, here we will have component of this omega 3 equal to omega 0 cos theta ok, we can write it like this and psi dot already we have as per our previous lecture, we have derived this h 0 divided by I 1. So, this is I 3 times omega 0 divided by I 1 and phi dot; this we have written this way h 0 cos theta. So, I 1 minus I 3, I 1 I 3 ok, this we can do for this is an approximation for small theta condition ok. So, h 0 we can use it. So, I 3 times omega 0. So, h 0 in magnitude why this is I 3 times omega 0, we can insert here in this place. This is cos theta this is cancels out, I 1 times omega 0 cos theta ok. So, thus the phi dot we get from this place.

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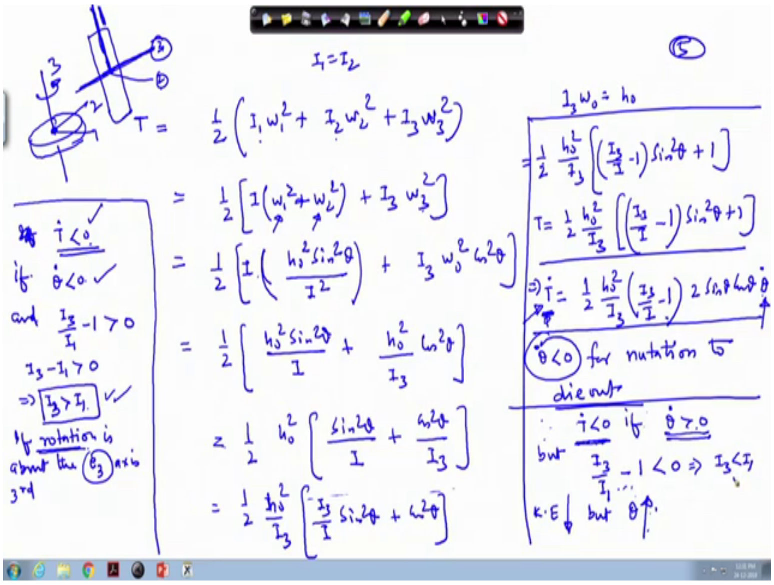

Now, we write the equation expression for the kinetic energy. This is 1 by 2 times I 1 times omega 1 square omega 3 square. I 1 and I 2 both are equal. So, we can take them outside and that this as I and insert the expression for omega 1 square omega 2 square ok. So, we are aware of this omega 1 square plus omega 2 square, this is h 0 square and omega 3 is nothing but omega 0 square, then cos square theta. From where we are getting this, we have inserted the expression for omega 1 and omega 2 in terms of h 0 ok. This, we have done in the previous lecture. Refer back to the previous lecture.

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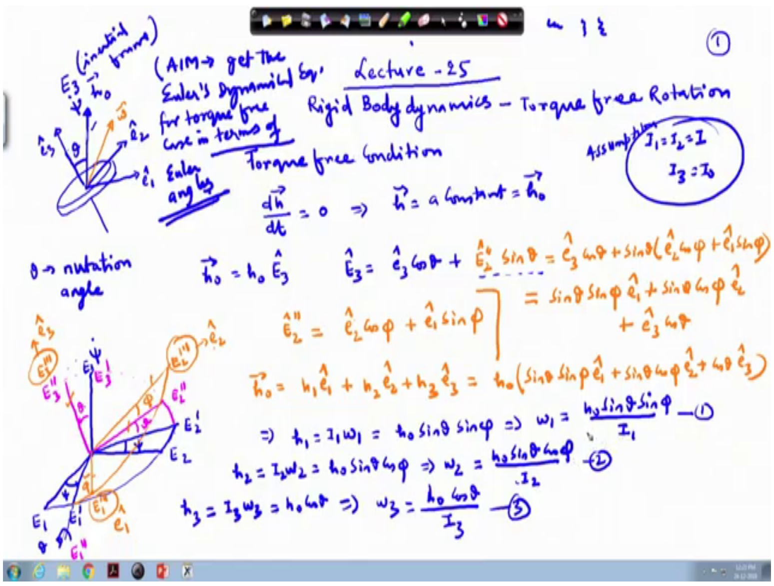

Here this is omega one omega 2. So, if we square them. So, cos phi sin square phi terms are there. So, this will be h 0 square sin square theta divided by I 1, I 2 they are equal. So, I 1 square we can write in terms of I 1 square. So, the same thing has been inserted here in this place. So, this we will have 1 by 2. And, so, if this instead of writing this as I 1 or I whatever we want we can write in terms of I also or this gets reduce to h 0 h square sin square theta divided by I plus I 3 times omega 0 which we have written as h 0 ok.

So, this we can write as h 0 square divided by I 3 cos square theta and h 0 square. We can take it outside and this will be sin square theta divided by I and plus cos square theta divided by I 3 and if we take outside the quantity I 3, so this will be h 0 square divided by I 3 times I 3 divided by I sin square theta plus cos square theta and we can rewrite this equal to 1 by 2; cos square theta we write in terms of sin square theta. So, this will be 1 minus sin square theta. So, this will become I 3 divided by I minus 1 sin square theta plus 1 ok. So, this is your t, t equal to 1 by 2 h 0 square divided by I 3 times I by I minus 1 sin square theta plus 1.

So, this implies t dot will be equal to if we differentiate this. So, 1 by 2 h 0 square divided by I 3 times I 3 minus divided by I minus 1 and this becomes 2 sin theta into cos theta times theta dot. Now, look for this situation. The theta dot quantity is present. So, if the system has to be stable ok, so theta dot must decrease theta dot should be less than 0 for nutation to die out; if you want to remove the nutation over a period of time, so this quantity should be negative. We can look otherwise also. So, we will I will copy this expression here in this place will work here.

If t dot is less than 0 means the energy kinetic energy is dissipating as we have just looked in to the previous case here, this particular case kinetic energy if it dissipates. So, this side is dissipating means t dot will be less than 0 and when this will be less than 0, if t dot is less than 0, if theta dot is less than 0 and I 3 by I 1 minus 1. This is greater than 0 because theta is a small. So, these quantities will be positive ok, sin theta and theta cos theta, they will be positive.

So, this implies that I 3 minus I 1 will be greater than 0 and this implies I 3 will be greater than I 1. So, if you are if the rotation is if rotation is about the E 3 axis or the third axis as we have shown here in this case ok, that is about the maximum moment of inertia here. For the disk is this is this one and for the cylinder case this should the third direction will lie here and 1 and 2, then we have to show the third and 2 and 3 will lie here in this direction this is the maximum moment of inertia case here this is the minor axis. So, we have to change the tag ok. Here, in this case, this is 1, this is 2, this is 3. So, I 3 is greater than naturally I 1 and I 2

So, for this case if you disturb it, so it says that this if there is dissipation of kinetic energy, if the inside there is some damper say inside the satellite, you have built some damper. So, that will dissipate the kinetic energy of the satellite and if and therefore, this rotation will be stable means the nutation will die out over a period of time. So, these are the way of killing the nutation. If the system gets disturbed for some or other reason ok, so and if you are built inside the damper, so that damper will dissipate the energy over a period of time. But for that, you have to ensure that you have set the satellite right in the beginning along, rotating along a proper axis. Only then this will happen, otherwise not.

So, here in this case, if you set the satellite to rotate about this third axis, so over a period of time as the t dot dissipates because of whatever reason it may be trans file antenna, it may be damper or whatever else. So, if theta dot will be less than 0, if this condition is satisfied. On the other hand, if t dot is less than 0, this is dying out ok. But theta dot is greater than 0 ok, theta dot say theta dot can also be less than 0, this can also be less than 0. If theta dot is greater than 0 but I 3 by I 1 minus 1 is less than 0 means the quantity here. This quantity turns out to be less than 0 ok, theta dot is greater than 0 ok. See the situation, this has become negative and theta dot is greater than 0 means still the there is damping in the say this kinetic energy is dissipating.

So, kinetic energy is going down ok. It is dissipating but theta dot is going up theta dot theta is going up because theta dot is positive. So, we should show here theta dot is going up theta is going up here the theta is going up means because theta dot is positive here in this case and therefore, theta will increase over a period of time. So, even if your kinetic energy is dissipating ok, but if this condition is there, this implies simply it implies that I 3 is less than I 1. So, if the rotation axis is the minor axis as it happened in the case of this explorer 1, this is explorer 1.

So, as it happened in the case of the explorer 1, it was rotating about this axis. So, here in this case if you write you are write showing this as I 3 and I 1 and I 2 along this direction. So, I 3 is less than I 1, I 2. So, and this is a perfect case for what we have discussed here; I 3 is less than I 1 and this implies that even if the kinetic energy is dissipating which happens because of the oscillation of this trans file antenna, but theta dot will being positive as we see from this equation and therefore, theta will build up; means, this satellite cannot keep rotating about this third axis. It is not possible

So, ultimately in the case of the explorer, it is so happened that initially it was rotating about this axis, but thereafter it started rotating about this transverse axis which is the major axis as I have shown here it in this place and a lot of things, we have learned from this particular example and say once the satellite is launched, there are rocket failure sometime, sometimes there is satellite failure. So, every time there is a learning. You do some changes in the system and sometimes it may fail. So, why that failure has taken place, so that is analysed, ok.

It is scrutiny is done by the expert team and they find out what the what are the wrong things inside the system. So, or you can say, what are the wrong things that has gone in the system. So, correct that and by correction. So, over a period of time this learning has taken place and nobody has inside right in the beginning that this thing is going to happen, but thereafter, the mathematical analysis another things based on the observation it is done and then the system is corrected.

So, thank you for listening to this lecture. We will continue in the next lecture.