Satellite Attitude Dynamics And Control

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Lecture - 23
Rigid Body Dynamics (Contd)

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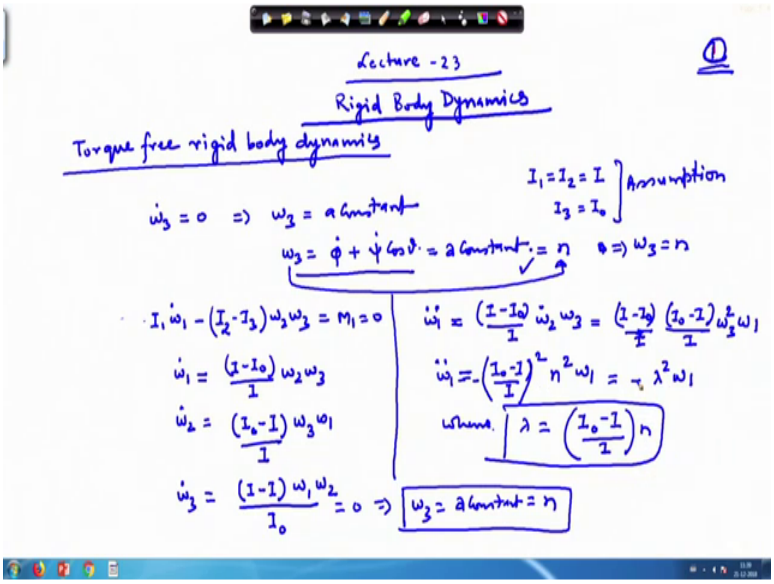

Welcome to the 23rd lecture. We will continue with Rigid Body Dynamics, we have been discussing in the last lecture ok. So, recalling that for the rigid body dynamics, we have used the Euler's equation and especially for this case where the it is torque free. So, what we have got we had three equations.

So, from the last equation, we got that the omega 3 dot, this will turn out to be 0 ok. And this implies omega 3, this is a constant. Moreover, if you remember that omega 3 already using the kinematics equation, we have written this as phi dot plus psi dot cos theta ok. So, this implies this quantity is going to be a constant.

Beside this we have derived the equation of motion in this format, where we have assumed that. So, whatever we have derived this was done under the assumption that I 1 equal to I 2 equal to I and I 3 equal to I 0, this is the assumption ok. So, under this condition we have got I times omega 1 double dot plus and let us write this omega 3 constant this equal to n. This is a means this omega 3 this equal to n, implying this is omega 3 equal to n.

So, under that condition if we look back into the previous equation, so it can be written as n times n square times let me write it in another way, we will start from the scratch and then write it. So, we had I 1 times omega 1 dot, this is the angular acceleration plus and will follow the sequence. So, I 1 then we have I 2 minus I 3 times omega 2 times omega 3 and on this side the M 1 which we have set it to 0, because it was torque free (Refer Time: 03:15) just I am recalling the previous thing.

And from there we have written this as omega 1 dot minus or this equal to I 2, we are writing as I and this as I 0 and I 1 then becomes I omega 2 omega 3. And similarly, omega 2 dot we have written as following the same kind of rule. So, here these are starting with 2, so it was here I 0, this is I 3 basically I 3, 2 3 and then 1, so this is I and omega 2, after this is 3, and then the 1 will come. And obviously here the I 2 is there, so I 2 will come here in the denominator which we are writing as I, so this is what we have written last time.

And omega 3 dot, this was taken from the Euler's equation. So, in that case omega 3 there after the I 1 and I 2 comes. So, I 1 equal to I and out I 2 equal to also I, so this will cancel out. And then we have 3 after this 1 and this is 2 divided by I 3, which is I 0 so, this is set to 0. And from there this is what we have recovered ok. So, this implies omega 3, this is a constant and which we are writing as n ok.

And we double different we differentiated once this quantity and omega is 3 is already constant. So, once we difference here, this remains constant. And from there we have got this as omega 1 double dot, this equal to I minus I 0 divided by I times omega 2 dot times omega 3. And omega 2 dot, we are inserting from this place, so this becomes I minus I 0 divided by I and I 0 minus I divided by I times omega 3 omega 1. And omega 3 from this place, because we have replaced to, so this becomes omega 3 square and then omega 1.

So, this way we are getting omega 1 double dot, this equal to now we will write in terms of I 0 minus I. So, this is I 0 minus I divided by I whole square, then omega 3 this is written as n square omega and this quantity we have written as and here there will be a minus sign ok. So, this is minus lambda square omega 1, where lambda we have written as I 0 minus I divided by I times n ok. So, the proceeding in the same way and we got omega 2 double dot equal to minus lambda square omega 2.

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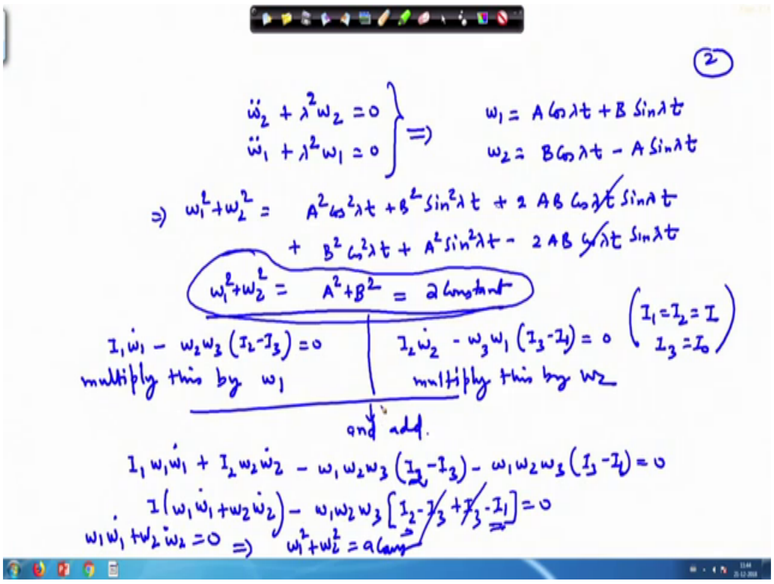

So, the second equation in the same way, we get as omega 2 double dot plus lambda square omega 2 this equal to 0. And the first equation we have omega 1 this equal to 0, and as I told you the solution to because this is a simple harmonic motion format the standard format.

So, we can write omega 1 this equal to A cos lambda t plus B sin lambda t. And omega 2 we have written as because it is a omega 2 will differ from omega 1 by 90 degree, but it is a perpendicular to that. And therefore, in this case if we put here phi by 2, so at the phase difference of phi by 2, so this becomes B cos lambda t minus A sin lambda t ok.

And if you a square and add, so this will be A square cos square lambda t plus B square sin square lambda t from the previous one and the first one 2 times A B cos lambda t times sin lambda t and from this one we get B square cos square lambda t plus A square sin square lambda t minus 2 A B cos lambda times sin lambda t ok. So, adding these two will cancel out plus minus signs are there and these two will add up.

So, A square we can take it as common, so this becomes A square plus B square ok. So, this is a constant, because A and B they are constant. So, omega 1 square plus omega 2 square, it turns out to be a constant. So, this is not the only way of doing this, this can be grouped in other way also. So, for this what we need to do that once we are writing this equation I 1 times omega 1 dot minus I 2 minus I 3 and on the right hand side it is 0, so multiply this omega 1 ok.

Similarly, we have I 2 times omega 2 dot minus omega 3 times omega 1 I 3 minus I 1 ok, on the right hand side there is no torque, so this is 0. So, multiply this by omega 2 and add. So, if we do this, so this will be I 1 times omega 1 times omega 1 dot and plus I 2 times omega 2 times omega 2 dot. And hear this term will be omega 1 times omega 2 omega 3 and now I 2 I 3 are there, so we need to replace them, this is I 2 minus I 3. Because, we have already assumed I 1 equal to I 2 this is the case of symmetry, we are assuming. If we do not assume case of symmetry, the result will be different. So, I 2 minus I 3 and then from this place again omega 1 omega 2 omega 3 I 3 minus I 1 ok, and this quantity is going to be 0.

And therefore, if we replace this by I, so I can be taken as common I 1 time omega 1 times omega 1 dot omega 2 times omega 2 dot minus here omega 1 omega 2 omega 3 this we can take as common and here this is I 2. So, I 2 minus I 3 and plus I 3 minus I 1 ok, this equal to 0. So, you can see that these and this they cancel out and I 2 and I 2 and I 1 they being equal so, this gets reduced to 0.

And therefore, we can write omega 1 times omega 1 dot plus omega 2 times omega 2 dot, this equal to 0 ok. And this implies that if we integrate it with respect to t, so this will be omega square plus omega 2 square, this is a constant. And this is what exactly we have got here also in this place ok. So, both way you can look into so the results are consistent in both the ways.

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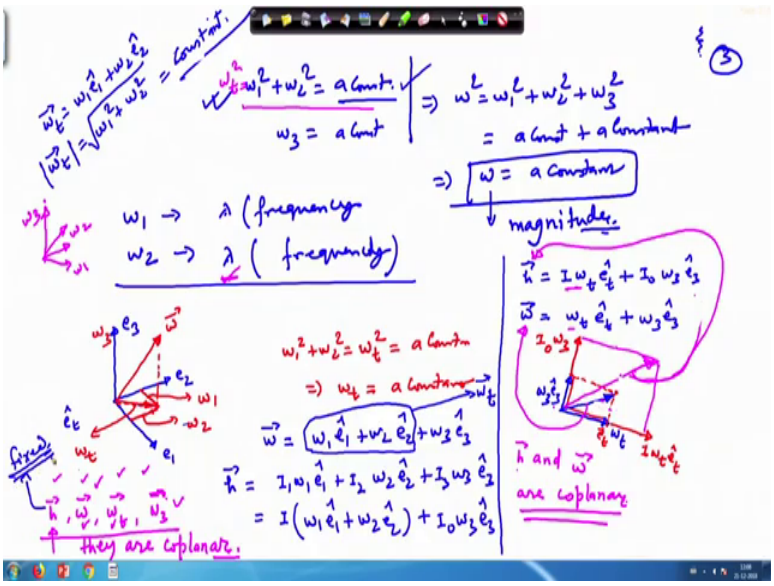

So, what we get that omega 1 square plus omega 2 square, this is a constant. And already we have observed that omega 3, this is also a constant. And therefore, this implies that omega square equal to omega 2 a square plus omega 3 a square, this will be a constant plus a constant. And this implies the omega is also a constant.

So, therefore omega turns out to be a constant omega 1 square plus omega 2 square, this is a constant ok. Now, this is constant in magnitude, but not in direction. As already we have seen that omega 1, this will oscillate at the frequency of lambda and omega 2 also this oscillates at the frequency of lambda. Therefore, we can draw the conclusion that omega will also oscillate with the frequency of lambda.

Now, we will look into this picture e 1, e 2, and e 3, this is the body frame. And let us say this is the vector omega, if we take the projection of this. So, this component this is omega 2, and this component is omega 1. So, omega 1 square plus omega 2 square, this equal to we can write this as omega t square which is a constant.

So, this implies that instead of writing omega 1 and omega 2, we can replace in terms of omega t so, omega t this is a constant. So, omega t it lies in the plane e 1 and e 2 to remember that omega we have described in terms of the components in the body frame, which are omega 1 and omega 2. So, your omega t, this is omega t vector. And if you want to show it in the vector form, so we need to put a arrow of this, but I am not showing like that only the components, I have shown here in this place.

Now, already we have written this like omega 1 e 1 cap plus omega 2 e 2 cap e 3 cap. And if we remember this h vector we are writing as I 1 times, because all diagonal terms are 0, we have taken the case of symmetry. So, I 1 times omega 1 e 1 cap I 2 times omega 2 e 2 cap I 3 times omega 3 e 3 cap. So, I 1 and I 2 though are they are equal, so are we can take it outside. And this remains as omega 2 e 2 cap plus and this I 3 we are writing as I 0 so, this omega 3 times e 3 cap, omega 3 is nothing but n.

So, h this can be written as if you look here in this place, this part if we write this as omega t like this ok. It was a magnitude of this if we write here, omega t equal to omega 1 e 1 cap plus omega 2 e 2 cap, you can see that the magnitude of this vector will be omega 1 square plus omega 2 square under root ok. This is a constant as we have written earlier, this is a constant.

So, from this place what we can try it that this is I times omega t ok, this is a magnitude. And let us consider that unit vector here in this direction is e t cap. So, this can be written as e t cap and plus I 0 times omega 3 e 3 cap ok. And this part omega also can be rewritten as omega 1 e 1 cap omega 2 e 2 cap is equal to omega t times e t cap and this part omega 3 times e 3 cap.

So, now let us look into the figure here. Like if we write here in this direction say, this is omega t ok, so omega t is here. And this is I times omega t means, you are magnifying it ok. So, one is this is omega t and further magnitude let us say this is I times omega t, so from here to here. And then multiplied by this vector e t, so it will get converted into will multiply here by e t cap and the here also as et cap to indicate these are the vectors ok.

Similarly, we have here in this place this is omega 3 and these two are perpendicular to each other, if you see the omega 3 is along this direction ok. So, we need to take perpendicular one, so for the perpendicular one, we have let us say this is omega 3. So, omega 3 times e 3 cap and the another vector which is I 0 times omega 3. So, I 0 times omega 3 let us make it here in this place. So, from this place it runs from this place and this also runs from this place from here to here and from this place to this place. So, I 0 times omega 3.

So, if we make a linear combination of this, so how it will appear? Vector addition it will give you for this blue one, this is the vector here ok. And this vector depending on this magnitude how much this is and how much this one is so you will get another vector, which may look like this from here to here and this place to this place. So, then this is your this vector and this vector is this ok.

So, what it implies that h and omega are co-planar ok, this is what it implies, because e 3 and it depends on e 3 and e t and it is a just different linearly ok, you can see this component omega t here this omega t. So, here it is I times omega t so, just referring in magnitude, while they are in the same plane ok. And therefore, they are bound to be co-planar, so this implies that h and omega they are co-planar ok.

So, what we can conclude here that h omega, then omega t and omega 3, they are co-planar. Now, already we have observed that vector omega 1 and omega 2, it is a oscillating at the rate of it is a rotating at the rate of this frequency lambda read in per second ok.

And therefore, omega 1 plus omega 2 square, this we are writing as omega t square ok. So, therefore this will also rotate at the frequency of this lambda ok, but from these two equations, these vectors need to be all the time in the same plane. So, what does it imply that if omega t is rotating ok, omega is rotating, because omega 1 and omega 2, they are rotating as a consequence of this omega t is rotating and omega 3 is always perpendicular to this.

And therefore, the combination of this which is the omega vector; omega vector will also rotate at the this angle of frequency of lambda. So, this implies that this is also rotating, this is also rotating ok, this is also rotating and therefore h must also be a rotating, because h has to be in the same plane ok. Here you can see that this is the h defined here ok, this is the h vector and they are bound to lie in the same plane.

So, if this vector is rotating which is the omega vector, here I should rather show it like this. And this one rather I should show like this ok. So, this is the omega vector and this is the h vector. So, if your h vector is rotating ok, so the if your omega is rotating, so that also implies that it h vector also must rotate means, the h vector lying in the same plane. We now go to the next page.

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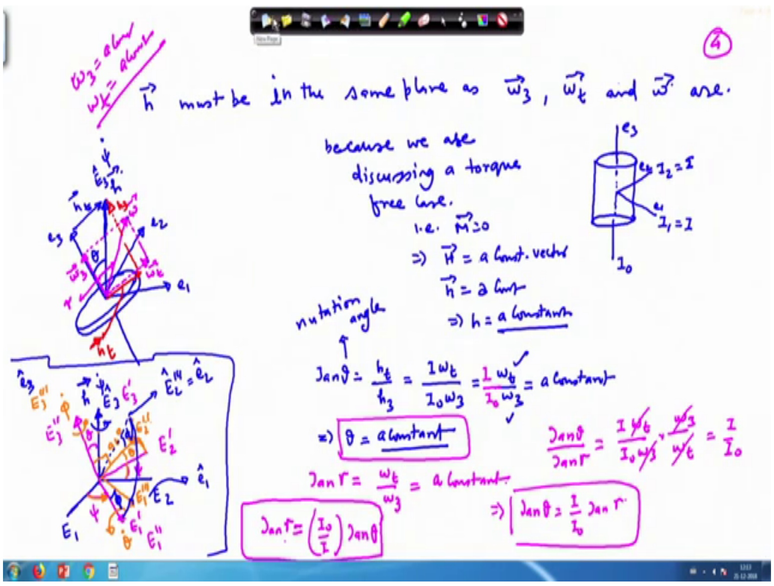

So, this implies h must be in the same plane as omega 3 omega 3 vector, omega t vector, and omega ok. So, let us consider the case here we have a disc or it may be a cylinder.

So, the case we have been discussing the torque free case, it can be represented either by cylinder or either by a disc.

As for the cylinder we know that along this axis, we have written this as I 0 and the other two axis we have written as I 1 equal to I and I 2 equal to I, it is the moment of inertia along the other two axis e 3, e 1 and this is e 2 ok. So, the same thing it applies in the case of the disc also.

So, what we assumed that let us assume that h is along this direction, because we are discussing a torque free case that is M equal to 0. So, this implies H will be a constant H equal to a constant vector or equally instead of capital notation, we have used here a small notation so, h is a constant and this implies there is a magnitude also remains here constant ok.

So, we have here e 1 axis and this is the e 2 axis, which is lying in the plane of this disc, and here we have this e 3 axis. Some where your omega vector is lying along this direction, so omega t will lie here in this plane and it can be shown like this. So, this is your omega t that implies that this component is omega 3 is along this direction as shown by this plane.

Similarly, h vector can be broken along these directions, so we will write the because they are all co-planar omega t, omega h and omega 3, they are co-planar. And therefore, the component of this h vector, it must lie in this plane itself. So, if they drop it, the somewhere shown by this red line, this is your h t ok and this component this will be your h 3. Obviously, h t is a combination of its component along the e 1 direction and the e 2 direction.

So, what exactly we have done here that given this initial reference frame E 1, E 2, E 3. And a body is there which is torque free and we are assuming that it is a angular momentum vector, it is along the E 3 direction. So, this is also your happens to be your E 3 direction or either we can show it like the h t cap. And we are assuming that the angular velocity vector which along this direction, so this simplifies the case you can assume it to be in other directions also no problem, but assuming this it definitely it simplifies the whole thing and as earlier we have done.

Let first we are giving rotation about this by angle psi. So, this is psi dot along this direction, so here psi dot will be lying along this direction, then once it comes here in this place somewhere here. This angle being psi, then you are rotating it by angle theta here. So, as a consequence of this, this will rotate from this place to this place.

So, here you have E 1 prime as we have shown earlier, E 2 prime we have shown here, and so once this is rotated this is psi ok and E lies along the same direction. And once we rotate by theta, so this will also rotate and here we will have theta. So, E 3 prime lies along this direction, E 1 prime, E 2 prime, E 3 prime and then E 3 double prime along this direction, E 1 double prime along this direction. So, once you are rotating about this, so this will rotate by this amount and here we have shown it like this.

So, once this rotates from this place to this place the corresponding thing, this angle is 90 degree, this is your theta. First we have rotated by psi, this rotates by psi ok. Then we have rotated by theta, so this rotates from this place to this place by theta. And this rotates from this place to this place by theta and then the final rotation you are giving about this line which is by phi. So, as a consequence of this phi dot is appearing here, and theta dot is appearing here in this place ok.

So, in the final configuration your E 3 triple prime is here, E 2 triple prime is here and ok. And this will rotate once we rotate about this phi, so this will go away from here ok. So, figure I am not showing here this will go away from this place, this will also go away from this place.

So, somewhere it will come near this point and this is your E 1 triple prime E 1 prime, this is E 2, this is E 2 prime, next rotation we have given. So, E 2 E 2 double prime E 2 double prime and E 3 double prime, it will go out like this. Let me try we can add just here, it will go like this. So, this and this they lie in the same plane. Say here this part, this part they lie here in the same plane, figure is getting complicated, this is phi. This will rotate from this position to this position, this is phi.

So, instead of dealing with this picture, which is getting very complicated. What we showed that this is my initial configuration E 3 vector is E 3 cap here is in this direction already and E 1 and E 2, this is the final orientation which we are showing by this part E 2 triple prime by showing it by E 2 triple prime, this equal to e 2 cap. And E 1 triple prime, we have shown this as e 1 cap similarly this one as a e 3 cap.

So, this configuration once we show it here ok, so you can forget about this figure. If you have confusion in understanding this for the time being just forget about this and just look at this figure. So, finally as we have discussed last time that once we rotate the body axis from the initial axis, so it will come here some what is the sort of orientation. And in this sort of orientation this is how your h vector is located. So, h vector is fixed, it is a non-rotating vector, but to your omega vector it rotates ok, because omega 1 and omega 2 it is a rotating and h is fixed vector. So, it is so happens that once this plane is rotating, the plane containing this omega t, omega and e 3 this is rotating ok.

So, we go back here on the previous one, they are co-planar. We have written them as a co-planar, but this one is fixed this is non-rotating, because it is a torque free case. So, it is a fixed in inertial place ok. So, here this is fixed in initial case place. And about this vector all these vectors are rotating and it, so happens that all of them remain co-planar ok.

So, finally what we can write that for this particular case, we indicate this angle as theta ok. So, tan theta we can write this as see the component here. This tan theta this is nothing but tan theta we are measuring from this vertical axis to this is called a nutation angle, theta is you nutation angle. So, how the tan theta will describe? Tan theta will be the component of h here in this direction, so which is h t h t along this direction, this is your h t divided by this quantity here, which is h 3 ok. And h t is nothing but going back on the previous page h t is nothing but I times omega t ok. So, h t is nothing but I times omega t and h 3 is nothing but I 0 times omega 3.

So, this implies omega t divided by omega 3. Omega t is a constant this is a constant and therefore this is turns out to be a constant and this implies theta is a constant. So, again and again I am telling you refer to the previous lecture, where this picture was made in a much better way here the in the sort of space, it has got much complicated. And repeating the same figure and again we wasted a lot of time over that. So, theta turns out to be a constant, this is one part.

Now, there is another angle involved, which is of interest to us which is the angle gamma. The gamma angle is between this and this, this is your gamma angle means between the omega 3 and omega vector. So, tan gamma we can writer as tan gamma you can see from this place, this is nothing but omega t; omega t divided by omega 3. So, omega t divided by this is here we have missed out on those particular term, this is I times I 0, this is constant. So, this is a this also turns out to be a constant. But, here they are not independent, they are related to each other.

So, how we can get this just divide this tan theta by tan gamma. And once we divided, so this is I times omega t I 0 times omega 3 and here this becomes omega t omega 3 will cancel out and we get I by I 0. So, this implies tan theta equal to I by I 0 tan gamma or in the other way tan gamma equal to I 0 divided by I times tan theta. And this relationship is of great importance, this where getting from geometry ok. So, just by showing the geometry, we have come to this conclusion. And obviously, we have used the information's like omega 3, this is a constant. Then omega t, this is a constant as per our theoretical work ok.

Now, if we look here in this equation, so if I 0 is greater than I, which happens in the case of disk ok. So, for the case of disc I 0 is greater than I and therefore this quantities going to be greater than 1. So, this equality will hold, if and only if gamma is greater than theta. So, we write it on the next page tan gamma equal to tan gamma this equal to I 0 by I times tan theta.

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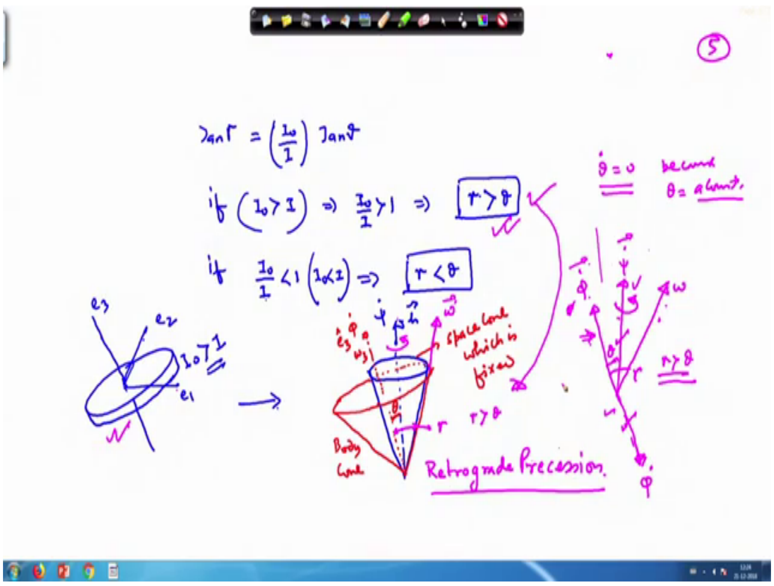

So, if I 0 is greater than I, this implying that I 0 by I is greater than 1, so this will imply that this quantity is greater than 1, so this implies gamma is greater than theta. On the other hand, if I 0 by I this is less than 1 ok, so this will imply that gamma will be less than theta.

So, we have two cases where I 0 is. So, equally we can write in terms of I 0 is less than I. So, I 0 greater than I this implies gamma is greater than, theta I 0 less than I this implies gamma is less than theta. So, what does this mean, what does this imply. Here take this case take case of the disc. In this case I 0 is greater than I, because the moment of inertia along this axis will be more than the other two axis this is e 1, e 2 and this is e 3.

So, moment of inertia along this direction is more. So, for this case it implies that gamma will be greater than theta, the situation is something like this as shown here; gamma is greater than theta. So, this is your gamma angle from this place see here from this place to this place, this is your gamma angle ok. So, this angle is greater than theta ok. The other one if I 0 is less than I, in that case gamma turns out to be less than theta means gamma will this angle will turn out to be less than theta.

So, here we have to be careful in describing the whole issue, the picture looks like something like this. For this disc, the motion it will appear as we will follow further, it will appear something like this. We have a cone here, fixed cone this is called, the space cone this is a fixed cone in the in inertial frame. And we show our h along this direction and psi dot also along this direction.

And this motion can be the motion of this particular disc this can be described as remember that this is your torque free case and the disc case initially inclined with the vertical by angle theta ok, which we are calling as a nutation angle ok. So, the motion of the whole disc can be perceived not only in terms of see the perception in terms of omega 1, omega 2, omega 3, it is a difficult how it is a rotating along the 3 axis it is a difficult to perceive. And therefore, we try to perceive this in terms of Euler angles which has physical meaning along which axis how it has rotated.

So, see here in this case first it is rotating like this, then it is rotating like and then it is rotating along this axis. Firstly, it has gone from this place to this place, then it is a rotating about this axis, then rotating about this axis. So, this is physically visualize able, but omega 1, omega 2, omega 3 these are not physically visualize able you would not be able to get anything from that.

And therefore, for this particular case the around this space cone we describe a body cone. This two ends are touching, this is called a body cone. And this blue one, it is called the space cone which is fixed ok. So, body cone you can consider that your body is towards the center of this cone ok. It is a the third axis, it is a lying along the center of this cone so, you have e 3 along this direction ok. So, phi dot is along this direction so, you can consider it is a spinning ok.

So, phi dot is here then e 3 is also here and omega 3 is also here. Then the angle between omega 3 direction and this one this is a theta angle ok. So, here we can show this as theta and omega vector here in this case it so happens, it we can show it by some other color omega vector it lies along this direction ok. And the angle from omega 3 to this place, this is your gamma. So, here in this case gamma is greater than theta as we have shown in this place.

And obviously, you see that theta dot will be equal to 0 because theta this is a constant. So, theta is a constant, therefore, theta dot it turns out to be 0 means now we have the situation that. The motion which we have shown about this line with the motion which we have soon about this particular line here this way this motion is not present, when theta is remaining constant want it has tilted and it is remaining like that. The only motion that appears there is constituted by phi dot and psi dot in terms of Euler angles which we can see here, it is a like this is your phi dot and this is psi dot.

But as you see that the omega is lying here on this side ok, so in this kind of situation, it will so happen that the sense of rotation means if we are taking this to be positive means it is a going like this anticlockwise ok, we have taken this direction to be positive, here it is like this ok. So, this direction we have taken to be positive, and omega it appears here in this direction. Omega is appearing somewhere here. You can see that if we look in the terms of Euler angles ok so, this is one vector and this is another vector. So, the combination of these two vectors should lie here in this place, but the omega is lying here as per our conclusion from this place ok.

So, therefore, this implies that these vector cannot be in this direction rather this vector has to lie here in this direction phi dot. Phi dot will be lying opposite to this. And as a consequence of this the summation of this vector and vector some of this and this vector this will be lying here in this direction. So, this kind of rotation where the phi dot is not having the same spins as the psi dot ok, the spin is not having the same says as the precision is called a retrograde rotation. So, this is retro grade or retrograde precession, it is called the retrograde precession.

So, from this geometry, we have been able to find it out this particular particularly you remember this angle here this is theta, this angle is gamma and where gamma is greater than theta. So, phi dot is bound to lie along this direction. It cannot be here in this direction ok. We initially we have shown that it is a lying here in this direction, but for this kind of disc it is a bound to go downward and vector some of this and this will give you this omega. And this omega, obviously, you can break along three free body axis e 1, e 2 and e 3. But here in this case especially this theta dot is absent and because of this the whole thing gets simplified. So, we continue in the next lecture.

Thank you.