Satellite Attitude Dynamics And Control

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Lecture - 22
Rigid Body Dynamics (Contd)

Welcome to the lecture number 22. We have been discussing about the transformation of psi dot theta dot and phi dot to omega 1 omega 2 omega 3 so, we will continue with that.

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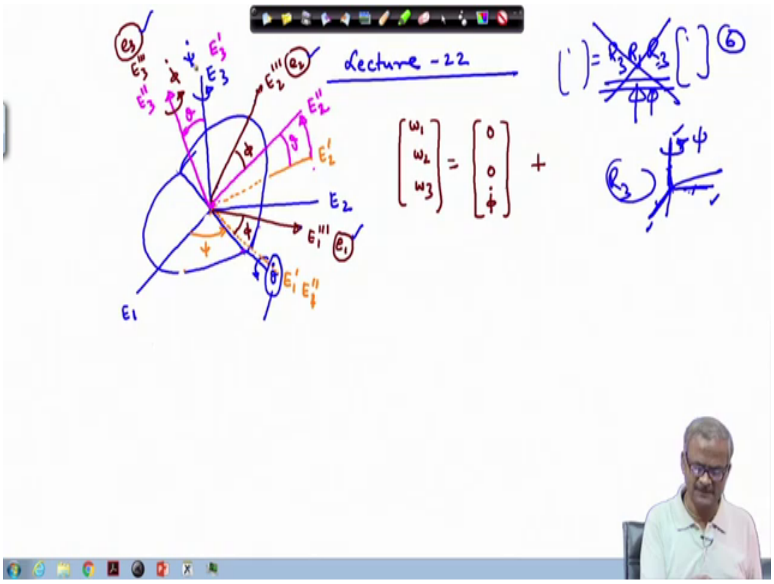

And I told you that we will do this through this figure also. So, let us say that this is one half of the circle and the other half is lying here. So, if you rotate it so, the rotated circle if I rotated by theta. So, the rotated circle it will look like this. So, I have rotated this line by theta angle ok. So, the rotation that rotation we have given about this line by theta. Now, I complete this first the triad. So, this is your E 1, this is E 2 and E 3 and the first rotation if we give you about this E 3 by psi. So, psi dot is lying here so, that does not change anything except that the access gets rotated.

So, here you will have E 1 prime and this gets rotated. So, I show it by dotted line here and here so, solid means this is below this blue line E 2 prime, then we need to do the correction here. So, we rotate it by this is rotated by psi here. So, this comes here and this goes here in this place and these two are perpendicular to each other. So, the next line this rotated this gets rotated from this place to this place so, we can show this line here itself.

So, this is your E 1 prime, E 2 prime goes here E 2 is here ok; next rotation we are giving by theta here itself. So, E 2 prime here will be E E 1 double prime will be here. This will come out and it will go on to this circle because, we are rotating here by theta and already we have rotated this plane by theta. So, this I will show by some other color. So, this goes from this place to this place so, it has turned up. So, it has gone up from here to here, this angle is theta. This line is below this blue line while, this red this pink line is in this plane itself.

So, E 2 this rotation while we are giving the E 1 double prime; so, here this will be E 2 double prime and this line will also rotate E 1 E 2 E 2 prime then we have given this rotation. So, we will rotation we will show it like this. So firstly, E 3 prime is along distance this direction itself and E 3 double prime will be along this direction as we have written earlier, this angle is theta here. Now, be it should be now clear very much clear that where these lines are lying. The last rotation we are giving about this line by phi. So, phi dot will lie here in this place and this line then it will rotate.

Now, this line will rotate and go to this position; it will come here. So, this is your E 2 triple prime and this will also rotate and say this comes here to this point E 1 triple prime. This angle is your angle phi, this angle is your phi. So, you can see that in which plan they are lying. Now, as we can see that we are interested in finding omega 1, omega 2, omega 3, this is my interest. So, as we can see that phi dot is already along the omega 3 direction. So, here each here E 3 prime here E 3 prime triple prime is lying along the same direction. So now, we tag it as E 1, this as E 2 and this as E 3 instead of using that E triple prime we are writing as E 1 E 2 E 3 which is the final orientation of the body access with respect to the initial access.

So, the phi is along the z direction so, it is a directly contributing to the omega 3. So, I need not rotate it there is no once you see in this frame the E 1 E 2 E 3 frame. So, already the phi dot is lying along the z direction and therefore, there is no need to convert it ok. Next we take up this theta dot, your theta dot is here this is theta dot. So, theta dot is here psi dot is here. So, this theta dot we need to convert. So now, look into this frame how many rotations we need to give to finally, reach to the body access which is a small e 1, small e 2 and a small e 3 frame.

So, if we give just one rotation so, we are reaching to the body frame and why we cannot add these 3 theta dot psi dot and phi dot directly because, these are not orthogonal to each other. Psi dot or theta dot and phi dot they are not orthogonal to each other. Had been they if they were perpendicular to each other, it would have been direct like what we have done earlier. We could have directly written as R 3 and here psi dot theta dot phi dot and we would have got the final omega 1 omega 2 omega 3. But, here in this case this is not valid because, these are not perpendicular to each other. And, this rotation rule R 3 R 1 R 3 you are applying this rotation this pertains to the orthogonal rotation.

Means, if the components you are taking or whatever the vector you are taking which you are trying to rotate if they are perpendicular to each other ok; making it triad perpendicular to each other then only you can use this rotation and convert to the other one. Like here if I give rotation so, I can use this here I am giving rotation by psi here ok. So, you can use this rotation R 3 because, here all the 3 axis are perpendicular to each other, but here in this case theta dot phi dot and the psi dot they are not perpendicular to each other. So, never do this mistake of adding them together with this rotation; it will be a totally wrong thing to do so.

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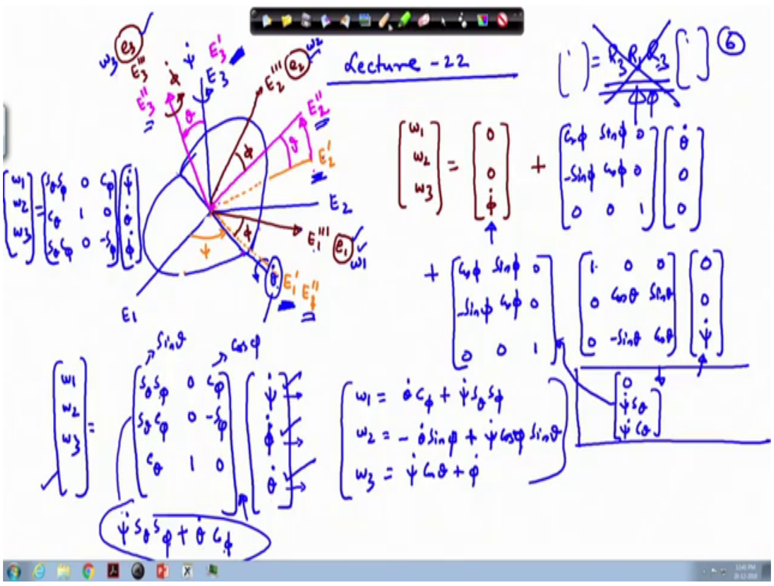

So, the next rotation this theta dot can be converted to the this e 1 e 2 and e 3 by just by one rotation because, here if you look that this theta dot will take this here as E 1 double prime E 2 double prime and E 3 double prime. This is my access from there I have to go to E 1 do triple prime E 2 triple prime and E 3 triple prime. So, that can be done just by one rotation about the z axis. So, if I do this so, means I have to give rotation about the z axis by phi. So, this will be cos phi sin phi minus sin phi cos phi and this is operating on theta dot. So, theta dot is along the x axis here. So, here in this case so, we put here theta dot rest others are 0. There is along this direction there is nothing ok, along the E 2; E 3 double direction along this direction this already phi dot we have taken care of; so, there is nothing to do with that.

So, theta dot is one component which we are trying to finally, convert to the E 3 triple prime E 1 triple prime and E 2 triple prime E 3 triple prime and E 1 triple prime. So, the last one then remains is psi dot. So, psi dot is a again along the z axis. Here you see here it is along the z axis and how many rotations we need to give that finally, we convert into E 1 triple prime E 2 E 1 triple prime E 2 triple prime E 3 triple prime. So, we need to give two rotations; how? Already if you look here this psi dot E 1 prime is here E 3 prime is here in this place, this is your E 3 prime. Here is your E 1 prime and E 2 prime is here. So, these are perpendicular to each other ok. So, in this access system E 1 double prime E 1 prime E 2 prime and E 3 prime in this system this is forming a triad rectangular triad, they are perpendicular to each other.

Therefore, if I rotate it by theta first so, this is my first rotation that I am giving then that rotation I am giving about the x axis. So, here I need to write it like this and this rotation is my theta. So, this is cos theta sin theta minus sin theta and this is cos theta ok. The next rotation then once we have rotated like this; so, it has got converted to E 1 double prime E 2 double prime and E 3 double prime. The last one rotation is to be given to bring it to E 1 triple prime E 2 triple prime E 3 triple prime just rotate it by phi and that rotation is about the z axis. So, we do it by writing like this ok. So, this is cos phi sin phi minus sin phi and cos phi. Now add all of them, here this is 0 0.

So, first solve the matrix and then write omega 1 omega 2 omega 3 separately and put it back in the form of matrix notation because, here psi dot is also there and phi dot is also there. So, we cannot put both of them along the z direction. So, it is better to do it in a way like we can write it as psi dot theta dot and phi dot ok. And, in no way this is indicating that this is the x direction and this is also the x direction and this is the y direction or anything. These are the 3 vectors on which we have three components of the angular velocity vector psi dot theta dot and phi dot which, if you operate by certain matrix it will get converted into omega 1 omega 2 omega 3.

And, that particular term we can write as S theta S phi 0 C phi; for this you need to just add it up. First solve this, solve this and put it in a simple format, linear equation format ok. This will be basically the in the algebraic format I mean in the algebraic format where, psi dot theta dot phi dot will appear and this will act as those coefficients. So, this is S theta is nothing, but sin theta and similarly this is cos phi. This notation we are following here to make it shortcut. This is S theta C phi 0 minus S phi C theta 1, 1 is here in this place, this is the second column and 0.

So, this gives you omega 1 omega 2 omega 3 and one I will verify here in this place; let us say that omega 1 I want to verify. So, omega 1 will be from this place I am writing here there is no component along the omega 1 direction, this part theta dot plus cos phi. So, theta dot C phi sin theta is 0 so, this is 0 this is 0; so, nothing contributes here. Now, we have to come to this place. So, this is little complicated, we have to first do this and then operate by this. So, first we need to work it out and this part I will write here in short this 1 here 0 0 0. So, the this vector will get resulted into 0, from here this is 0 0 psi dot S theta and from here this is 0 0 and this is psi dot C theta and this has to be operated by this ok.

So, once we operate it so, cos phi this will be 0 and the other term will be sin phi times psi dot plus psi dot S theta times S phi and the third term will be 0, because this is 0 here ok. So, what we get here? The first term you can check from this place whether its a matching or not. So, if you multiply here this row with this column what we see that this is S theta S phi times psi dot we will write in the here. This is the psi dot and then plus this term will be 0 and this is C phi, phi dot times C phi. So, check here is it same this is psi dot here we have written it in the wrong way; this is psi dot phi dot phi dot phi dot and here this is theta dot.

So, going into this S theta S phi psi dot this part is 0 and this is C phi theta dot phi dot times C phi. So, here we need to correct this also, this becomes theta dot times C phi theta dot times C phi. So, same way you can expand it and write this kind of terms and once you put these kind of terms in this matrix notation so, it will get converted into this. So, on the right hand side we will have psi dot phi dot this is ok, this is and this is theta dot and these terms are also theta dot here this term are together. So, S theta and S theta times C phi, this two are together ok, this two are together. This is the second column, this is the third column, this is the first column; similarly, the first row second row third row. So, this way omega 1 omega 2 omega 3 we can get from psi dot phi dot and theta dot.

So, you are basically getting here 3 linear equations in terms of theta dot psi dot and phi dot, then only you will be able to represent it. If it is non-linear equation in psi dot phi dot theta dot where the matrix representation will not be possible in terms of theta dot psi dot phi dot. So, for your omega 2 it can be, if you check from this place it will appear as theta dot sin phi plus psi dot cos phi times sin theta and omega 3 is psi dot cos theta plus phi dot. So, this you can solve for theta dot psi dot and phi dot in terms of omega 1 omega 2 omega 3 and vice versa. So, already we have this relation and if we invert this matrix, we get psi dot phi dot and theta dot ok.

Or, either you can solve this linear equation directly and get here psi dot phi dot and theta dot. If you want to change this, let us say that rather than writing here in this format you want to write it as omega 1 omega 2 omega 3. We want to write in terms of, here instead of writing psi dot psi dot is ok, here we want to change it make it theta dot and this part we went to make phi dot here in this place. Then the changes you need to do that these two rows will get exchanged ok; that means, that here you will have C theta 1 and 0 here in this place and S theta C phi S theta C phi will come downward, this will be 0 and minus S phi will be here in this place.

Similarly, this term because psi dot is in its original position so, we do not have to change it. This is S theta S phi 0 and C phi, that remains as it is S theta C phi and its. So, this way you can do the conversion. Now, we need to get psi dot psi dot theta dot phi dot. So, mark this change that I have interchange this position of theta dot and the phi dot. So, if you interchange it the rows are also interchanged.

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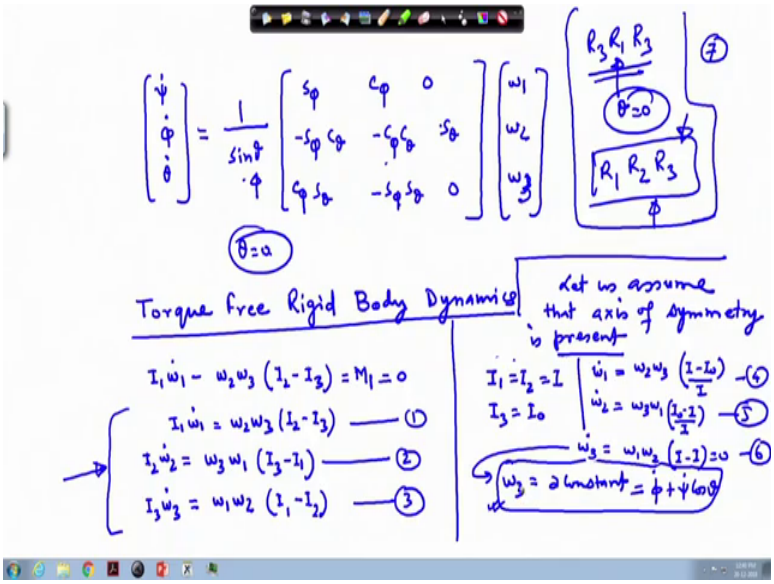

And, if we want to write here the psi dot and phi dot theta dot ok; so, by doing the matrix inversion or by solving the linear equation which I am skipping because, it will take time. This you can do as homework. So, you can see that if theta equal to 0 you get the singularity problem ok. As I was telling you earlier that, if this angle is a small so, this blows up and you will have trouble working with this matrix notation. However, you will not have trouble if you work with this linear equation and theta is equal to 0. So, you will not face that particular trouble ok.

So, while working with matrix notation you have to be careful with this R 3 R 1 R 3 in this rotation; if this is pertaining to theta rotation and if this happens to be 0 means, we get the trouble here in this place. And therefore, this kind of whenever you are thinking that in your particular system you are dealing with this angle, if you are trying to use this kind of rotation R 3 R 1 R 3. So, you should be careful that if R 1 happens to be a small rotation; so, you should avoid this rather than using this you should use like the first rotation giving about the R 3, this is a second R 2 and then R 1 like this. So, then you will not face this kind of trouble, but here again certain kind of other kind of trouble can be there.

So, due course of time; obviously, those things will appear right now we should not worry. So this concludes our rotation and you can do all these things as a homework. It is necessary to do certain exercise which I will put in the form of tutorial also. Next week go into the torque free rigid body dynamics means already we have used the Haller's equation. So, we continue with Haller's equation torque free, but here in this case the torque is not there, toque free rigid body dynamics. So, Haller's equation we have written as I 1 times omega 1 dot minus omega 2 times omega 3 times I 2 minus I 3, we are writing in a circular way.

As I have told you earlier this will equal to the torque acting on the system either this way it by T or M perhaps I have used the notation M. So, I will write here M is the moment and this quantity will be 0 ok. If it is torque free means this is 0. So, this implies I 1 times omega 1 dot this equal to omega 2 omega 3 times I 2 minus I 3. So, let us make this as equation 1. Similarly, you will have I 2 times omega 2 dot, if the torque is 0 so, only term will appear there and in the circular way we are writing omega 3 omega 1 I 3 minus 1 and into equal to 0. So, this is you your equation 2 and the last equation is 3 times omega 3 dot, this will be omega 1 times omega 2 I 2 minus I 1; omega 1 3 1 I 1 minus I 2. This is I 1 minus I 2 (Refer Time: 25:20) I 1 minus I 2 this is equation number 3.

Now, this kind of torque free rotation you get in the case of the satellite not always, but in certain cases. As we will see that the gravity gradient we will discuss later on the lecture. So, those are also present. If you are considering the case of a torque or a gyroscope, like you are playing with a torque ok. So, if you are playing with a torque it is not a torque free motion, the gravity torque is acting on that. So, here this particular one discusses the ideal case which happens in the case of some of the satellite where, there is no external torque acting on the system it so, happens ok. So, what we assume that let us assume that axis of symmetry is present because, that simplifies the case this case quite a lot.

And, it makes it easy to handle ok; if it is non-symmetric case it will be complicated you would not be able to get any picture from this ok. So, what we assume here that I 1 equal to I 2 this will write as I and I 3 we will write as I 0. And therefore, our equation of motion if we insert here in this place this gets reduced to omega 1 dot, this equal to omega 2 omega 3 I 2 minus I 3. So, I minus I 0 divided by I this equation number 4. Omega 2 dot this is from this equation, this will be omega 3 times omega 1 and I 0 minus I divided by I, this is equation number 5.

And, similarly omega 3 dot that gets reduced to omega 1 omega 2 and I 1 and I 2 both are equal to each other so, this vanishes. So, I minus I so, this equal to 0; so, omega 3 dot equal to 0. So, directly from this place what we get that omega 3 this is a constant, that is if you go on the previous page and look here omega 3 equal to phi dot plus psi dot cos theta phi dot plus psi dot cos theta. So, this quantity vanishes omega 3, this is omega 2. So, the angular velocity and along the third body axis, along this axis here you can see along this axis omega 3.

Omega 3 we are writing in this, this is omega 3, this is along omega 2 and from this place we have omega 1 ok. So, omega 1 is along this direction, omega 2 is along this direction, omega 3 is along this direction. So, omega 3 value does not change with time, this is what it says that if you have the torque free case of the rigid body omega 3 will remain constant. Means, there is a relationship between the phi psi dot and this is related by this theta.

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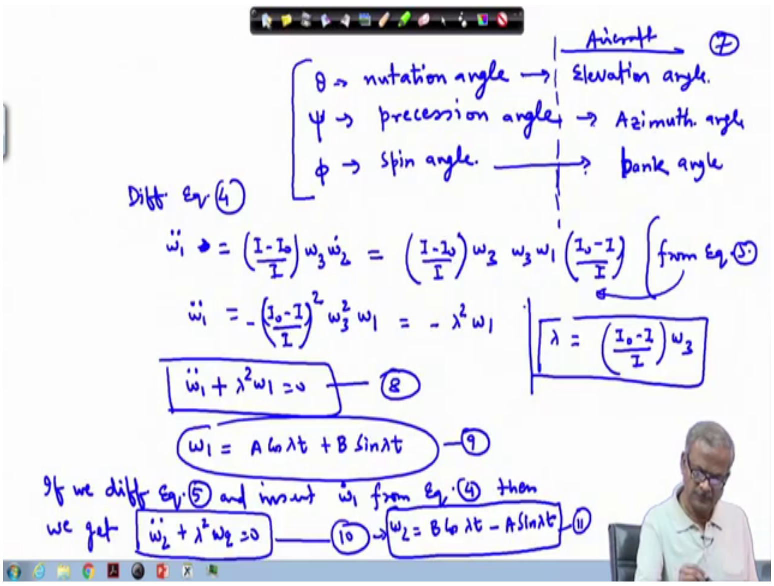

As I have told you earlier this theta, this is called a nutation angle, psi dot its called the precession angle and this psi is called the precession angle not dot and phi this is called the spin angle. In the case of the aircraft the same thing is called as the elevation angle, this is for aircraft. This is precession angle instead of this precession angle, we write this as the azimuth. And, this spin instead of this phi elevation azimuth and then elevation and this we can call as the bank angle. In the case of the aircraft things are miscomplicated ok.

There is deep difference in the bank angle, bank about the body axis, bank about the velocity axis. It is a different terms and the whole things gets quite complicated and it is out of context here so, we are not going to discuss that ok. So, with this what we get from here omega 1 dot equal to omega 2 omega 3 I minus I 0 divided by I. If we difference this ones so, we will have omega 1 double dot plus equal to omega 2 omega 3, omega 3 is a constant remember. So, we this will not be differentiated only this term will be there. So, I minus I 0 divided by I omega 1 dot I minus I 0 divided by I omega 3 is a constant so, it remains there and this is omega 2 dot. So, omega 2 dot omega 3 is a constant, I minus 0 divided by I I minus I 0 divided by I.

Now, in this equation put omega 2 dot omega 2 dot is present here; so, insert it from the equation number 5. So, omega 2 dot is omega 3 omega 1 omega 3 omega 1 and then I 0 minus I divided by I; I 0 minus I divided by I from equation 5. Differentiate equation 1 differentiate equation 4, equation 4 and this we have inserted from equation 5 ok. So, this gives me omega 1 double dot equal to I 0 minus I divided by I whole square that minus sign times omega 3 square times omega 1. This we can write as minus lambda square omega 1. So, if you look here in this format this equal to 0 ok. So, name this as question number 7.

This is a simple harmonic motion equation ok, this format is of simple harmonic motion equation motion. So, it says that omega 1 will vary as A times cos lambda t plus B times sin lambda t. Similarly, if you differentiate omega 2 dot means, if you differentiator equation 5 and insert omega is constant. So, insert omega 1 from equation number 4, here in this will get the same type of equation ok. So, this we differentiate equation 5 and insert omega 1 dot from equation 4. Then we get omega 2 double dot plus lambda square lambda 2 equal to 0. So, we get the same type of equation.

And, here also the lambda square is the same thing as we have written here, here lambda square this is nothing, but I 0 minus I or simply say the lambda equal to I 0 minus I divided by I times omega 3 this is your lambda. So, this is also the equation of simple harmonic motion. So, it says that omega 1 and omega 2 they are harmonic in nature ok, they are oscillatory and also we see that omega 1 omega 2 they are perpendicular to each other because, they are along the perpendicular direction here omega 1 and omega 2. So, these are two are the perpendicular direction to each other. So, rather than stretching it, it is very easy to write from this place. So, the solution of this can be written as, you can write omega 2 this equal to the differ why the phase angle phi by 2. So, if we insert here phi by 2 so, this gets reduced to B cos lambda t minus A sin lambda t. So, this is the solution.

So, they are not independent, they are related by this relationship because, the phase difference is phi by 2. This is your equation number 11.

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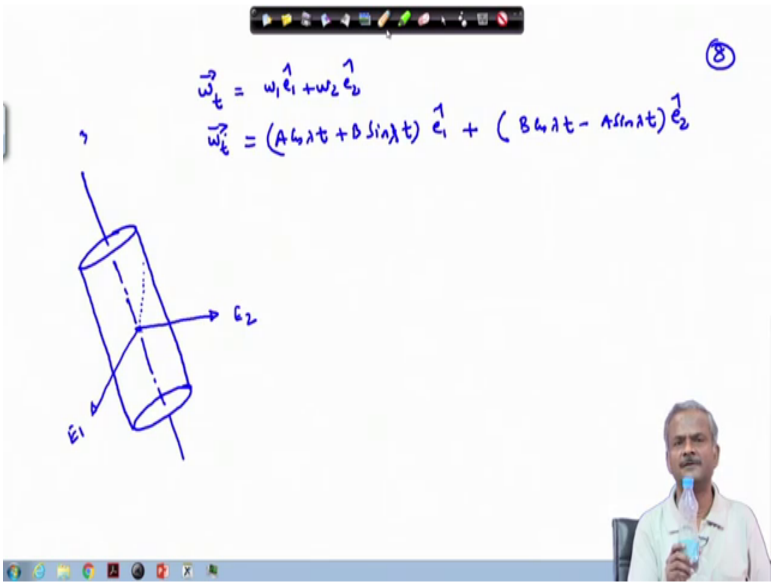

So, we can define a vector omega t omega 1 e 1 cap plus omega 2 e 2 cap and then we can write here A cos lambda t plus B sin lambda t times e 1 cap plus B cos lambda t minus A sin e 2 cap. This is your omega t, what does it mean by this omega t. If we go here in this figure so, omega t lies in the plane E 1 triple prime ok, it lies in this E 1 triple prime and E 2 triple prime. And, this is very useful in the case of the we are discussing the case of a rigid body which is symmetric ok. So, this case we can look from this place say this is a cylinder. And, this is the axis of the cylinder and we have here E 1, direction then E 2 direction and this is E 3 direction.

So, what we can see from this place that ok, we will change the notation little bit, instead of writing here this as of what we want to do that say I have this water bottle. So, consider this as a cylinder, this is a water bottle and we can consider this to be a cylinder.

If this body is spinning about this axis and simultaneously rotating like this, its spinning and rotating like this ok. So, if I consider this vertical to be the initial frame z direction and from there its rotated by theta angle ok. And, then spinning like this about this axis and simultaneously about this z axis it is going like this, as it happens in the case of the torque. If you have ever seen a torque or a toy gyroscope so, you will see that this keeps rotating on this axis and also it does like this. So, this motion is precession motion while, this motion is called a nutation motion ok.

This is a spinning motion, an angle from this to this from this vertical to this it is called the notation angle, from this position to this position it is a notation angle ok. And if there is variation in a angle ok, if it is varying like this it is going up and down whatever so, this is called a nutation motion. So, precession motion is this ok, this is the spinning motion while this one is the nutation motion. So, here in this case it so, happens that the nutation angle which we have written as the theta it remains constant ok. As you will see, that this angle will turn out to be constant. And, based on that tenth we will do certain relations, we will get certain relationship between psi and phi dot. Because, here we as we see that if it is rotating spinning about this axis and also it is a precessing.

So, what will be the relationship in the torque free case, not in the case of the torque; he here you are discussing in the case of the torque. So, in that case if the cg is lying here so, it will get affected by the gravity ok. But, in the case of a rigid body which is from free from the external torque, it will not get affected by the external moment. This is what it means, it is a free from the external torque means, there is no external moment. And, then we are looking into how this rigid body behaves. And, as a problem later on I will take the case of a torque or pay; you can consider that to be a gyroscope where, the external torque is present. So, in that case how the equation of motion it changes and what are the consequences. So, we will stop today here and we will continue here in the next lecture.

Thank you.