Satellite Attitude Dynamics and Control

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Lecture - 21
Rigid Body Dynamics (Contd)

Good morning and welcome to the twenty first lecture. We were working off with the rigid body dynamics and in that context; we worked with the Euler's theorem. So, we derive the Euler theorem. Today, we will look into the kinetic energy of the rigid body.

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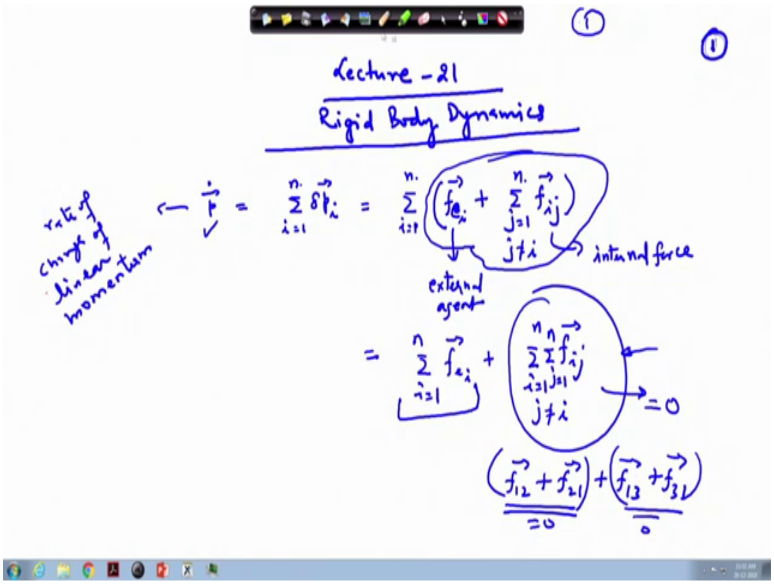

So, before we do something about the kinetic energy, let us look into what we have done till now and so, what we have written that p dot, which is the rate of change of linear momentum ok. This we have written as, this is for the whole body and this is for the individual particle and where the sum is over i equal to 1 to n.

Now, as you know this consist of the external forces and the internal forces and that we can write as i equal to 1 to n and on the ITS particle, you have the forces acting, which can be written as fe i this is the external force acting on the ITS particle and plus the internal forces acting on the ITS particle. So, internal forces will be from j equal to 1 to n, where j not equal to i and f ij.

So, this is the force on the ITS particle due to the JTS particle and here this is only due to the external force this is the external, external agent, while this is due to the internal forces, which arise from the interaction of the different particles ok. So, if you write it this way so, you can check it that i, this will be i equal to 1 to n fe i plus summation and this summation over j equal to 1 to n j not equal to i. Now, if you look into this term, we have not written this particular term while deriving the Eulers equation. We have used only this term for deriving the Euler equation, because this term vanishes.

As you can see from this place, if we expand it so you will get terms like f 1 2, f 2 1, f 1 3, f 3 1 and from Newton's law you can check that this kind of terms they are equal and opposite to each other. So, this will be 0, this will be 0. So, over all this term is equal to 0 and therefore, this vanishes. So, the effect of the internal forces, we have already neglected while deriving ah, the equation of motion for the rigid body ok.

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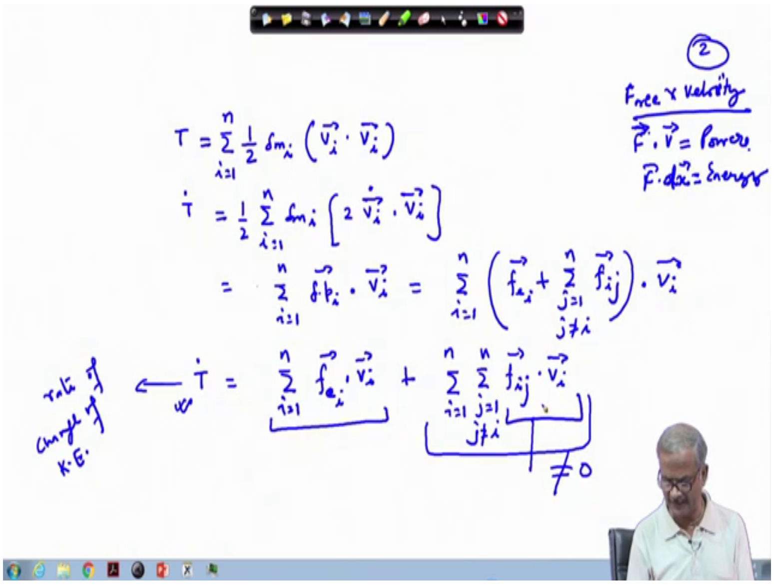

While it does not affect the equation of motion of the rigid body ah, but if we look into the ah, this kinetic energy of the rigid body so, in that case the internal forces, they do play their role and it will be clear from once we write this equation as follows. As you know that kinetic energy, this will be 1 by 2 times delta mi is the mass of its particle and, you have the v i is the velocity of the ITS particle. So, this is the kinetic energy of a single particle and you need to sum it over i equal to 1 to n.

So, what we can check from this place that T dot equal to 1 by 2. We can take it outside delta mi and this will be two times v i dot v i or whatever way you want to v i write on the side. It does not matter, because it is a dot product. So, 1 by 2 1 by 2, this cancels out and we get here delta mi and delta mi times v i dot we will write it together, this is nothing, but your this quantity from here to here this is your delta pi dot v i ok.

Now, delta pi we can from the previous part, we can fetch it from this place, this is the internal part, this is your delta pi ok. So, we can take it and insert here in this place. So, this is i equal to 1 to n and delta pi equal to fe i plus j equal to 1 to n, j knot equal to i fij and then dot v i.

So, this will be i equal to 1 to n fe i v i and plus so, this is the rate of change of kinetic energy on the left hand side, rate of change of kinetic energy. On the right hand side, this is the power term, the force into velocity, force into velocity or the speed basically, this is your F dot v. This term is power term and if you write simply F dot dx or dr whatever. So, this is your energy term. So, as you can see this is the rate of change of energy. So, this becomes a dx by dt. So, that appears as v here and the other term.

Now, unlike the previous week where we are dealing with the linear momentum term, as we have looked here in this place the v i is not present and therefore, this distance are 0, but here in this case it so happens that, this is the dot product is present here fe i and feij dot v i. So, dot product is there and this therefore, this term is not equal to 0. So, you can see that the rate of change of energy of the body, it also depends on the internal forces, because fij this is the internal force.

So, unlike the equation of motion, the equation of energy, the rate of change of the kinetic energy, it gets affected by the internal forces also. So, you need to take into account like in the case of the gases or maybe for some flexible body, we are the interaction of the particle that is why changing the energy, a structure so that this term must be accounted.

So, we will close this here in this place and now, we will look into derivation of the kinetic energy. Ah perhaps, we have not still done as of today I do not remember exactly, but certainly it appears to me that I am not done. Therefore, we go into the that particular topic to the kinetic energy T.

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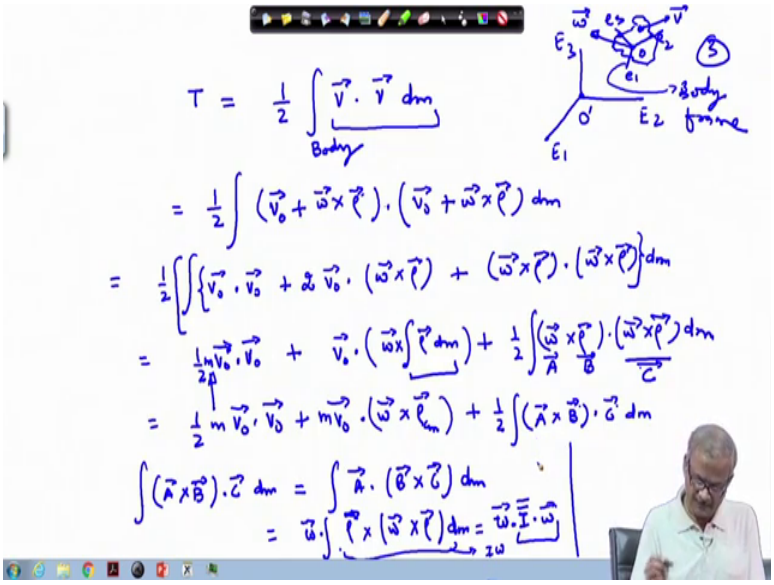

This can be written as, as we have shown earlier 1 by 2 times v dot v dm this is the initial reference frame o prime E 1, E 2, E 3, the body frame. This is O e 1 , e 2, e 3 and there is a rigid body here ok. Some particle is there then velocity of this particle is v and this particle mass is delta m. So, the kinetic energy then can be written like this. This is for the kinetic energy 1 by 2 times, this term that gives you kinetic energy of this small particle.

So, we integrate it over the whole body and once we integrate it, we get the kinetic energy of the body. Now, we can break it up, this will be, this v will be the velocity of this point to O, with respect to this and then velocity of this particle with respect to this point O. This is the body frame, we have fixed here. This is the body frame, as previously we have done. So, this is omega, where omega is the angular velocity of you can show it like this. This is the angular velocity of this frame and rho is the position of that particle with respect to this point O. So, we need to take the, the other term also and if we expand it v 0 plus, the other term will be v 0 dot omega cross rho times 2 and plus.

So, if we say this term is not dependent on the mass, so we can take it outside and this can be written as m times v dot 0, v 0 and dot v 0 ok. So, this dm has been integrated and we are inserted here as m, here in this place. This term 2 2 will cancel out and we get here v 0 dot. Again this omega, omega can be taken outside the integration sign, it can be written like this omega cross rho dm and plus 1 by 2 times and we know that this quantity present here rho is nothing, but this integration, this term rho will be rho cm and multiplied by m. So, m we will put here in this place. So, m time v 0 dot omega cross rho cm and this term we need to rewrite it in this little better way. Let us write this omega as A this as, this rho as B and this term as C ok. So, these are the vectors here A B and C. So, this terms will work out let us copy it here.

So, it is ah, it can be written as A cross B dot C dm. So, we will work, this term only here A cross B dot C dm. This can also be written as A dot B cross C dm and if we inserted back here, in this place. So, this becomes omega, if we put A is nothing, but omega. So, that becomes omega and it can be taken outside the integration sign, because it does not depend on this dm or the mass B is rho. So, this rho will be put here in this place cross and C is omega cross rho.

And in great details we have worked out this term inside the integration sign and is a nothing, but your ok, this term is your I double bar dot omega. This is the moment of inertia term means this term, it reduces to 5 times. This is nothing, but your I omega, which in the inertia dyadic notation, we have written as I double bar dot omega ok. This we can put it back here in this place. If we put it back, though it equation will get simplified. And therefore, T gets reduced to 1 by 2 m ∨ 0 dot ∨ 0 plus v 0 m times v 0, m times v 0 dot omega cross rho cm and plus plus this term.

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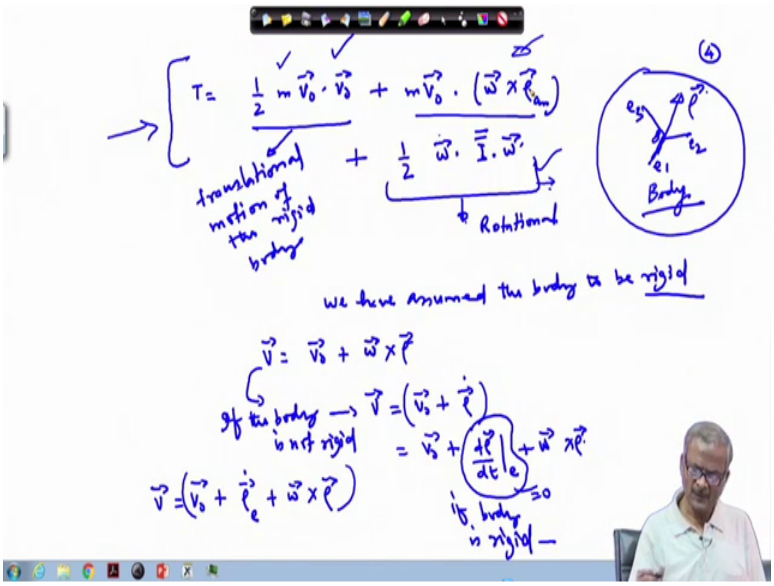

So, this term the quantity under the integrand, we have shown it here. So, this term goes and here we can write as 1 by 2 I double bar omega dot I double bar dot omega.

So, this term is your, the rotational term. This is, because of the linear or this is, because of the velocity or the motion rectally translational motion of the body. This is due to the translational motion of the rigid body v 0 dot v 0 omega. What we have done here that we have assumed the body to be rigid.

So, that way we have simplified the things like v, we have written as v 0 plus omega cross rho, but as we know that this, if the body is not rigid, if the body is not rigid this can, v can be written as then v equal to v 0 plus rho dot. We have to write it this way. Now, rho dot as you see that this is a rho vector in the e frame e 1, e 2, e 3 and this is the rho vector here.

So, if we use this then, we get the extra term ok. So, in that case we have to expand it like this, this will be d rho by dt with respect to the e frame plus omega cross rho. So, if the body is rigid this term drops out this equal to 0 if body is rigid. If body is not rigid then this term will be there and you will get some extra term here in this case, that you can do yourself ah, because we are not going to discuss in this whole course on the flexible body, where the d rho by dt with respect to the e frame will be present ok. This is O here and this is your body frame.

So, for hardly we will get any chance to discuss this complicated case, in that case if you are willing to introduce this also. So, use this whole term. So, you can write in that case v equal to v 0 plus rho dot with respect to e, you can write it like this. Here, indicating this is with respect to the e and then omega cross rho. So, there is a extra term here present and then in this equation instead of these two terms, we will have 3 terms and once we expand it, so we get some extra terms, which we need to work out.

So, all the terms he will not be able to solve exactly like we have done here, but here in this case for the rigid body, it can be solved easily and the final result that we get is here in this case, we have written here this is your kinetic energy of the body. So, this is the translational kinetic energy, this is the rotational kinetic energy and this is a coupling term coupling term, which appears from the v 0 and, because of the rotation if omega is 0 so, you can see that this term will also vanish, this term will also vanish and you will be left only with this translational term. If the v 0 is 0, so this term drops out, this term drops out and this is your pure rotation term. So, this is a pure translational term, this is a pure rotational term and this is a mixture of the rotation and the translation term ok.

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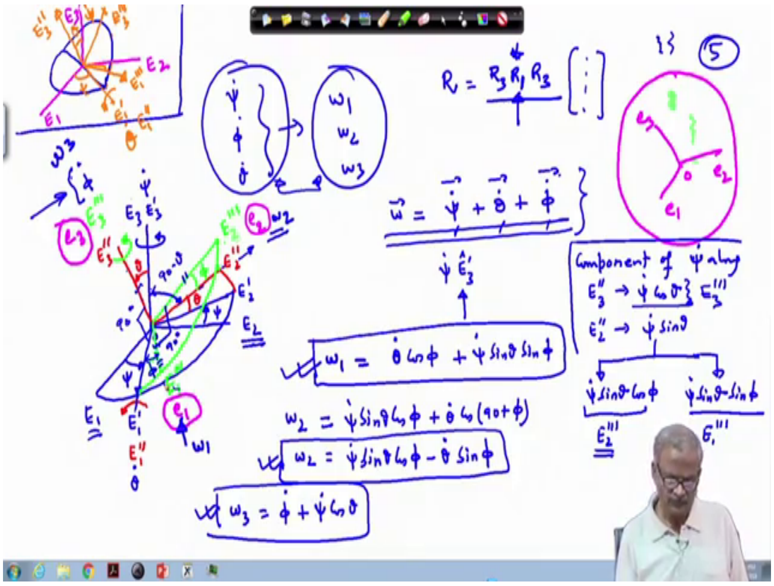

So, now, we have done almost whatever was required for our work of going into the rigid body dynamics. Now, we look into the one again the. If you remember that we have derived the relationship between psi dot phi dot and theta dot and omega 1, omega 2, omega 3. So, this we did purely from mathematical point of view ah, taking into the metrics consideration, we look also this through the, because later on we are going to utilize it. So, I will discuss it here, in this place giving it sometime.

So, having some physical perception of this it is good to have that. So, let us consider the case of a body. Here, I have a this. This is a frame. Now, we will choose this is point O let us say, we will choose the rotation type R 3 R 1 R 3. As I have told you that if the third rotation, this is a small. This rotation about the first axis, this kind of rotation where R 1 is the rotation about the X axis here, in this case, we can tag this as the 1, 2 and 3 or ah, if we write in terms of they E 1 E 2 E 3 cap.

So, we will do it in terms of E 1 E 2 E 3 cap, because those are the unit vectors and we will need it. So, if the rotation about the one access for here it is a very small. So, in that case that gives rise to singularity as we will see it later on and earlier also we have discussed, but I have not shown you the excli, this in the explicit term and therefore, that becomes a problematic thing, but if this angle to the rotation about the first access it is a large. So, that will not be a problem.

So, here in this case we have ah, if this rotation is large or good enough. So, this sequence of rotation can be used. Now, let us look into this place that the first rotation we give, we will operate this rotation on certain metrics or whatever, let us say that here we have E 1 h m. This is capital E 1, capital E 1, capital E 2, capital E 3 and these are the unit vectors here ok. Along this three direction, now we give the first rotation about this axis anti clockwise, the right handed rotation ok, this by psi.

So, if we have rotate about this axis, this axis will remain here in this place, while this axis will come to this place, it will go from this place to this place and this will go from this place to this place, so this rotation will be written as psi, this rotation will be written as psi. So, this is basically a rotation in the same plane means this rotation has been given. This is rotating in the same plane; this is not out of the plane nothing. Now ah, so, here we can represent this as; so, this cap I will remove for the time being for the convenience, this cap I will remove ok.

So, that the representation remains easy here I will write this as E 1 prime, this point as E 2 prime and this remains here itself, so E 3 prime. Now, the next rotation we give about this axis E 1 prime. So, let us say that once we get this rotation here by theta so this will move out and this will go out of this plane and here from here to here, it goes like this right angle theta. This also it will rotate from this place and it will come to this place. This is the angle theta.

So, in the new position, because this rotation is here about this line, so this becomes E 1 double prime, we will write this as E 2 double prime and this is E 3 double prime. Now, the third rotation we choose to give about this axis. So, if we rotate about this axis to E 3 triple prime will be here and then this will rotate. So, this rotates now, this is the X axis and this is the Y axis of the rotated frame.

So, it will rotate in this plane and it will go out like this. So, it will go out from this place to this place and here this is your E 3 triple plane and this particular line then it will come from this place to this place. So, this angle is your phi and this angle is your psi. Notice that psi and phi, they are not in the same plane, if you consider plane of this page. So, that is the E 1 E 1 E 1 prime. So, here in this case your E 1, this is triple prime. So, E 1 triple prime comes out along the Z direction. So, this is out of this plane.

So, from this figure what we are interested in that we want to write omega as a function of psi dot plus theta dot plus phi dot. So, one way of doing it will be that we write unit vector along the psi direction, theta psi dot direction theta dot direction and phi dot direction. So, here in this case those unit vectors in the psi dot direction is your here is your psi dot ok. This is your psi dot theta we have rotated about this by theta. So, theta dot will be along this direction and then phi we are rotating by phi about this.

So, this will be your phi dot will be here along this direction. So, I will remove this notation here theta just keeping it theta dot here in this place ok. So, theta dot psi dot phi dot and this is the psi dot. So, either we need to write unit vector along this direction, this direction and we can do this there are some alternate way also. So, we will do it in the alternate way. So, as many as three ways we can consider doing it here; one way of through the matrix analysis, we have already done three more ways we can do.

So, out of that I will try for the two ah, the rest two and the third one I will not do. Third one will be this one that you just take the unit vector along the psi dot direction, theta dot direction and phi dot direction, which I have indicated here and then insert here in this case. So, the psi dot will be like the psi dot and unit vector along the E 3 cap direction. So, this is E 3 cap and E 3 prime both are in the same direction. So, you can write it like this. So, find out this E 3 prime vector.

Now, I have to write here something more that say, this E 3 triple prime I will write this as E 3. So, this is your final body axis here, sorry, this is we have to change it. This is a correction here, this will be, this is E 2 triple prime. This is E 2 triple prime here, it is E 3 triple prime. So, here in this place also we need to change it, this will be, this will be, we will write as e 2 and here in this place we will write as e 1 and in this place we will write as e 3.

So, instead of using E 3 triple prime we are using this as the e 3, e 1 and e 2 and this constitutes e 1, e 2, e 3, e 1 e 2 e 3, this constitutes your body frame. So, you can consider capital E 1 capital E 2 and capital E 3, this is your initial frame and from there you have rotated by these angles and you are reaching into a final configuration. So, the final configuration is being termed as the body frame.

So, now, let us look from this place. So, omega 1, omega 1 is the, you have to get the components along this direction, this is e 1. So, omega 1 will be along this direction. Omega 2 will be here as, as you can see that this is the two direction here. So, omega 2 will be here in along this direction and here omega 3 will be along this direction. So, omega 1 we need to take the components of psi dot, theta dot and phi dot along the, along this direction, which is from O to E 1 triple prime and in this direction, we have the e 1, this is written as the e 1, e 2 and e 3.

So, let us write this. So, first we will take component of this theta dot is along this direction shown by this blue line E 1 prime. So, here we write it. So, this is theta dot, is theta dot cos phi ok, we taking component of this along this direction ok. Now, phi dot is along this direction. So, this vector and this vector they are perpendicular to each other and will have no components, because you can see that this vector has been rotating about this line ok.

So, this vector from here to here, it remains perpendicular, the this angle is 90 degree. So, phi dot will not have any component along this direction, what about the psi dot, this is perpendicular. Now, if we move it like this. So, the next rotation, we are giving if you rotate about this. So, this also remains perpendicular to this, I cannot show here in this figure, but this red line, this one and this line they are perpendicular to each other if I show it using any marker here, so it will get blurred ok.

So, this 2, this one and this one they are also perpendicular to each other and therefore, there is no component along this direction ok; however, because this E 1 triple prime or the e 1. It has gone out of the E 1, E 2 plane, it has gone out of E 1, E 2 plane and therefore, it will have component along this direction, to get the component of this along, this direction we need to work out. So, the next part we have to do. So we will break it systematically. So, psi dot, this angle is directly not known to me. So, I cannot break, it here along this direction.

So, what we do that the angle from this place to this place this is known to me. This is 90 minus theta. So, I will take component of psi dot along this red direction, which is E 2 double prime and along this direction, which is E 3 triple E 3 double prime. So, along this direction you have the component. So, I will right here in this place component of psi dot along E double prime 3, this is psi dot cos theta and along the E 2 double prime will be psi dot cos 90 minus theta so that becomes sine theta ok.

Now, this is along the this red line here ah, this is along basically along this line. So, this can be broken along this line and along this line, these two lines are perpendicular to each other this line and this line, these are perpendicular to each other and in between this line is line. So, along the E 2 triple prime direction the, this can be broken along two directions.

So, we will have psi dot sine theta and from this place to this place, this angle is phi. So, that becomes cos phi. So, this is along the E 2 triple prime direction and the other one psi dot sine theta. So, here this is given by cos phi this part will be sin phi. So, that becomes sin phi and this is along the E 1 triple prime direction.

So, it from here, we can see that the total thing that we get along the omega 1 direction is psi dot sin theta times sin phi. So, this is your omega 1 ok. So, we can see that without using matrix notation also. We can work it out and this kind of picture it is a essential ok. Without this you will face many difficulties, if you have this clear cut view, it will be very easy.

So, what we here, we have done, the whole thing we can now, I will do the whole calculation here itself, but something I want to show you. This picture, it can be represented as, this is a capital E 1 capital E 2 capital E 3.

Then once you are rotated, it means this plane you are rotating by theta. So, here this angle is your theta and what we have shown here as E 1 prime. So, E 1 prime is lying along this direction. This angle is psi and then E 1 prime, you are rotating and E 1 prime is rotating from this place and E 2 prime will E 2 will also rotate from this place, so E 1 prime. Next rotation you are giving about this axis itself. So, once you have rotated. So, you have rotated by theta so, theta dot have, you have shown along this direction psi dot along this direction, here you have the psi dot. So, next E 1 double prime is lying along this direction itself, this is your E 3 prime and E 3 first E 3 double prime and E 3 prime is here, itself along this side fraction, which I am not showing here, then the rotation shown by this green line. This takes place in this plane, the upper one, which is shown here by the blue line.

So, if you give this rotation say this comes from this place to here. So, this is E 1 triple prime and E 2, which I have not shown. Here, E 2 there will be E 2, then E 2 prime and E 3 triple prime. So, E 3 triple prime, it will lie along this direction E 3, triple prime. So, this is often called this, this line is particularly called the nodal line ok.

We will take up this figure separately, to make it more evident, here for the space is less, but let us first workout this particular part, otherwise in the on the next page again, I will have to refer to this page again and again Omega 2, now, omega 2 is along this direction ok, omega 2 is here. So, omega 2 components we have to take. So, already if you see E 2 triple prime, E 2 triple prime, this component is here. So, psi dot sin theta cos phi. So, this is the component of psi dot along the E 2 direction.

Now, what other component we can take theta dot is here ok. So, theta dot is lying here in this plane itself. So, we can take component of this. So, from here to here this angle is phi from here to here this angle we have written as phi and the angle from this place to this place this is 90 degree. So, the total angle from this place to reckoning from this line to this line, this is 90 plus phi. So, we will have the component of theta dot cos 90 plus phi.

So, this becomes theta dot psi dot sin theta cos phi minus theta dot sin phi. This is omega 2 and check that the omega 3 is along this direction and psi dot phi dot is along these directions. So, this line is perpendicular to this particular line, this is perpendicular to this also, because last rotation you have given about this. So, this is perpendicular to this red line and also to this green line ok. Therefore, psi dot will not have this phi dot will not have any component either along this red line or either this green line ok.

Therefore, there is no contribution from this phi dot the lastly we have omega 3. So, omega 3 is along this direction ok. So, here phi dot is already there. So, phi dot plus what else. So, psi dot component will arise, rest others we will not contribute, because they are perpendicular to this theta dot is perpendicular to this. So, theta dot will not have any contribution, only contribution we get from this place which is psi dot and then along the this direction, the component already we have written along the E 3 ok.

Here, we have written this part E 3 double prime and E 3 triple prime it is along the same direction. This is also along the E 3 triple prime direction. So, therefore, picking up from there this is psi dot cos theta. So, this is your omega 3 and these three are very important for us. So, summarizing we have omega 1, written omega 1 as the theta dot cos phi times phi dot psi dot sin theta into sin phi, omega 2 equal to psi dot sin theta then cos phi minus theta dot sin phi, omega 3 equal to phi dot plus psi dot cos theta.

So, this is final and this is correct ok. Now, the same issue we could have done it some other way and there are we can follow the matrix notation in the sense that I will go on the next page and so, we will stop here. Ah then the next lecture, we will take up this other part and finish it.

So, thank you very much for listening.