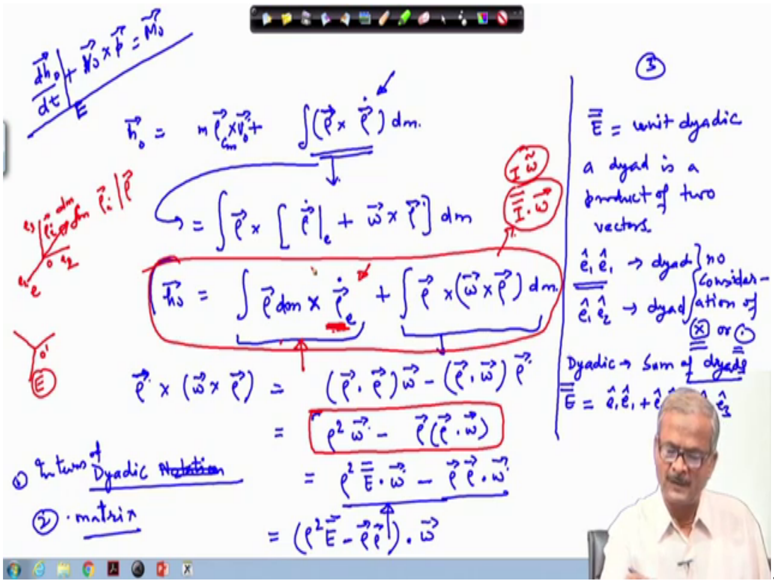
Satellite Attitude Dynamics and Control
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology, Kharagpur

Lecture – 19
Rigid Body Dynamics (Contd)

Welcome to the lecture 19th, we have been working with the Rigid Body Dynamics and last time we looked into how from the angular momentum equation the inertia matrix emerges. So, instead of discussing the mathematics of a (Refer Time: 00:29) nation matrix right in the beginning, I just waited for that I present the present about the angular momentum and from there the automatically, how the inertia matrix will emerge and why it is important that will come out.

(Refer Slide Time: 00:51)

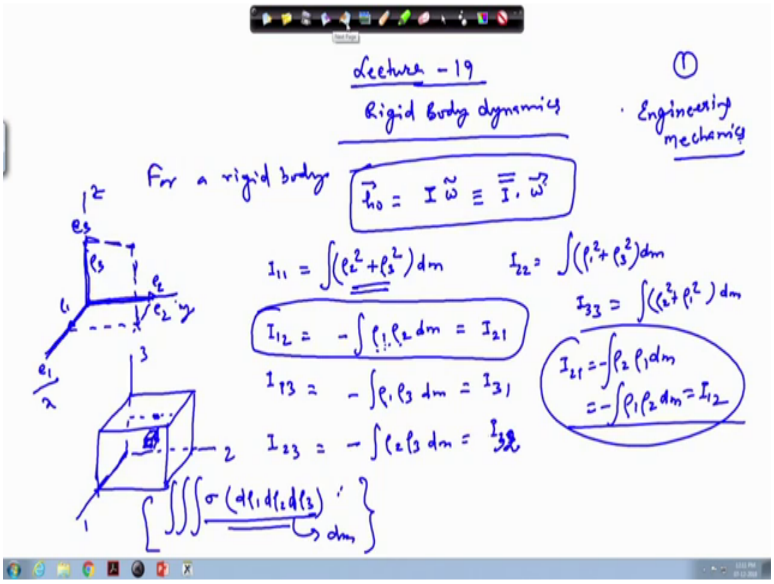


So, what we have looked last time that we got this equation h 0, we derive this equation. And, for this part we showed that this quantity is equal to, I times I double bar inertia dyadic dot omega or the same thing we have also written in terms of I times omega tilde either in the matrix notation or the inertia dyadic notation.

So, we will continue with this particular part, now if you look into this part. So, here this is rho dot e. So, this is with respect to the e frame. So, if you have a non rigid body if this is your e frame, here this is your capital E frame which is the inertial frame, this is a o prime and this is o. So, the mass you we are considering or dealing with this delta m or dm whichever notation we follow. So, if this is the vector here rho i or rho. So, if this vector rho i or rho, it changes with respect to this body that is with respect to this body frame e 1 e 2 and e 3.

If these changes position with respect to this, only then this exist otherwise this term will be 0. So, for a non rigid body this term will be 0, for a this for a non rigid body this term will not be 0 and for a rigid body this quantity will manage this particular quantity and therefore, this term will be eliminated.

(Refer Slide Time: 02:45)



And therefore for a rigid body, we can write h 0 this equal to I times omega tilde and equivalently we can write in terms of I double bar times dot omega, this is in inertia dyadic and this is the inertia matrix. So, now the expression which are entering here; so, we have obtained it from these quantities. So, this we have integrated.

So, if you look into this first one. So, here the first one is rho 2 a square rho 3 a square. So, the first one is I 11 we have written as rho 2 a square plus rho 3 a square and this is the integration of this term. Similarly I 12 this is rho 1 rho 2 dm. So, all the terms in the inertia it can be written in this and this way and I 13 is minus rho 1 rho 3 dm equal to I 31 I 23 rho 2 rho 3 dm is equal to minus I 32 this is plus here (Refer Time: 04:25). So, the way we have written we have observed the minus sign here in this place and the other terms can also be written I 22 equal to rho 1 square plus rho 3 a square dm and I 33 rho 1 square dm.

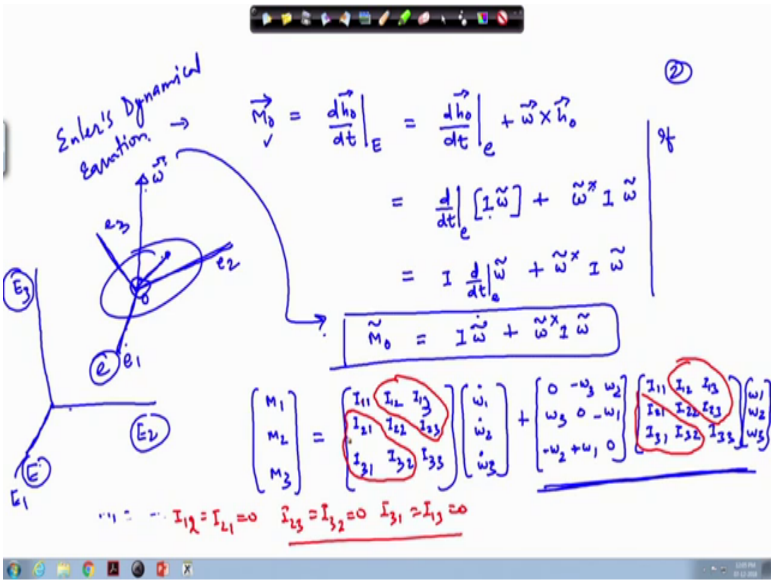
So, if this is your body frame where you are writing e 1 e 2 and e 3 this is the first direction, second direction and the third direction; so, rho 1 if you look into this quantity. So, this is rho 2, you are taking the moment about this axis I 11 indicates the moment about this axis. So, rho 2 a square plus rho 3 a square you have any point here located ok. So, this distance from here to here this is your rho 1, this distance from this place to this place this is rho 2 and vertically you have if you drop a perpendicular here. So, these distances rho 3; so, I 11 indicates rho 2 a square plus rho 3 a square dm.

So, you have integrate with respect to this rho 2 this one and this one ok. So, this is in the say the y z plane. If you write this as the x, if you write this as the y and this as the z; so, this is the integration in the f z plane. So, the mass distribution accordingly you have to take. If you know the mass distribution how the say here in the case of the a cube let us consider a cube. So, I can have this direction this is the first direction, this is the second direction and this is the third direction. So, I can have a mass here located and for this mass then properly we have to model it ok.

So, we have to consider a small element of this, which will be indicated by this d rho 1 d rho 2 and d rho 3. So, that is the volume of this small element which is located at certain distance from the second axis, third axis and the first axis and then say sigma is the density of this material here. So, this constitutes your dm ok. So, this is what will enter here in this place and sigma is the density of this material and thereafter you have to put here along with this rho 2 a square, rho 3 a square. So, integration will go over rho 2 and rho 3. And further at the end you can integrate over rho 1 or either even in the beginning you can integrate over rho 1.

So, take care of your integration sequences, that you should not create any error in that ok. So, this part I will not take up, but rather because this is going to take a lot of time discussing with the inertia matrix, you can it is a part of your basic mechanics. So, either you can look into the book by Irving Shames this the engineering mechanics book by Shames or either ER Johnston. So, this way we have got this equation and this equation from here, then will be able to get the next part which is your M 0.

(Refer Slide Time: 08:23)



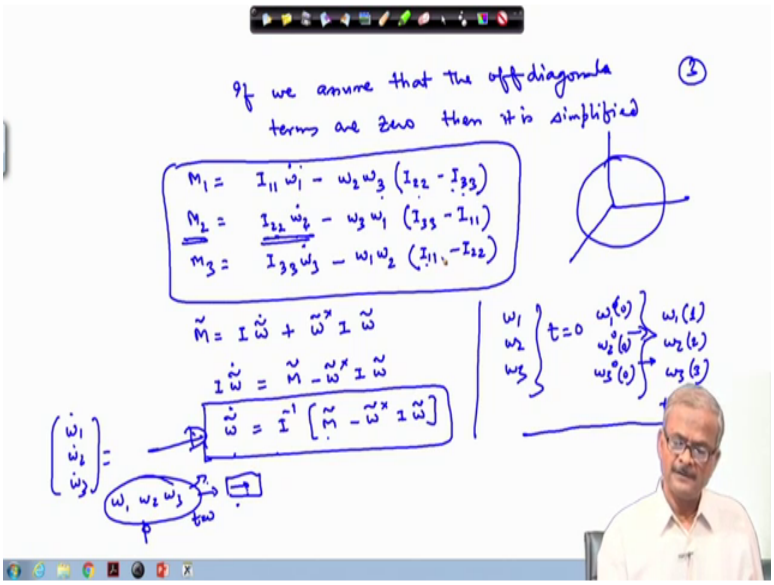
The moment equation and this moment equation when we get it this particular equation it so, called the Euler's Dynamical Equation or Euler's Rigid Body Dynamics Equation. So, M 0 this we have written as dh 0 by dt and this is with respect to the E frame. So, as indicated earlier this can be written with respect to the first the small e frame and then omega cross h 0; and how do we do this? We have already discussed this in the transport theorem in the mechanics. So, I will not repeat it here again. Now h 0 you know and therefore, d by dt we can write, this is with respect to the e and if you write in terms of the matrix notation. So, this is I times omega tilde rather you can write in terms the inertia dyadic and here this quantity this will get reduced to omega tilde cross times h 0 which is I times omega tilde.

Now, if your body is rigid. So, if the body is a rigid and this is your e frame e 1, e 2, e 3 then the distance of a particular particle from different taxes it remain fixed with respect to only the e frame this is your e frame. So, this particular distance will remain fixed from all this places. And therefore, the I this will not change and this you can take it outside, and write this as d by dt omega tilde plus omega tilde cross and this is nothing, but omega dot. So, omega tilde is the rotation rate of this rotation of rate of this frame with respect to the inertial frame.

So, this is your inertial frame e 1, e 2 and e 3. So, with respect to this say this is your omega vector ok. So, omega as this body rotates it is a angular velocity may change. So, right now the omega vector is indicated here in this direction sometimes afterwards it may change its direction. So, I will return back to this and here this is your M 0 tilde. So, this is your Euler's dynamical equation. And if you expand it you can write it like M 1 M 2 M 3 and obviously, this is with respect to the point o, this is your point o. So, I will drop this notation o here and simply write in terms of this is I 11 I 12 I 13 again here I 12 this is you can write either in terms of I 21 or I 12 no problem I 31 I 32, I 33 omega tilde is omega 1 dot omega 2 dot omega 3 dot and then omega tilde cross here which is nothing, but the skew symmetric matrix numerous times we have written it here.

So, I will just write it plus omega 1 here and this is minus omega 2 and here plus omega 3 and then put here similarly the I 1 times omega tilde which is omega 1 omega 2 omega 3 and then you can expand it. So, if you expand it the first term it will appear as I 11 now we have to obviously, multiply and this multiplication sorry if we have to operate this matrix multiplication and it will be a longer one, but it will get simplified if we assume that if we assume.

(Refer Slide Time: 13:37)



If we assume that the off diagonal terms, terms are 0 then simplification takes place then it is simplified and then how it appears? This is M 1 and the next term will be I 11 go back here. So, all these terms I 12 I 13 these are the off diagonal terms you are on this side these are the off diagonal terms on this side these are the off diagonal terms.

Similarly, here in this place we have to ok. So, if you consider that this terms are 0 that is I 11 I 12 equal to I 21 this equal to 0, I 23 equal to I 32 this equal to 0 and I 31 equal to I 13 this equal to 0. So, if these off diagonal terms are 0. So, you considered this multiplication the first one will be I 1 times omega 1 dot similarly here this simplification which you can do it yourself ok. So, this will appear as this is simple matrix multiplication you can do it yourself and this will follow a particular sequence which. So, getting into this equation you need not derive it every time and you need not remember it. It is a just a cyclic equation which you can follow the way I am writing here.

So, see here first we are writing M 1. So, this is the one first term and the second term the 2 and 3; this is one then this is the second and third is coming and similarly here the second and third is coming and there is a minus sign here in this place. Similarly we will have M 2 equal to I 22 times omega 2 dot minus now this is 2 here corresponding to this. So, here this will get as omega 3 and then omega 12 then 3 then 1 ok. So, we will go in the same sequence. So, this will be I 33 minus I 11 similarly your M 3 this will be I 33 times omega 3 dot minus after 3 comes 1. So, this is omega 1 and then 2 and similarly we will have I 11 minus I 22. So, your equation gets reduced into this simple format provided your off diagonal terms are 0. If the off diagonal terms are non zero then you need to consider this whole equation expandit and then you can work it.

But if you try if you do the same thing with the MATLAB it is a very easy because the MATLAB is meant for the matrix operation and once you have written in this matrix format, it is a very easy to work with this. If this torque is this is the external torque acting on the body and this torque is about point o this is about this point ok. So, if this torque is known therefore, if we transport this whole thing on the left hand side, if we pick up this whole part from this side and put it here on this side and then will be able to integrate this equation. So, in this full format also we can integrate or either in the format which we are having here in this place also we can do we can integrate.

So, our equation is basically M equal to I times omega tilde dot and this plus what we have written omega tilde cross plus I times omega tilde. So, here we can write this as I times omega tilde dot equal to M tilde minus omega tilde cross I times omega tilde and omega tilde dot is equal to I inverse I times omega tilde. And you can say that this is the this is in the this is a differential equation and here on the left hand side you have the omega 1 dot, omega 2 dot and omega 3 dot and on the right hand side these are the terms which are present here.

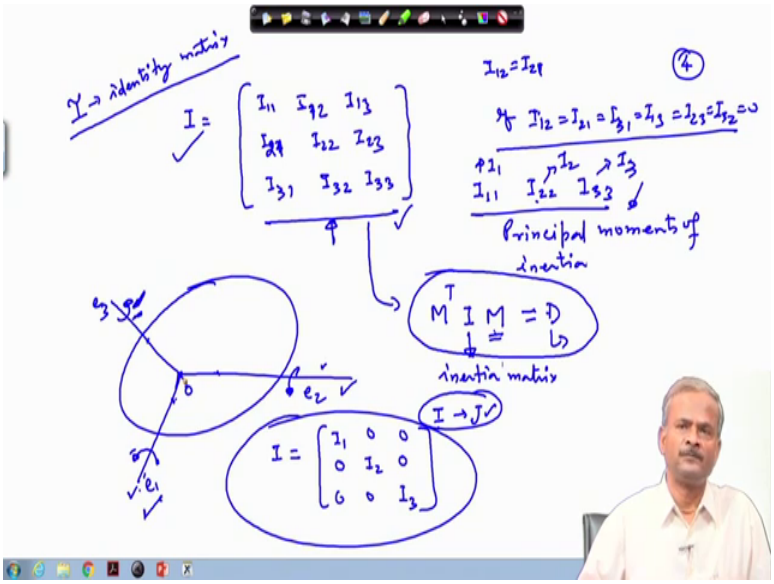
And therefore, this can be numerically integrated we can do the numerical integration. So, if you are a starting with say omega 1 omega 2 omega 3 value at time t equal to 0 and let us say that value is omega 1 0, omega 2 0, omega 3 0. So, next instant if you update say after one second how much it will be. So, that you can write as or here we can put as 0 0 to indicate this as a time t equal to 0. So, next time it will be omega 1 omega 2 omega 3.

So, from here we will get the angular velocity of the body at the time it t equal to one second ok. And already we have looked into the kinematics of rotation. So, using that then this omega 1, omega 2 either you can work in terms of quaternions or either you can work in terms of this can be done in terms of the quaternions or this can be worked out in terms of the other angles. So, of in terms of quaternions its a easy quite easy because that is in the algebraic notation. So, you can see that that things I have started emerging that, if I am giving a rigid body and a certain torque is applied to that body.

So, how do we are start? We are start with integrating the equation of motion which is the eulers dynamical equation if we integrate it, this particular equation if we integrate it. So, from here will get the omega 1 omega 2 omega 3 at the next instant and at the present instant we have the omega 1 omega 2 omega 3 this is available. So, this can be used to this is at the present in instant say that t equal to 0. So, this can be used to propagate the kinematics. So, we will propagate the kinematics at the next instant. So, your Euler angles or Euler parameters or the quaternions, they will be propagated from one instant to another instant and then again use the updated value of the omega to propagate the quaternion.

So, this way we can keep propagating. So, we will take one particular once we come to the controls part. So, at that time we will take one program in the MATLAB and then look into this how this particular thing it is a done ok. So, we have discussed about the Euler’s dynamical equation, now the inertia matrix we need to also discuss little bit about the inertia matrix.

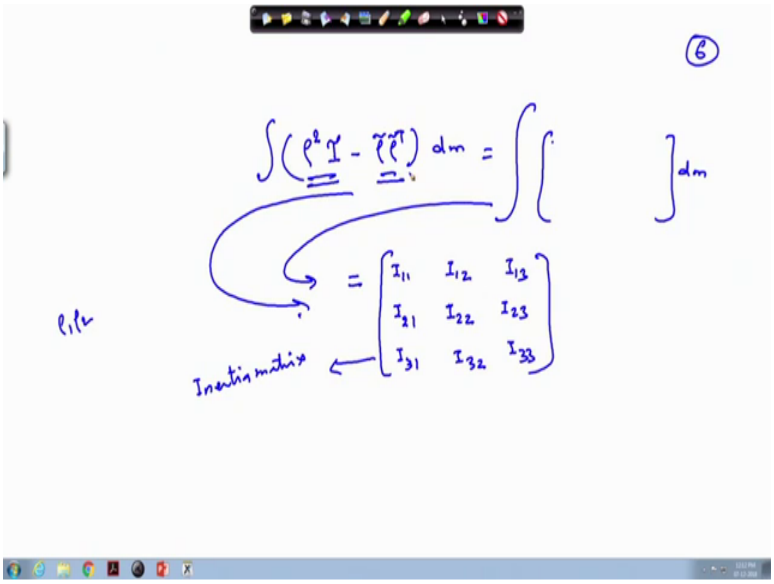
(Refer Slide Time: 21:43)



So, the inertia matrix we are writing as I this I 11, I 12, I 13 I 12 or I 21 I 22. So, as I told you that if the off diagonal terms are 0 and only the diagonal terms remain then this diagonal elements, if I 12 equal to I 21 31 equal to I 13 these are 0 ok. Already we have seen that, this inertia matrix is symmetric means I 12 equal to I 21 this directly obtains from this equation ok. I 12 equal to I 21 why because if you change the order of 2 (Refer Time: 23:04) this is a scalar this integration is over the scalar quantities.

So, if you change the order of this I 2 and I 1 if you make it does not make the difference and therefore, if you are writing I 12 as rho 1 rho 2. So, I 21 as rho 21 rho 2 times say the I 21 if you write as rho 2 times rho 1 that minus sign here dm. So, this is nothing, but rho 1 times rho 2 the order changed and this is equal to I 12. So, the off diagonal terms they are the same means the inertia matrix is symmetric and that I have shown you mathematic through by using this equation ok.

(Refer Slide Time: 23:49)



This terms is symmetric, this term is symmetric and therefore, the whole thing will symmetry this is a matrix this is a matrix and therefore, if this is symmetric this is symmetric. So, addition of these 2 symmetric matrices will also be a symmetric matrix ok. So, it is a very simple and other ways of doing of are also there, but let us confine our self to this only ok. So, if these off diagonal terms are 0, the terms then I 11, I 1 I 22 and I 33 these are called the principal moments of inertia these are the principals principal moments of inertia. So, in a rigid body I have any rigid body, I have fixed this frame into this which we have written as e 1 e 2 and e 3 this point we have taken as o.

So, I can compute this matrix for any rigid body, if the mass distribution is given its always possible either numerically or analytically it is always possible that we can get this I matrix. Once we have got this I matrix, then this can be diagonalized also means that if I operate on this I matrix this is the inertia matrix by here m is an orthogonal matrix. So, if I operate on this. So, this will get diagonalized this is a diagonal matrix. This is not identity matrix remember this is inertia matrix inertia matrix and identity matrix we are writing by this notation this is identity matrix. Many of the books instead of I they will use it as J, but J is mostly confined to the polar moment of inertia. So, of I am not using J, I will confine myself to the I and for identity matrix I will use this notation.

So, what says. So, I will write some points about the symmetric matrix what are its properties. So, of one way of this property one of its properties is that this can be diagonalized where M is an orthogonal matrix. So, if we diagonalize it. So, this diagonal elements are nothing, but then the principal moments of inertia and in that case instead of writing this I 1 I 11, we write this as I 1 this will write as I 2 and this will write as I 3. So, we have drop that notation I 11, I 22 that can be along this direction this is for this is the one is a product of inertia and this is about this axis ok.

So, we have taken it as. So, let us go on the next page. So, once we diagonalize it our process, then gets simplified and this I diagonalize matrix in that case it will it is written as I 1 I 2 and I 3 this is 0 0 ok. So, this is the diagonal matrix. Now what is the benefit of this using this particular one? In many cases it may so, happen that your access they get decoupled means the dynamics if you look back here in this equation and suppose that I 22 and I 33 they are all same means this is the case of a sphere if we take the case of a sphere. So, all these quantities will be equal for this sphere the moment of inertia along any direction it is the same.

So, of this will be same and this gets here you get only this terms and then it becomes and this motion we call as the decoupled means any torque along the direction 2 this is moment along the second direction. So, it will affect only the angular velocity along the second direction, it will not affect the angular velocity along the first direction or the third direction provided these quantities are equal. So, of projecting the inertia matrix in terms of its principal moment of inertia it is a very useful and the control design also gets simplified otherwise it so, so happens that you are trying to steer along one of the axis and the system is getting actuated along the other axis also, but sometimes it may happen that if there is coupling between the different axis. So, it is advantageous.

Say if we have control available only along the first direction and the second direction and third direction we do not have any control available. So, of and if the dynamics is not coupled to this axis, this 2 facts dynamics it is not couple to this x dynamics means if I apply torque along this and the motion takes place only along this axis. And there is no torque available along this axis, then we are in trouble we can never control along this axis, but if there is coupling then it becomes helpful in that cases. So, some of the cases will face that, if the system is coupled we are able to handle some of the complex cases where we do not have the enough control moment or the moment along all the axis.

But in some of the cases it may so, happen that if I have control moment only along 2 of the axis, it’s we have what I wanted to say that, in some of the cases where we do not want that the coupling is along all the axis. If it is along all the axis, then we get into trouble that my other axis all always getting perturb. So, it depends on situation to situation which will take count take all this issues as we proceed and progress further. So, of thank you very much and we will continue in the next lecture.