Satellite Attitude Dynamics and Control
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Lecture – 17
Rotational Dynamics - Rigid Body Dynamics (Contd)

Welcome to the 17 lecture on Rotational Dynamics.

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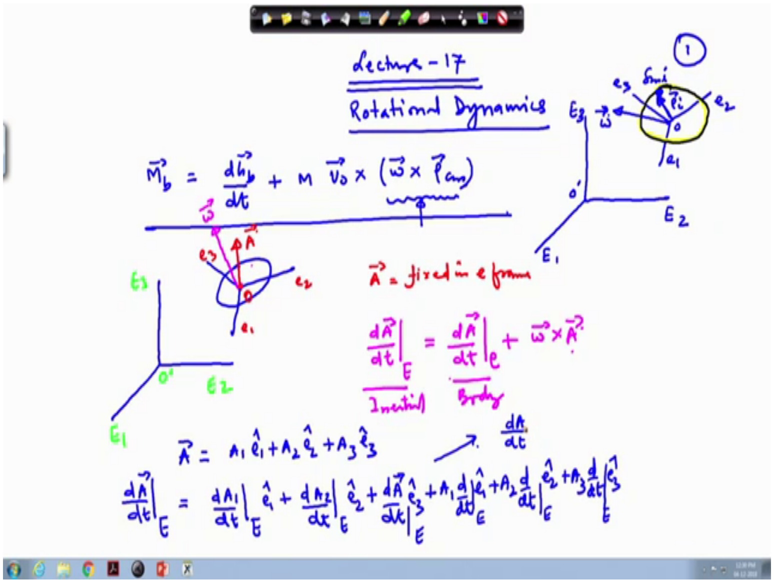

So, if we remember that in the last class we derive this particular equation ok. So, one particular equation which may be of concern that this omega cross rho m we have inserted there so from where it is a coming. So, first we will do it in little bit details because this is very important and basic to our derivation and thereafter will pick up this again to consider the rigid body dynamics. And, I do not remember exactly that how much in details we have discussed this particular part earlier.

So, let me complete it here ok, this is any vector A which is fixed in e frame ok. So, this is e 1 e 2 e 3 and this one is e 1 e 2 e 3, this point is O prime and this prime we are taking as O. So, this vector is fixed in this body, this a rigid body in which it is fixed and this rigid body it is moving with angular velocity omega. It may be say if even it can be the centre of, if this is a rigid body. So, you can consider this A to be the centre of mass location. This vector is directed to the centre of mass location.

But it can be any arbitrary one ok, this is the angular velocity vector. Now what we are interested in this dA by dt in the frame E. And, it can be proved that this dA by dt this can be written as in the frame e which is the body frame this is a body frame this is the inertial frame and omega cross A. And if A is fixed in the body and therefore, this part will drop down and only this term will remain, this is what we have used in this result in the previous lecture.

So, this A vector we can write as A 1 times e 1 cap A 2 times e 2 cap where A 1 A 2 A 3 these are the components of vector A in the body frame. Now we take the derivative of this vector with respect to E frame ok. So, you know that this will be dA by dt by simple your calculus e 3 cap plus now we have to take the derivative all this points. So, A 1 times d by dt e 1 cap with respect to the e frame, generally A 2 times d by dt. All this derivatives are with respect to the e frame, these are with respect to e frame A 3 times d by dt with respect to e 3 cap. This particular equation then this can be written as dA by dt or we will write it on the next page.

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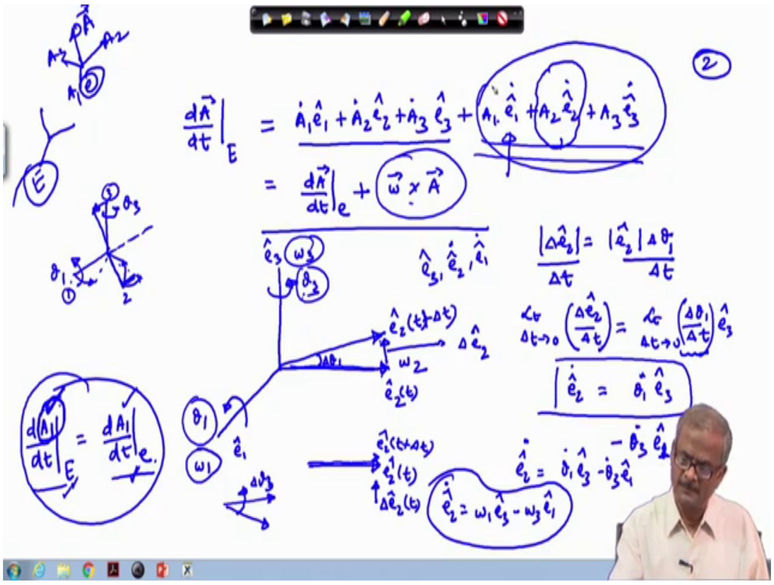

These with respect to the E frame now, the quantity that you are getting here dA 1 by dt with respect to the E frame. A 1 is the scalar quantity ok; so what it says what does this mean and A 1 we have already taken along the body axis, there is a vector A is here. This is e frame and this is E frame capital E frame and it's components along this 1 is A 1 here this is A 2 and here it is A 3. So, this d A 1 by dt this will be equal to d A 1 by dt e, what is the reason?

This is the rate of change of the length along A 1 and that with respect to the E frame and the same thing with respect to the small e frame this is the inertial and this is the body frame, how this is possible? Thing is very simple, say yours shape fan is hanging from the ceiling of the roof and suddenly one blade starts contracting in length or either it starts expanding in length. Say in 1 second it goes from 1 meter to 2 meter, means in 1 second it has changed to 1 meter to 2 meter length, isn't it?

So, if you are sitting on the fan and you are rotating along with the fan so, you will see that it has doubled in 1 second ok. So, this is a scalar quantity ok, now the same quantity its length 2 meter if seen from the inertial; frame it will remain same isn't it, if the temperature there it is 5 degree. So, that will also be the 5 degree because it is a scalar quantity, it is not a vector quantity.

So, the fan blade if becomes one to 2 meter in the body frame, it will also true be true for the inertial frame, means the same length in both the places. So, here also it will from the ground you can tell that this has become 2 length, 2 meters. So, this is because of the scalar this particular scalar that we can write it in this way. So, the our equation then gets reduced to, we can write this as A 1 dot and that with respect to the body frame. So, will eliminate that sign for the body frame and simply write this as A 2, A 3 dot A 3 cap and the other part we have A 1 times e 1 dot plus A 2 times e 2 dot where dot indicates d by dt.

This part this is what we write as d A by dt with respect to the e frame. So, whatever you are seeing in the e frame it will exactly with the same in the capital E frame also, the small e frame and capital E frame, the other part this will be equal to omega cross A and this we need to just have a look of this, for this is very useful later on also will require this. So, say this is the e 1, e 2 and e 3 direction and let us say this is omega 1 component of the angular velocity here, omega 2 here and omega 3 here.

Now, if rotation is given about this particular line about here. So, what will happen to this e 2 vector? These are the unit vectors here. So, it will go from this place to this place, magnitude of this remain same as unity.

So, this is e 2 t if you are write then this will be e 2 t plus delta t and. So, after time delta t these is the situation and let us say that this changes by theta 1 or delta theta 1 and this is the vector which has changed. So, this is delta e 2, delta e 2 cap we can indicate it like this. So, delta e 2 cap magnitude we can write as this magnitude this length means the e 2 cap magnitude times delta theta ok, for a small values of delta theta. So, here you have done it by delta theta 1 and now if we divided by delta t on both side. So, it will look like this ok, if we take the limit the delta t tends to 0. So, delta e 2 delta t, here we are taking magnitude, but what we are looking for we are looking for quantities like e 2 dot e 1 dot e 3 dot, we are looking for this quantity.

So, I am interested in this part. So, what this part will be? So magnitude wise if you look here so if I take the limit. So, here in the limit this will be limit delta t tends to 0, this will be e 2 magnitude is 1. So, this is delta theta 1 by delta t and what will be its direction. So, if you see that this is the vector e 2 t and if delta theta delta t tends to 0 delta theta will also will become very small and it will almost it will lie over the same vector. So, e 2 cap t plus delta t it will here in this direction. So your, the change this delta e 2 it will be perpendicular to this e 2 cap vector; that means, this vector is along the change that has taken place which is along this vector which is the e 3 direction.

So, here we have to multiply by e 3 cap. So, this says that e 2 cap this will be given by theta 1 dot times e 3 cap, but the picture is not over ok, here still we have to do something. This is only one part here due to the rotation you have given by theta 1 ok, which is related to omega one you are considering. Now this vector can also change if you give a rotation about this direction, let us say this is theta 3 which is related to omega 3 ok. So, and if you look for that part so this is taking place in the page of the plane, now if you rotate about this is coming out of the face of the plane and this 2 are in the page of the plane.

Now, if you rotate about this. So, this vector will go inside the page of the plane means if I show e 2 like this. So, e 2 is going inside the plane and this changes taking place because of theta 3 and we have to indicate this angle by delta theta 3. So, the rotation will be by delta theta 3. So, if you would look from say if I show it like this so 1 rotation I have given about this ok.

So, this is going like this and this part accordingly it will go from this place to this place the other if you rotate about this one by theta 3, here you are rotated by theta 1. So, this will go from this place to this place and this will come from this place to this place. So, we can see that we have another motion here because of this theta 3.

So, for the motion of this particular axis it will take place because of theta 1 and theta 3 and in which direction the change will take place just like this. So, this is your vector, see this is your say the 1 direction. So, you can see that and this is the 2 direction this is the first direction, second direction and this is the third direction. So, this change will be along perpendicular to this e 2 vector and so which is negative in the negative direction of the A 1 means the e 2 cap also will result from theta 3 dot ok, but that is in the e 2 e 1 cap direction with negative sign, it is going here in this direction. So, this comes with a negative sign.

So, this way means e 2 dot it will be consisting of theta 1 dot times e 3 cap minus theta 3 dot times e 1 cap or either we can write this as e 2 times omega 1 times where omega 1 is the magnitude e 3 cap minus omega 3 times e 1 cap ok, this is dot this is dot here. So, this particular part you have to understand it properly and without that the things will not be complete. So, maybe if I draw for you another picture to make it more clear on the next page.

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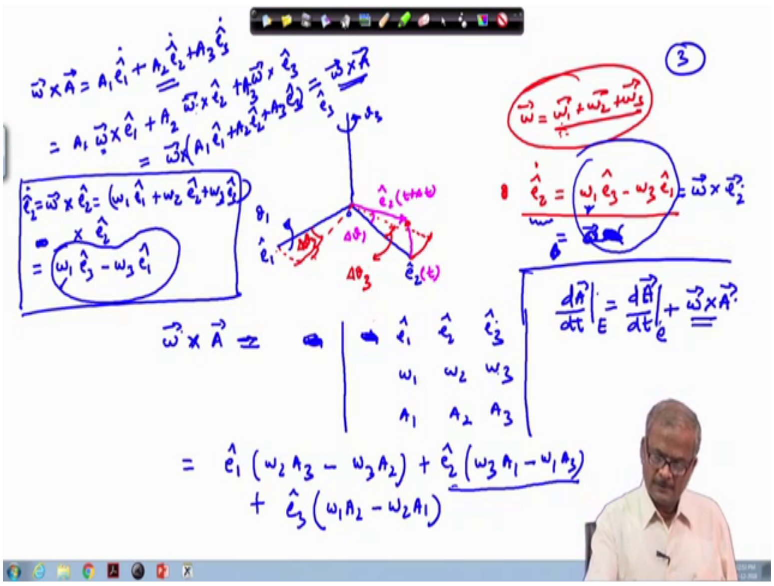

Say what we have done that we have this frame here, this is e 1 this is e 2 and this is e 2 and here this is e 3 giving this anticlockwise rotation by theta 1 here and another rotation anticlockwise about this by theta 3. So, because of this rotation, because of this rotation it goes from this place to this place. So, I will show it in another color it comes from here to here and this takes place in this plane, 1 and 2 and 3 frame. So, this is your e 2 t then this will be e 2 cap t plus delta t, now this angle you are showing as delta theta 1.

Another one you can show because of this motion this will be along this direction and then this angle from here to here this will be your delta theta 3 and here this will also rotate by delta theta 3. So, they are 2 motions involved ok, one along this direction another along this direction. So, of the change will take place. So, the changes that takes place it is a because of that changes that take place is because of the e 2 rotating because of the rotation about the this e 1 axis and also about the e 2 axis and if you remember that in the rotational kinetics we have discuss that for a small rotation the order of rotation does not matter ok. But for larger rotation the order of rotation will matter.

So, the angular velocity it is a you can sum them like omega one omega 2 omega 3 you can sum them as a vector because this consist of very infinitesimal rotation over infinitesimal period of time. And therefore, they are commutative and the already we have done a lot of discussion over this. So, I do not want to go further and spend time over this part, but the thing is that this theta the e 2 dot, the e 2 dot we have written this we have written as has on the previous page omega 1 times e 3 cap minus omega 3 times e 1 cap. So, omega 1 times e 3 cap minus omega 3 times e 1 cap.

The same way we can write for the other one also ok. So, if you look from, if you look into this equation omega cross A what we have written there ok. So, in that equation go back again here this particular part. So, here you see that this part we are writing as like this and where we require this particular equation e 1 dot e 2 dot e 3 dot ok. So, omega cross a this is omega 1 we can write this is in form of the e 1 cap e 2 cap e 3 cap and then omega 1 omega 2 omega 3 and then A 1 A 2 A 3.

So, this particular part then this becomes e 1 cap omega 2 A 3 minus omega 3 A 2 plus e 2 cap times omega 3 A 1 minus omega 1 A 3 plus e 3 cap times omega 1 A 2 minus omega 2 A 1 ok. Now check this part, this is your e 2 dot. So, e 2 was nothing, but go back here this part we have taken up. So, omega cross a we are writing like this where e

2 dot cap. So, omega cross a in this if you look for this particular term, this is a 2 times e 2 dot cap, let me copy the whole thing here of A 1 times e 1 dot e 1 cap dot A 2 times e 2 cap dot plus A 3 times e 3 cap.

Say if the same thing if I write this in the form of omega cross I want to express it in a proper way, let us look first here in this place then will come to that. So, this is A 1 times e 1 dot. So, this is omega 1 or the of this omega cross omega cross e 1 cap plus A 2 times omega cross e 2 cap plus A 3 times omega cross e 3 cap. So, e 1 dot is nothing, but your omega 1 times e 1 cap that you can check ok, here we provide the check in this place omega cross e 2 cap, what this quantity will be this we have omega 1 times e 1 cap plus omega 2 times e 2 cap plus omega 3 times e 3 cap cross e 2 cap.

So, if we take the cross product so this will be omega 1 times e e 1 cross e 2 this is e 3. So, this is e 3 cap and then e 2 cross e 2 that will be 0 and e 3 cross e 2 that will be e 3 cross e 2 is minus e 1. So, this is omega 3 times minus e 1. So, look here in this place this quantity is nothing, but we are writing as e 2 cap dot. So, e 2 cap dot this in nothing, but your omega cross e 2 ok, this is verified here in this place omega 1 this is omega 1 here omega 1 is present here omega 3. So, of what we have got here exactly omega 1 e 3 cap in this 2 places, if you look here in this particular equation and here in this place.

So, omega 1 is the first term in both this places in this place and this place omega 1 is present e 3 e 3 this is the same thing omega 3 omega 3 e 1 cap e 1 cap it is here. So, this quantity e 2 dot cap is nothing, but omega cross e 2. So, this way the order terms can also be written. So, you have to go in the same way and find out all the terms and if this omega we take it outside because A 1 is a scalar. So, we can write this as A 1 times e 1 cap plus a 2 times e 2 cap plus A 3 times e 3 cap and this is what your quantity and the cross sign here.

So, this is omega cross A. So, the rate of change of the vector d a by dt in the inertial frame e this you are writing as therefore, d A by dt this with respect to b and plus or here in this case we have use the smaller e. So, we will continue with the small e sign and then omega cross A which obtains from this place. So, omega cross a what we have written here it is a times A 1 times e 1 dot A 2 times e 2 dot for each of them we have already expanded so concluding.

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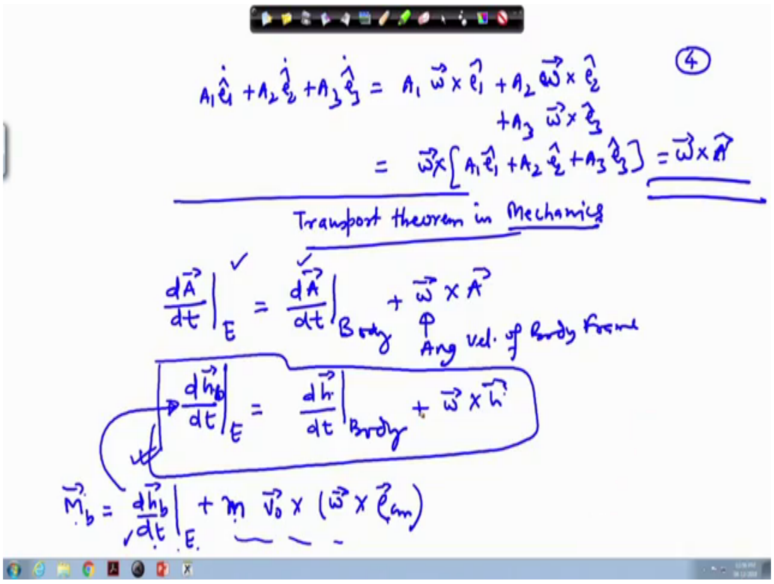

So, what we have A 1 times e 1 cap dot plus A 2 times e 2 cap dot e 3 cap dot is equal to A 1 times omega cross e 1 plus a 2 times omega cross e 2 plus A 3 times omega cross e 3 cap and we have taken this omega outside. And then written this as e 1 cap e 2 cap plus a 3 times e 3 cap which is written as omega cross A and this is what I, already I have told you this is call the transport theorem in mechanics. And, this is a very important conclusion and quite often you will require while you are discussing about the rigid body dynamics you will need it again and again in almost all the places.

So, remember this particular equation that d a the rate of change of any vector with respect to with an inertial frame it can be written as d A by dt with respect to the body frame, this is the body frame plus omega the angular velocity of the body frame. This is the angular velocity of body frame with respect to the inertial frame almost all the places you will need it and therefore, say if I replace this by h ok. So, h is a vector which you are writing about the find to o of the body frame already we have discussed.

So, this with respect to E will be written as dh by dt omega cross h. So, we will utilize this equation again and again for our purpose and what earlier we have developed there we have written as M which is the M b which is the torque this equal to dh b by dt with respect to the E frame that is with respect to the inertial frame. And then we have written m times where this M and this M is not the same this is the mass. So, maybe we can indicated by the small this small M and then of course, we have the other terms like the

V 0 cross omega cross rho c m and if the rho c m becomes equal to 0. So, this term drops out and this is the term which is dh b by dt which is present here in this place.

So, we still we are here in this place, after removing this term then we get this term and this term can be further expanded in this and there is advantage in writing it like this and what is the advantage will continue discussing in the next lecture. So, thank you very much for listening we again meet in the lecture number 15.

Thank you.