Satellite Attitude Dynamics and Control
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Lecture – 14
Rotational Kinematics (Contd.)

Welcome to the 14th lecture, we will continue with whatever we have been discussing in the last lecture.

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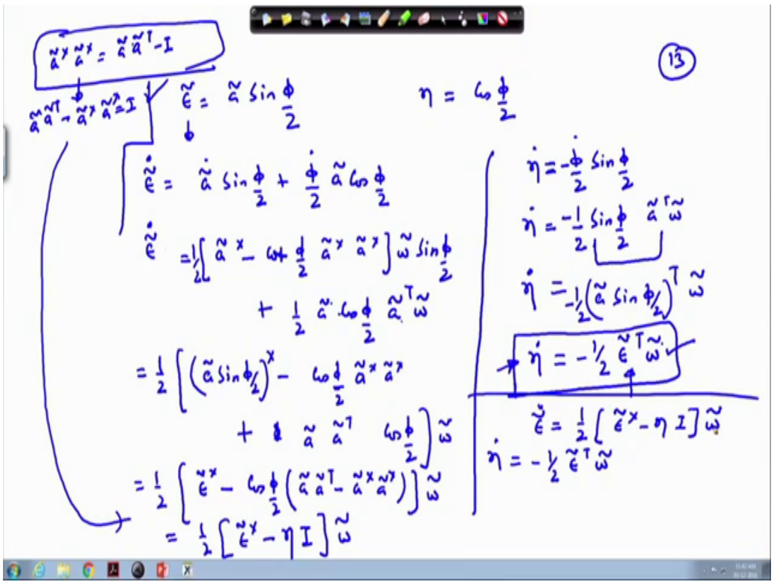

1 by 2 times hectare equal to. So now, we get that rate of change of this epsilon is either the quaternion or the Euler parameter. So, this will get reduced to a tilde dot sin phi by 2 plus phi dot divided by 2 a tilde sin gets cos this is cos and eta dot becomes phi dot divided by 2 phi by 2 with minus 1; phi dot is nothing, but a tilde transpose times omega tilde as proved earlier. So, we use that result and here this is 1 by 2 with minus sign. So, therefore, eta tilde dot now this quantity we can merge together and write this as a tilde times sin phi by 2 transpose times omega tilde and this is minus 1 by 2.

So, minus 1 by 2 this quantity is nothing, but epsilon. So, this have epsilon tilde transpose times omega tilde so, this is our eta 2 ok. So, this is 4th element of the quaternion or as the Euler parameter the rate of change of this is given by this contest. So, if your omega is known and epsilon is known that is the quaternion or the Euler parameter the vector part. So, then you will be able to estimate this eta dot; similarly we have to write this here this is epsilon dot we have already written here and this we can write in a proper way to meet our need.

So, a tilde dot we have already got this is 1 by 2 times a tilde cross minus cot phi by 2 times a tilde cross times a tilde cross times omega tilde then sin phi by 2. And then this term 1 by 2 times a tilde times cos phi by 2 and then phi dot ultimately we will write in the format a tilde transpose times omega tilde ok. And then we rearrange the terms so, 1 by 2 common 1 by 2 is common here. So, we take 1 by 2 outside the bracket sign we will take here this inside. So, this is a tilde times sin phi by 2 cross and omega tilde is here.

So, omega tilde we will keep it outside and rest other things will pull it inside. So, we have sin phi by 2 already we have included here and this term can gets reduce to cos phi by 2 a tilde cross a tilde cross, omega tilde we are keeping it outside and then sin has been merge with this. So, this has resulted in this cos and then this particular part. So, this is 1 by 2 we have take already taken outside this is a tilde times a tilde transpose times omega tilde cos phi by 2 and outside we will have omega tilde. So, as we see from this place this quantity is nothing, but epsilon tilde.

So, this of epsilon tilde cross minus cos phi by 2; so, cos phi by 2 we take this outside the bracket and here we write this as a tilde times a tilde transpose this omega tilde is not there because we are keeping it outside. So, this is removed and then from here this minus sign a tilde cross a tilde cross this equal to omega tilde. And if you remember that we have use the result earlier where we have written that a tilde cross a tilde cross equal to a tilde transpose a tilde transpose of a this is a tilde transpose minus i.

So, if you look from this place. So, its gets reduce to a tilde a tilde transpose minus a tilde cross a tilde cross this will be equal to I. So, we can utilize this particular equation here in this place. So, that gives you 1 by 2 epsilon tilde cross minus cos phi by 2 this is nothing, but eta. So, we write it eta and this is the we pick up this I from this place. So, this is I times omega tilde. So, our final result is epsilon tilde dot this equal to 1 by 2 times epsilon tilde cross minus eta times I times omega tilde times. So, and eta dot this equal to minus 1 by 2 times epsilon tilde transpose times omega tilde.

So, if you are working with the quaternion's or either the Euler parameters. So, you will be using these 2 equations to propagate the system kinematics means, the rotational kinematics ok. So, omega will we known to you and this omega this enters also into the dynamics of the system.

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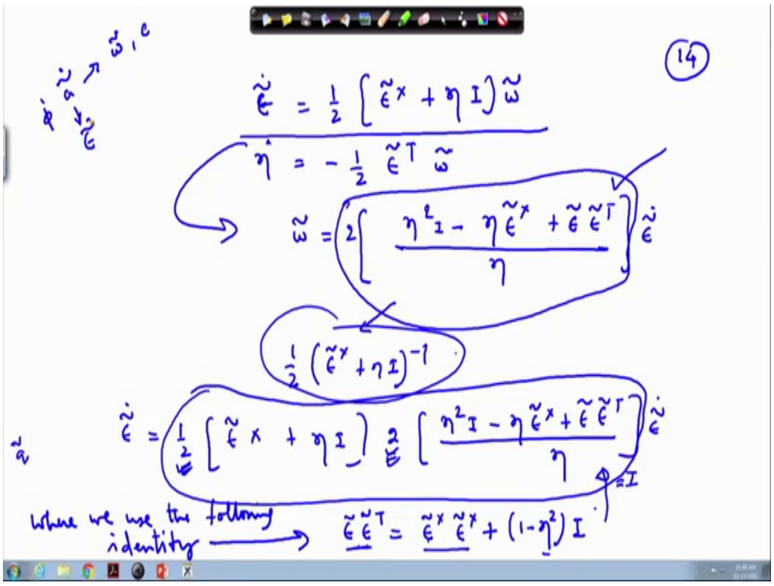

So, we have proved almost the required things; now once we are given this epsilon tilde dot this equal to 1 by 2 we go back here, here this should turn out to be plus. So, we have to just look for this plus a tilde a tilde cross a tilde a cross a equal to a tilde trans minus I ok. So, these quantity if we write it like this.

So, this equal to a transpose a transpose minus a transpose a cross this equal to I and here we have here this quantities already plus we are taking this a transpose a tilde times a tilde transpose. So, this is with plus sign so, this is minus sign so, that goes here so, this is a tilde cross here this comes with the plus sign. So, here we have the plus; now with these 2 equations we can get back omega tilde for that.

So, epsilon tilde dot this equal to 1 by 2 times epsilon tilde cross plus eta times I this is 1 equation we have and eta dot we have written as minus 1 by 2 times eta tilde transpose times omega tilde. So, if we invert this for inverting it is a very long procedure and, but it can be inverted and the simple thing will be justify the once I write this I will tell you how to do this you can verify it.

So, the quantity which is appearing here this quantity, this is nothing but inverse of 1 by 2 times epsilon tilde cross plus eta times I. So, if this is inverse of this quantity ok. So, if we take the product of this so, that must get reduce to an identity form. That means, that suppose that we write it like this epsilon tilde this equal to 1 by 2 epsilon tilde cross plus eta I times omega tilde. So, omega tilde just replace from this place. So, this is 2 times eta square I minus eta times epsilon tilde cross divided by eta.

So, this 2 2 will this 2 and this 2 will cancel out and if you take the product of this quantity on the here we have the epsilon tilde dot. So, it will get reduce to identity. So, this whole part will be equal to an identity. So, and you will get here epsilon dot equal to epsilon dot. So, which will be then which is the correct thing ok. So, this indicates the inverse of this matrix which we have not going to do here you just you can try it yourself because, this is not very difficult. Its simple and already we have done we have carried out proved many identities.

So, those identities you can utilize and solve this problem ok. So, where we use the following identity? So, what is that identity? So, this is epsilon tilde times epsilon tilde transpose this will be equal to epsilon tilde cross epsilon tilde cross plus 1 minus eta square I had it been a a a a here in this place. So, this quantity would have been equal to 1 and we would have got just I here, but here this is epsilon which is the Euler parameter these are Euler parameter or the quaternion's ok. For the quaternion we use the notation q tilde which we have not using here. So, if we use this identity so, then you will be able to prove that this quantity is equal to I means, this is just an identity matrix ok.

So, far we have looked into the kinematics and we will be able to if you use these results whatever we have derived you can go forth back on forth from the quaternion notation to the Euler notation. And Euler notation to the quaternion notation means from phi dot a tilde dot we can go into the omega tilde and c or and back on forth we can just keep jumping from one place to another place. So, as and when what is required we can prove it. So, of instead of a tilde here we are using the epsilon a tilde for this particular part the quaternion part.

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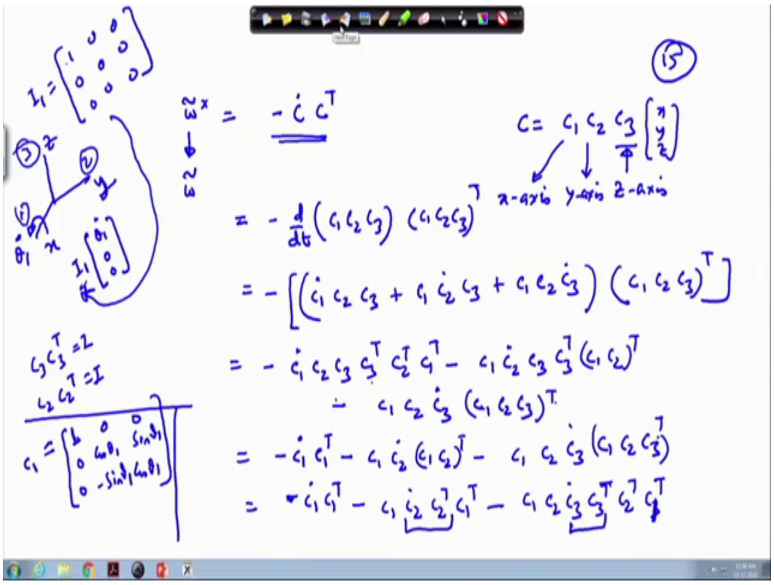

Now, few of things which will come in the future also, but right now I want to target it because this is the right time to discuss about this omega tilde cross this equal to minus c dot c transpose. And from here what I am trying to do that I will try to get this omega tilde from this place. So, if this quantity is known you can get this and so on ok. So, now looking on the right hand side let us suppose that c we right as c 1 c 2 c 3 means on any vector if we have the vector x y z. So, this is first is a being rotated about the z axis this is the z axis rotation then this is being rotated about the y axis and then the rotation is being given about the x axis.

So, if we give this rotation so, this gets reduce to minus d by d t c 1 c 2 c 3 and c 1 c 2 c 3 transpose and if we expanded c 1 c 2 dot c 3 plus c 1 c 2 times c 3 dot and here again we have c 1 c 2 c 3 transpose and we can break it and will get 3 terms. So, minus c 1 dot c 2 times c 3, if we multiplied with this; so, this will be c transpose take the transpose of this c 2 transpose c 1 transpose minus c 1 times c 2 dot times c 3 times c 3 transpose c 1 times c 2 transpose minus ok.

So, this quantity this is identity if we look c 3 times c 3 transpose this is the rotation matrix. So, this is identity so, we have c 3 times c 3 transpose this equal to I c 2 times c 2 transpose this equal to I. So, the first one gets reduce to minus c 1 dot times c 1 transpose, the second term here will be c 1 times c 2 dot this gets reduce to identity.

So, this is c 1 times c 2 transpose minus c 1 c 2 c 3 dot c 3 transpose. Now we use some results like if we look into the c 1 dot. So, c 1 dot is nothing, but your c 1 is nothing, but your let us first take c 1. So, this is c 1; c 1 is nothing, but your rotation about the x axis means 1 0 0 0 0 and here if you are writing in terms of theta 1. So, this will be cos theta 1 cos theta 1 sin theta 1 minus sin theta 1 ok.

So, this is rotation about the x axis. So, that this implies if we write the same thing in terms of means you have x y and z. So, you are rotating about this axis by this is your theta 1 dot along this you are rotating by theta 1. So, this theta 1 dot is pointed along this direction and therefore, the c 1 dot it can be written as c 1 dot is nothing but I 1 times theta 1 dot 0 0. There I 1 is what this I 1 quantity is we will continue to write here in the I 1 is 0 what is this I 1 is 1 0 0 0 0 0 if you multiply this with this one.

So, you can see that only theta 1 will result ok. So, I 1 picking up from this place and multiplying by these vectors; so, this will just result in theta 1 dot. So, this is a vector which is lying along the first axis here we have written as x y z, but basically this is the 1st axis the 2nd axis and this is the 3rd axis. So, c 1 dot we can replace in terms of. So, what we are going do now we utilize the result if you remember that we will have to go to the next page there is no other way here.

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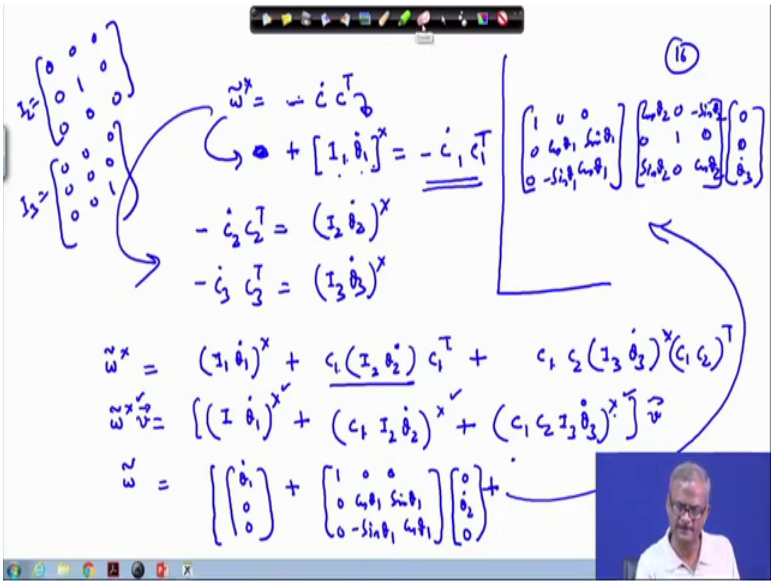

If you remember that omega tilde dot we have written as going back this is omega tilde cross this omega tilde cross omega tilde cross equal to minus c times c transpose. So, following c dot times c transpose. So, following this notation what we will have this is the rotation about the first axis ok. So, this will get reduced to theta 1 dot this quantity then the left hand side we are writing here this equal to minus c we have using 1 c 1 transpose with minus c.

So, for we will write it like this plus sin on the left hand side. So, I 1 times theta 1 dot we know that omega is directed along the one direction and put here the cross. So, this is the quantity. So, minus c 1 dot c 1 transpose we can replace by this, in the same way we can replace the other quantities like c 2 dot times c 2 transpose this will be equal to I 2 times theta 2 dot cross and minus c 3 dot times c 3 transpose this will be I 3 times theta 3 dot cross we basically we are using this equation. So, this helps us to reduce the previous calculation in terms of theta 1 dot theta 2 dot theta 3 dot. So, going into going back into the original equation we have been writing.

So, omega tilde cross this equal to look back minus c 1 times c 1 dot times c 1 transpose. So, this quantity is; so, will pick up the first quantity this one. So, from this place this is I 1 times theta 1 dot cross next quantity this is c 1 times c 2 dot and similarly we need to break it of this quantity let us expand it here c 1 transpose c 2 dot c 2 transpose times c 1 transpose minus c 3 dot this is whole transpose.

So, c 3 transpose c 2 transpose c 1 transpose so, this quantity is we are going to replace. So, c 1 c 1 here and in mid between we have this quantity. So, minus sign will absorb with this. So, we get here plus c 1 times I 2 times theta 2 dot times c 1 transpose. Then the next term we have c 1 times c 2 times similarly c 3 and c 3 dot with minus sign we will absorb here this will be written like this.

Now this quantity it can be written as c 1 times I 2 times theta 2 dot cross. Similarly this quantity can be written as c 1 times c 2 times I 3 times theta 3 dot cross when this you can check what we are derive the earlier identities. Now from here we can put it into the format like the first one then gets reduce to already we have discuss about this. So, each of them can be reduce after removing this cross sin.

So, these are the 3 cross sin on the right hand side and this is the cross sin here. So, if we eliminate this so, we can write this as omega omega because, this is operating on certain vector. So, here if suppose if you operate it on vector nu. So, here also I will have to operate on vector nu. So, this will be omega this cross sin can be taken out of the bracket and we can be element all together this cross sin and nu from both side. So, omega tilde gets reduce to I times theta 1 dot. So, already we have written this will be this quantity plus c 1 times I 2 times theta dot.

So, cross already its gone and c 1 is the quantity which we have written as 1 0 0 0 0 cos theta 1 sin theta 1 and then sin theta 1 with minus sign cos theta 1 and here this quantity is 0 theta 2 dot times 0 and plus this quantity here. So, if we if so, we cannot write here this third part, we will write it here in this place 1 0 0 cos theta 1 cos theta 1 minus sin theta 1 sin theta 1 and this rotation is about 2 so, second axis. So, this is 1 0 0 cos theta 2 minus sin theta 2 and sin 2 and cos theta 2 and then this is being operated on theta 3 dot.

So, this is I times theta 3 dot so, I times theta 3 dot is nothing, but 0 0 theta 3 dot. Similarly the I 2 nothing, but 0 this I 2 0 0 0 1 0 and I 3 its 0 0 0 this is just knowing and the I 1 already we have written. So, if you now reduce this quantity so, omega tilde will get reduce to now we have to multiply multiplying and simplifying.

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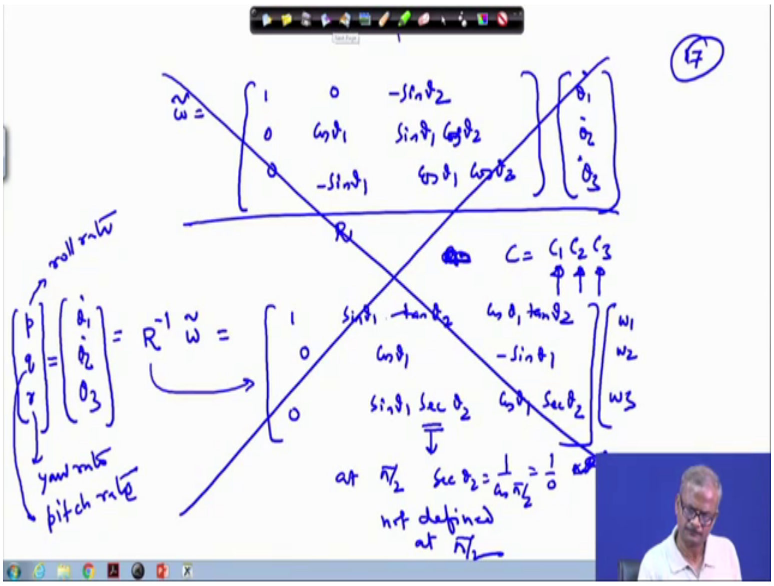

So, omega tilde can be written as 1 0 0 cos theta 2 cos theta 1 times cos theta 2 that is we have combined this term this term and this term together. So, first what you have to do first you have to take this product after we have taken the product. So, the whole product will come in terms of theta 1 dot theta 2 dot theta 3 dot and then again that product has to be the whole term has to be written in terms of the matrix as we have written here so, you can break it. So, this excise you can do it yourself ok.

So, remember that this is the result of the rotation where we have indicated this rotation by c equal to c 1 c 2 c 3 means first rotation about the z axis then about the y axis and then about the x axis. If you change the order of this rotation so, this matrix is also going to change it will not remain the same ok. So, and we will see that in the future the same thing can be done in other way also just from the geometric, we can do it here just we have use the mathematics. And purely from the geometrical point of view we can deal with it if we take the inverse of this; let us say this matrix is we write this as a matrix as the R matrix, the rotation matrix.

So, R inverse times omega tilde this will give you the theta 1 dot theta 2 dot and theta 3 dot and it’s a quit often very common in the aircraft at least p q r to write this as means the about the first axis y or the x axis the rotation by p the y axis rotation by q. And, this is the R axis rotation by R this is call the roll rate this is call the yaw rate and this particular 1 is equal to the pitch rate. Here in the for the space craft it will be convenient to work this theta 1 theta 2 theta 3.

Because, this notation once we got defined it will not be confusing and this can be written as once sin theta 1 times this times tan tan theta 2 then cos theta 1 times tan theta 2 cos theta 1 times sin theta 2. And you can see that this sin theta 2 this is at phi by 2 this sin theta 2 becomes 1 by cos phi by 2 means 1 by 0. So, this is not defined this is not defined at phi by 2. So, this inverse; inverse matrix this is the R inverse this is not defined that theta equal to theta 2 equal to phi by 2.

So, this is the similarity faced ok. So, here we have omega 1 omega 2 omega 3 we converted it in from. This is we have done written the wrong notation here so, we need to correct it [FL] cut [FL] cut [FL] notation [FL].

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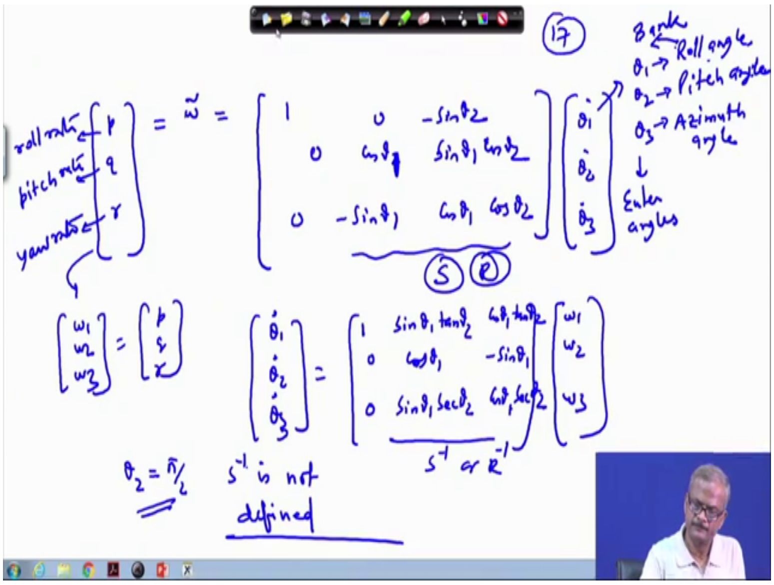

Omega tilde this we have 1 0 minus sin theta 2. So, once we combine the matrix ok. So, this can be written as like this 0 cos theta 1 sin theta 1 times cos theta 2 and this is 0 minus sin theta 1 times cos theta 1 times cos theta 2 and times theta 1 dot theta 2 dot and theta 3 dot.

So, the theta 1 theta 2 what they are appearing here; so, theta 1 theta 2 and theta 3 these are the Euler angles and on the left hand side what is appearing as omega 1 omega 2 omega 3 this is nothing, but your p q r ok. So, these we are writing as this is nothing, but the roll rate this is the pitch rate and this is the common notation for this ok. And this is the yaw rate terminology used for this p q r or the same thing you also you can write as omega 1 omega 2 omega 3 equal to p q r; well on the right hand side these are the Euler angles.

So, this is call the assumed angle this one is call the pitch angle and this one its call the pitch and this is the roll angle roll pitch and azimuth angle or the proper notation for this is the bank angle for the case of air craft. So, of here in this case if you want you can just say that this is the roll pitch and yaw angle, but this is not correct term for that. So, correct term is azimuth angle pitch angle and the bank angle and if we invert this. So, from here we can get this as theta 1 dot theta 2 dot theta 3 dot in terms of omega 1 omega 2 omega 3 and this inversion is left to you to do it yourself in the future again we are going to deal with this equations cos theta 1 times tan theta 2 and 0 cos theta 1 minus sin theta 1 0.

Say theta 2 times cos theta 1 times sin theta 2 theta 2 equal to phi by 2 S inverse is this is nothing but if you write this as S or R. So, this indicates S inverse or R inverse ok. So, S inverse is not defined because at this time this blows up it goes to infinity 1 by 0 only this is cos sin theta 2. So, 1 by 0 there that becomes infinity ok. So, we have covered the kinematics of the rotational part little bit thing that is remaining.

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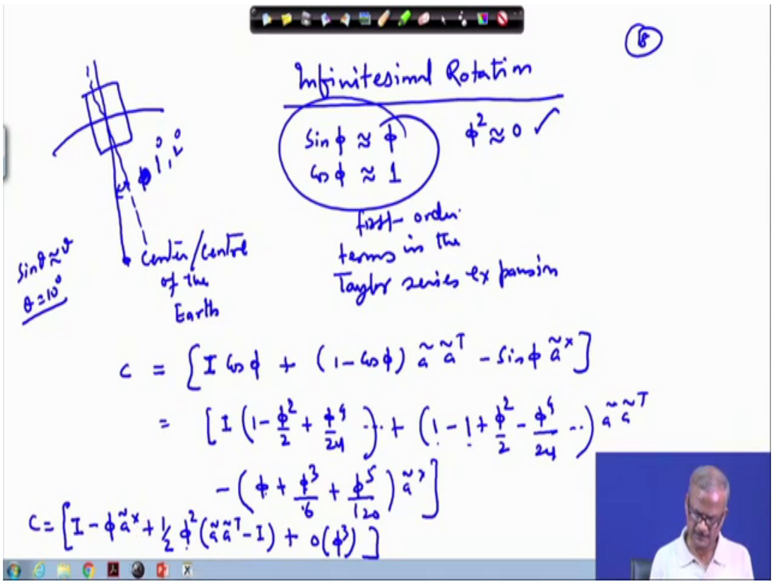

That is about the infinitesimal rotation; this implies that the cases we are sin phi can be approximated as phi and cos phi will be approximated as 1 ok. And if we have the phi of square term so, these will approximated as 0 means only these are the first order term we will consider first order terms in the Taylor series expansion and we delete the higher order terms of.

So, there are cases we are the say the satellite is in the orbit and this is what is required that you point over the center of the earth center both way we can write of the earth ok. So, you need to precisely point the satellite towards the center of the earth, but is so, happens that your satellite is just pointing say away by 1 degree; 1 degree or 2 degree something like this and you want to do the precise control.

So, at that term time you can go for this kind of notations though it. So, happen that sin theta this can be approximated as theta till theta equal to 10 degree ok, 10 12 degree you can approximate but you have to just check that how much according to your need how much error your doing by assuming this kind of approximation.

So, we just quickly go through this part and may be in another 10 15 minutes we will wind it up. So, c equal to I cos phi this is the rotation matrix 1 minus cos phi a tilde times a tilde transpose. So, already we know that this rotation matrix if we rotate about vector a by angle phi. So, we now if we approximate it so, this gets reduce to I times you can see that in the Taylor series expansion this will be phi square by 2 plus phi to the power 4 by 24 and higher order terms.

And 1 minus 1 plus phi square divided by 2 minus phi to the power 4 by 24 and so on ok, times a tilde times a tilde transpose and this particular part. So, sin expansion then will be phi minus phi plus phi cube divided by 6 phi 5 divided by 120 times a tilde cross ok. So, if we ignore or either if we collect that terms.

So, you can see that this and this the cancel out and we can collect the term and if we collect and you write the things like this I minus phi times a tilde cross plus 1 by 2 times phi a square a tilde transpose minus I plus terms of order phi cube. So, justly you can see that if phi a square if I ignored it. So, this will be I minus phi times a cross.

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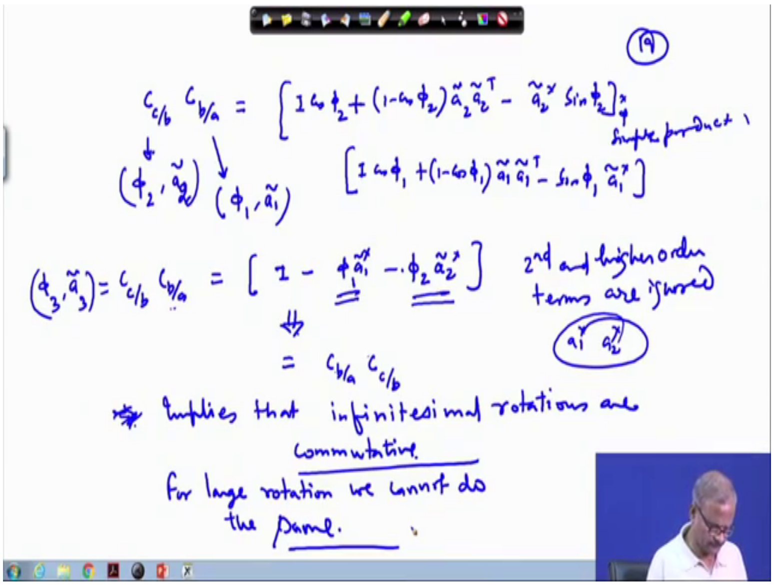

Now, if we have 2 rotations like; first rotation we are giving from a to b and the next rotation we are giving from say c to d though the notations we have used earlier I do not remember that a to b and perhaps e and something we have used, but right now we will go like this. So, this is a to b and b to sorry you will write this as b to c ok. So, this is c to b this rotation we will indicate by phi 2 and a 1 while this rotation will be indicated by this is a 2 phi 1 and a 1 tilde ok. So, if we multiply these 2 matrixes.

So, this will I times cos phi by 2 minus 1 minus cos phi 2 times a 2 tilde transpose this is plus here minus a 2 tilde cross times sin phi 2 and then we have to multiply this with this is not cost product this is just simple product. I times cross phi 1 plus 1 minus cos phi 1 a I tilde times a 1 tilde transpose minus sin phi 1 times a 1 tilde cross.

So, if we multiply it this will get reducing into the format it will come in this format phi times a 1 tilde cross phi 1 times and minus phi 2 times a 2 tilde cross ok. So, we are higher order terms 2nd and higher order terms order terms are ignored. So, this is c b e times c b a means, this is the equivalent rotation equal to phi 3 about certain vector say a 3 tilde. Already we have discuss about all these things so, if we look here in this part. So, once we have ignore the 2nd order terms. So, here there is no like the terms like a 1 cross times a 2 cross something like that so, those are non commutative term.

Here there is as such note on so, this implies this implies that this can also be written as c b slash a times c this is again here we have written it wrongly, we need to correct it this is c c slash b means a to b and then b is rotatant to the c position. So, this we can correct here also. So, this can be written as c b a and c c b means either you change the order of this matrixes, this comes first and this comes afterwards this will not get change, it will remain same.

Because, here the terms like a 1 cross and a 2 cross they are not present; if this terms are together then we can flip it, but here in this case flip it and this also implies. This implies that implies that infinitesimal rotations are commutative; that means, you can change the order of the rotation, but for large rotation you cannot do the same. For large rotation we cannot do the same then you will be doing the blunder if we look for the last part that is remaining.

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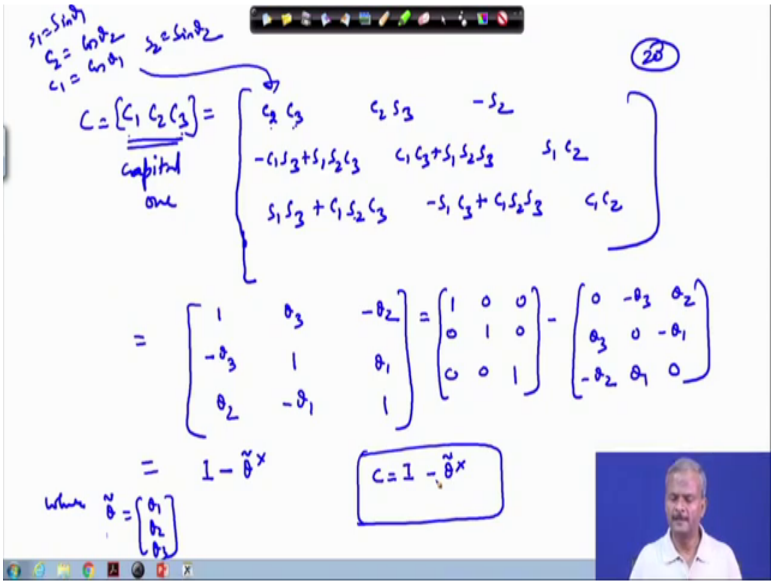

If we are looking for C 1 C 2 C 3 and we have the infinitesimal rotation. So, as already we have written this matrix. So, I will just write it first here in this place will write C 2 as see here this is small C 2 I am writing here this is capital bigger C 2 this is in the capital notation the capital once. And if I write this C 2 means this is cos theta 2 if I write C 1 this is means cos theta 1 similarly S 1 while indicate sin theta 1 and S 2 will indicate sin theta 2.

So, we can write here compactly. So, this is C2 C 3 these are the matrixes this C 1 C 2 C 3 are matrixes while these are scalars as per this then c 2 S 3 minus S 2 for infinitesimal rotation now you can see that already we have stated that if theta is a small. So, this can be cos is can be approximated as a 1. So, this term gives you 1, here this is 1 and this is theta 3 here minus theta 2 ok, here in this place this is theta 1 time theta 2 this is 2nd order term so, this is ignored this is 0

And therefore, this term gets reduce to 0 what about this term this is minus S 3 c 1 is 1. So, this is minus S S 3 is equal to minus theta 3 and this term again this term is 0 and this term is 1. So, here we get 1 about this term this is theta 1 and here this is this term is 0 c 1 and c 3 they are 1 1. So, this gets reduce to theta 2 and here if we check this is this term is 0 and this is minus theta 1 and this is 1 1. So, this get reduce to 1. So, you see that this one is here so, this can be written as 1 1 1 0 0 0, then it minus 0 minus theta 3 theta 2 then theta 3 minus sign will be taken outside 0 minus theta 2 theta 1 and this is 0. So, that gets reduce to I minus this is your is q symmetric matrix.

So, we use a notation for this as theta tilde cross this is not omega tilde here. So, this is because these are the angles ok. So, will writer in terms of theta tilde cross where theta tilde this equal to theta 1 theta 2 theta 3 ok. So therefore, the theta tilde cross this written in matrix notation. So, this will be nothing, but q symmetric matrix. So, what we get as for the finite infinitesimal rotation the rotation matrix gets reduce to I minus theta tilde cross and this is a very useful rotation this result which will be using while discussing the stability of the satellite.

So, thank you very much, we have finished the rotation and the kinematics of the rotation and next we are going start with the rigid body dynamics. So, next class onwards so, we will be continuing with the rigid body dynamics ok.

Thank you.