Satellite Attitude Dynamics and Control
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Lecture – 13
Rotational Kinematics (Contd.)

Welcome to the 13th lecture. So, we will continue with the previous lecture whatever we have been discussing.

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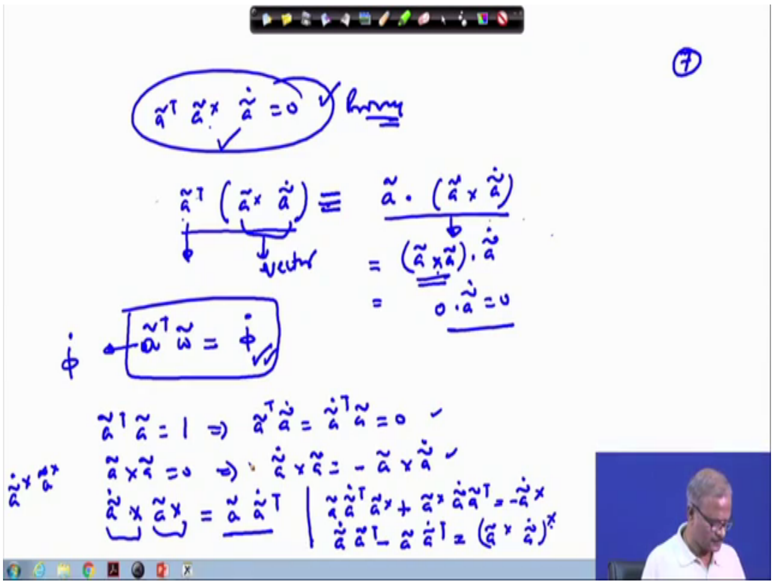

So, we need to prove this for proving this we can write this is equivalent to this we can write as a tilde see here this is a inner product, this is a vector, this is vector. And so this vector and this vector and here this is there is a transpose. So, that makes it a inner products. So, we can write this as a tilde dot times a tilde cross times a tilde dot. This can be rewritten this can be written as a tilde across a tilde dot. We take this dot product from here to here.

So, there will be exchange of this dot will become cross and this cross will become dot. Now if we use this so this part is now we; obviously 0. So, 0 times a tilde dot this will be equal to 0. So, what we see that this quantity is 0 and therefore, the previous expression we have been working with this gets reduced phi dot ok. So, the already we have written from the geometry that this would be equal to phi dot which in the turns out to be equal to phi dot ok.

So, we need a number of identities to prove few things and those identity are like last time we have discussed this is equal to 1. So, this imply a tilde times transpose times a tilde dot equal to a tilde dot transpose times a tilde this equal to 0. And then a tilde cross a tilde; obviously, this quantity will be 0 and this will imply a tilde dot cross a tilde equal to minus a tilde cross a tilde dot ok.

If we differentiate it the two terms will be differentiated right hand side is 0 taking on the right hand side we get this quantity. And then we have another identity a tilde dot cross times a tilde cross this usually we write as a tilde dot the cross is written off because this indicates a matrix similarly here this part indicates a matrix. So, this is a tilde cross of this part will be equal to a tilde times a tilde dot times transpose and this can be proved ok.

Similarly, we have a tilde times a tilde dot transpose times a tilde cross this can be proved. And lastly we have a tilde dot times a tilde transpose minus and all of them can be proved ok. So, it will be better that we do this as part of the tutorial otherwise I feel can we cover it today, let me try ok.

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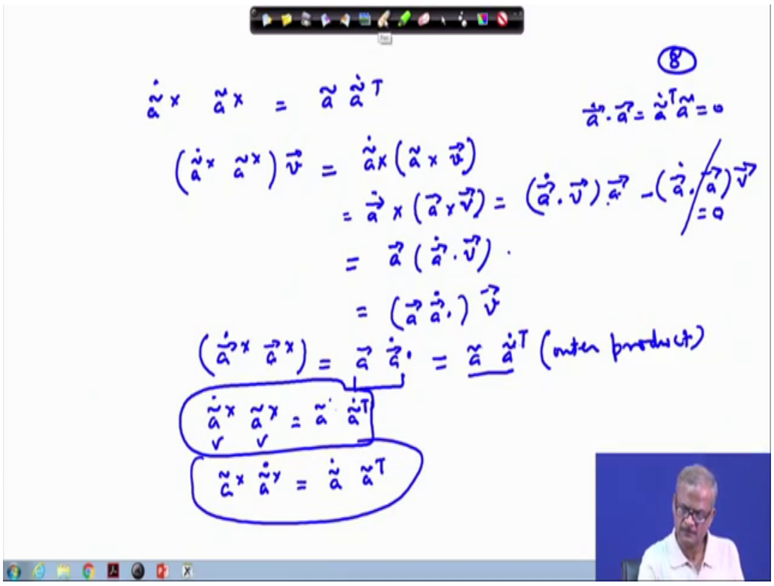

So, we will pick up this a tilde dot cross times a tilde cross this we need to prove that a tilde times a tilde dot transpose ok. So, for working this what the trick we can apply that just write this as let us assume that this is operating on a vector nu ok. The left hand side we have taken we have to prove that this is equal to the right. So, what it indicates that this quantity is equal to this is a tilde from here this is a tilde cross nu or either if you want to write it in the form of vector. So, equally you can write in terms of vector it is not a problem so; that means, that we can write this as a dot cross a cross a cross.

And then if we break it up a dot v times a minus a dot a, we write it in this way. This quantity we know that this quantity is 0 a dot a already we have because this is the a dot a this is nothing, but a tilde dot transpose times a tilde which we have already written as 0 ok. Now, we have to look in to this quantity. So, this we can write as a times a dot nu this is operating on nu vector, now remove from both side this is nu. So, this is a dot cross times a cross this will be equal to a dot and this quantity is nothing, but your a transpose a or in terms of if you write it a tilde transpose times a tilde and here we have dropped out this part.

So, we this part here this here means this place. So, this a times a dot and here this is the dot. So, a transpose so this quantity is equal to a tilde transpose times a tilde dot. So you can see that dot was here on the front one the first one it has gone to the second one here there was no dots, so dot is here. So, this way we have proved that a tilde dot cross a tilde cross this will be equal to a tilde transpose times a tilde dot. So, if we use this technique so we will be able to prove many such identities.

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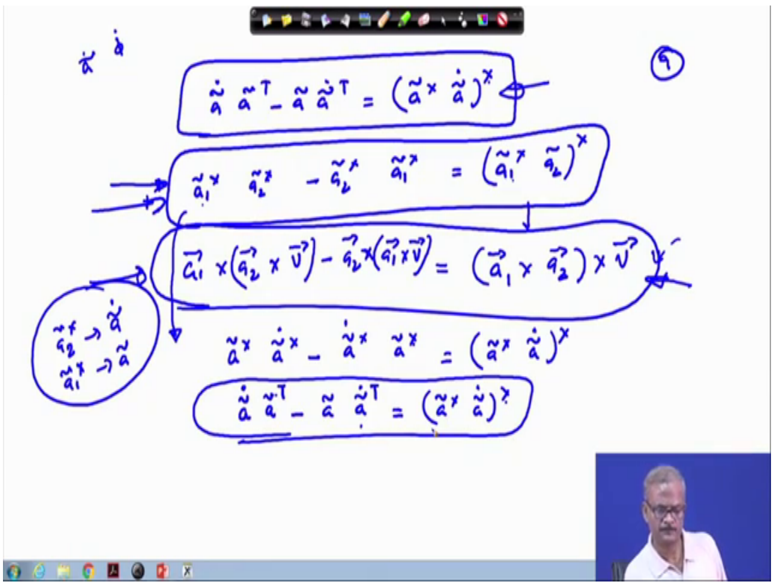

So, next we have the next identity we have a tilde dot times a tilde transpose minus actually this identities are required while, we deal with getting the reverse thing; means if we are looking to get a dot and phi dot from omega. So, at that time these things will be required. So, the next we need to prove this quantity and the way of doing is the same as we have done earlier, but here this particular expression it is a expansion of certain term that we need to write first; just [FL] [FL].

Now what we will look in to this is the identity we use. So, if you look into this particular part. So, this is nothing, but if you look in the vector notice on this is nothing, but a 1 times a 2 cross it is a something like this say if you are operating on vector v ok. So, right hand side like this and on the left hand side the same thing will appear as a 1 cross a 2 cross v minus a 2 cross a 1 cross v. So, this is an identity this is part of vector triple product ok.

So, we use this identity here to solve for this. So, if you look in to this if we replace a 1 cross by already previously what we have done that see here this is a 1 tilde cross and a 2 tilde cross ok. So, we will replace this a 2 tilde cross here by a tilde dot and a 1 tilde cross by a tilde. So, if we replace in this equation and then we see that what the results comes. So, here using this identity we can replace this by a 1 by a tilde. So, this a tilde cross times a tilde dot cross minus a tilde dot cross times a tilde cross and on the right hand side we will have a tilde cross.

And this part then we have a 2 is nothing but a tilde dot and then whole cross. So, if we go back and look in to this a tilde cross times a tilde this is turning out to be this. On the other hand if the same quantity is if you write it like this the things will get reversed and you will have a tilde dot it will appear like this again here we have done the mistake we need to correct it. This is a times we have already done this part.

So, here this is transpose, here this is dot ok. So, this is a times this is the outer product outer product. Again here also we need to correct it so this is our outer product ok. So, if we utilize this result so we can utilize it here so this will get reduce to a tilde dot times a tilde transpose minus a tilde transpose times a tilde dot. This dot will turn up this is transpose this will be equal to a tilde cross times a tilde dot cross. So now go back and what here in this places this is what we are looking for a tilde dot a tilde transpose a tilde times a tilde dot on the right hand side we have this ok.

So, this is an identity which we have proved using the vector triple product properties. This is another identity from the vector triple product we have just picked up and put here in this format ok.

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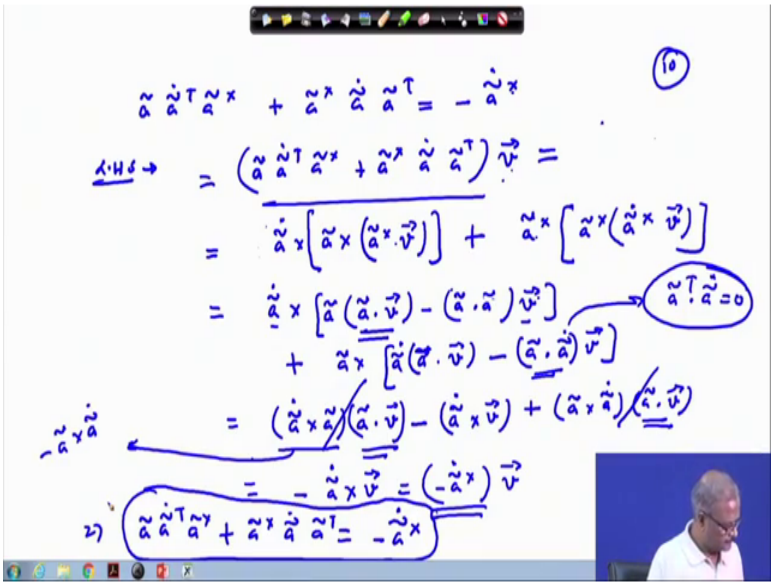

So, for lastly we have that a tilde times a tilde dot transpose a tilde this is the last identity that we need to prove ok. So, we are start in the same way that we take the left hand side first, and multiply on both side by we multiplied this by nu. So, we can write this as a tilde turns a tilde transpose a tilde transpose and this operates on this vector nu equal to minus a tilde dot cross nu. And later on again we will take out nu from both the side and see the right hand side equal it is satisfied or not.

So, we will remove this part let us seeing that this is the left hand side we are operating on nu the vector nu and then re write this quantity. So, this quantity is nothing, but a tilde dot cross using the previous one we have derived a tilde cross times a tilde cross and this will operate on a tilde cross. So, this is say it is something like this and this is going to operate on this nu and plus the same way here a tilde cross and a tilde dot. So, this is a tilde cross times a tilde dot cross and this will be operating on this vector and then break this if we break it a tilde is a tilde dot nu times a tilde minus tilde transpose ok.

If we have done this is fine we have converted this so this is ok. Now this is a dot a and this our nu and plus a tilde cross a tilde dot nu times a tilde dot minus and then; obviously, attached to be operated by this cross product. So, what we get on here this is a scalar quantity. So, therefore, we can write as a tilde cross a tilde times a tilde dot nu minus a tilde this is your a tilde. Now this quantity is 1. So, here what we get just nu plus now one more thing here we are missing where we break the brackets. So, it will operate on this 1.

So, this will be a tilde cross nu times a tilde a tilde this is 1. So, a tilde dot cross nu this and this plus again we have to break here in this. So, a tilde cross a tilde dot ok. This and this we have taken here in this place this quantity is the nothing, but a tilde transpose times a tilde dot. So, this quantity is 0 already we have observed it ok. So this part gets reduce to 0 and this is what we have here this are the 3 parts we are having.

So, over all we have now we look in to this part this is this is both same these two are same and here the dot comes after words dot comes in front. So, this is nothing, but minus a tilde cross a tilde dot this quantity. So, this and this they cancel out and what we get here minus a tilde dot cross. So, this we can write as minus a tilde dot cross. Now remove from we operated on vector v. So, if we remove it. So, this is what we get.

So, from here this is equal to ultimately we can write here this equal to minus a tilde dot cross v ok. We can see that we can remove this v and this is what the result we were looking for so this is proved. I will remove it finally, not to confuse that we are putting in the beginning. So, therefore this implies a tilde times a tilde dot transpose a tilde cross plus this equals to minus a tilde dot cross. So, these are the some of the results which are very useful while reducing the matrices ok.

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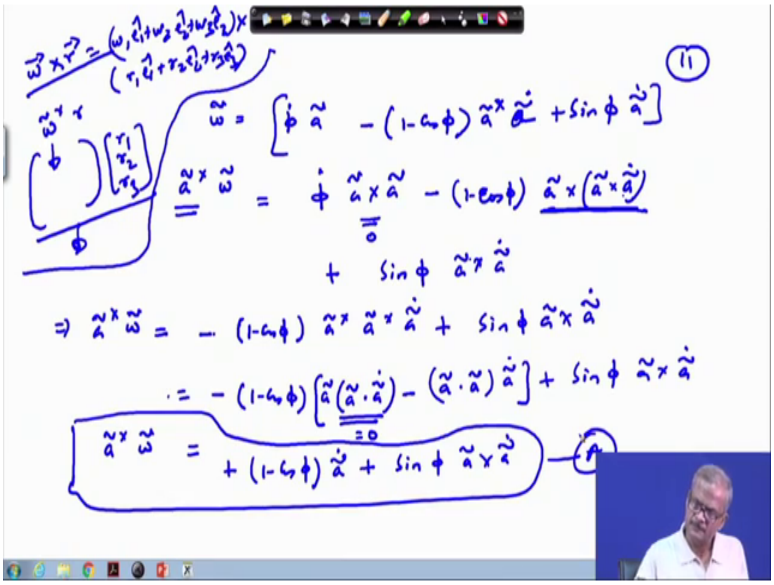

Already we have dot omega tilde this is nothing, but phi dot times a tilde minus 1 minus cross phi tilde cross times a tilde dot plus sin phi times a tilde dot. So, from here then we can get this phi dot and a tilde dot also. So, for this what we need to do that operate on both sides a tilde cross omega on both side we just operate while this a tilde cross. So, if we do that this quantity will be equal to 0, 1 minus cos phi times a tilde cross a tilde cross a tilde dot. This is the quantity here and plus sin phi times a tilde cross times a tilde dot.

So; obviously, this quantity here vanishes this implies 1 minus cos phi remember we can either write like this or either we can write like this ok. If we write it like this indicates that this is a matrix is skew symmetric matrix is skew symmetric matrix and this is operating on this vector. And if you are writing here in this format so we are just working with the vector notation and we can do the vector operation here. So, this is the difference if I write it up so that indicate that this is the matrix, if I write it in this way this indicates vector. So, here either way if you proceed result will be the same because they are the same thing like if I have omega cross r ok.

So, this is a vector product and the same thing I can write as omega tilde across r means here we are dealing with a skew symmetric matrix and then here we are writing r 1, r 2 r 3 or either write your in this format omega 1 say e one cap plus omega 2, e 2 cap plus omega 3 cap and then cross product with r 1 times e 1 cap plus r 2 times e 2 caps plus r 3 times e 3 cap. If you do this you will get a vector here also I will get a vector, but here you get here in the matrix format and here in the original vector format. But result of both of them will be identical that will not seen with different ok. So, it does not make difference whether we write it like this or either write it like this ok.

So, this part and then we have sin phi a tilde cross a tilde dot and we need to expand it further, so that it gets reduced into a simplified format this part we can expand. Now here if you look into this is in the vector triple form product format. So, we can write this as this is a tilde times a tilde dot and this is a tilde minus a tilde dot a tilde times a tilde dot this quantity is 0. This quantity equal to 0 therefore, this gets reduce to 1 minus cos phi and this minis minus and gets plus and this quantity is 1. So, this is a tilde dot plus sin phi so this is our a tilde cross omega tilde ok.

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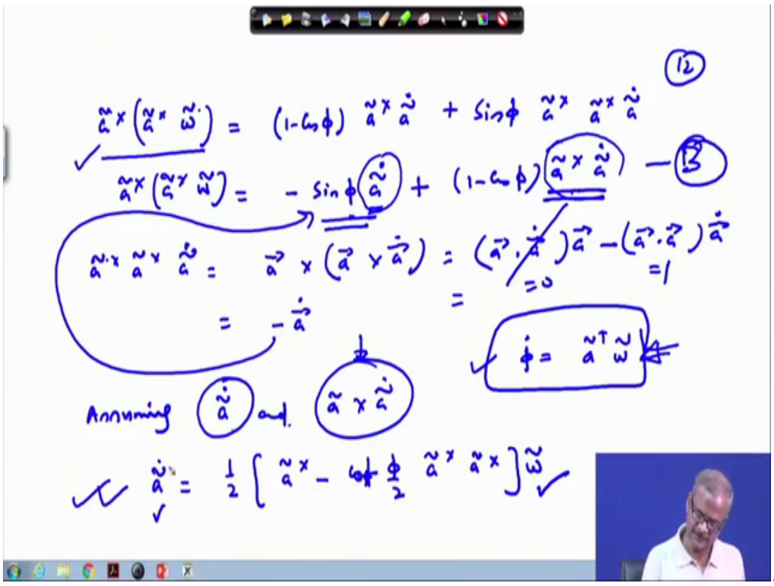

Same way this a tilde cross times omega tilde we have already done that again if you operate on this by this one a tilde cross. So, this will be one minus cos phi cos phi and the whatever the previous results we have got. So, a tilde cross times a tilde dot plus sin phi a tilde cross times a tilde [FL] ok. So, this is left hand side we have multiplied by this right hand side also gets multiplied by this. Now this part we can simplify and this part can be written as minus sin phi times a tilde dot and plus 1 minus cos phi times a tilde cross times a tilde dot. So, what we have done that a tilde cross a tilde dot.

So, this is nothing, but your a cross a cross a dot we have done earlier same thing a cross a cross a cross a dot this is what we have done earlier also now here this is a cross. So, we will do it here this quantity then this is a a dot a minus this quantity is 0. Because this is the unit product of yes we have already proved this only this parts this equal to 1. So, therefore, this is minus a dot minus a dot and therefore, this we can write from this place if use this here. So, this our things got simplified now this is the equation B. This is equal to a tilde cross a tilde cross omega tilde if we solve A and B for assuming phi dot and a tilde dot if we assuming a tilde dot and these are the two variables involved here a tilde dot and this one.

So, you assume this if you look here in this one this is your one variable, this is your another variable on the right hand side and go on the previous one here also if you look a tilde cross a tilde times a tilde cross dot and here a tilde dot. So, in both the places the in the both the places the same thing is appearing ok. So, if we solve it assuming a tilde dot and a tilde cross a tilde dot to be two variables and you remembered at this will be equal to minus a tilde dot times a tilde.

So, assuming these to be two variables and if we try to solve it we will get a tilde dot. This is the one of the variable which what we are looking for we are not looking for this one we are looking for this so a tilde dot this gets reduce to 1 by 2 a tilde a tilde cross this is known to us. Because this is vector about which we are giving rotation this is called phi by 2 times a tilde cross times a tilde cross and omega tilde.

So, from omega tilde then we are able to re try a tilde dot. So, vice versa we can go either from this direction to this direction or either from this to that direction and phi dot this will turn out to the a tilde transpose times omega tilde. This already we have done because omega is known. So, if we free multiplied by transpose this get reduced to phi dot this previously we have proved it. So, we are not repeating here in this place. So, this is another important conclusion. So, this can be utilize for reducing the quatrains or either the Euler rates to compact format which is very useful ok. So we will continue in the next lecture.

Thank you.